A Companion Reader to Polchinski's String Theory

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November 20, 2020

Introduction to these Notes

If you are anything like me, then you like to work out almost all the details when reading a physics text book. I find that this is the only way that I can really understand fully what I am reading. Of course, most text books cannot give all the calculation details, or they would be many times thicker than they already are.

From experience I have found that I understand things best if I force myself to write out everything in glorious detail. These Notes are a result of such an effort.

Joe Polchinski's two volume set on String Theory has become an instant classic on the subject. Like many other authors, Joe often only gives a brief sketch of a derivation - if at all, or he assumes that the reader already has a certain knowledge of the material. That may not always be the case. I set it up to myself to understand Joe's book in depth and work out as many details as possible. Other people have already published solutions to the exercises in the book, so I won't bother doing that, except when they are directly needed for an understanding of the main text. As an aside, it is also when you work out many of the details that you realise how well written the book is. In many cases you find sentences that seem innocuous, but that are, as you realise many pages later, not innocuous at all.

These notes are organised along the chapters of Joe's book. Per chapter the Notes are given per page and usually per equation. References to equations in Joe's book are given in round brackets, ( ). References to equations in these Notes are given in square brackets, [ ].

I claim no originality whatsoever in these notes, and even less correctness. All errors, and I am sure there are plenty of them, are entirely mine. Some open issues that I have not been able to resolve are summarised at the beginning of the relevant chapters, and detailed in the main text. If you want to help improve these Notes, either by correcting errors, changing, adding material, or answering open questions please contact me on hepnotes@hotmail.com. The latest update of these notes will also be available on my website, hepnotes.com.

But first and foremost, enjoy these Notes and enjoy Joe's book!

Stany M. Schrans

References Used

Obviously these notes are about Polchinski's two volume book on string theory [1, 2]. There are some other textbooks that may, at times, be useful to consult. Of course we start with the classic of Green, Schwarz and Witten, known as GSW [3, 4]. Zwiebach has written an introduction on string theory for advanced undergraduates [5]. It proceeds slowly and has a lot of details not often present in other textbooks. The book by Blumenhagen, Lüst an Theisen [6] is also a nice companion to consult as it often contains more details than Polchinski. There are two more books on String Theory that I need to mention as they tend to get a lot of attention. Becker, Becker & Schwarz [7] is geared towards more recent developments in string theory. It does contain some introductory material overlapping with Polchinski, but this is happening at a rather breakneck speed. If that book is at break-neck speed than the book by Kiritsis [8] is at relativistic speed. Its scope and detail is so vast that I consider it more an encyclopedia than a textbook; it covers so many topics that it does not have the time to explain too many things.

It has been said by some that you can learn string theory without any (or at least with very little) knowledge of quantum field theory. I strongly believe this is not true and a good understanding of field theory and the Standard Model is important. For this, I refer to two classics in the field: Peskin & Schroeder [9] is well worth a detailed study and Zee [10] is recommended for those with less patience. As an alternative I can offer my own notes on quantum field theory [11]; these are mostly based on Peskin & Schroeder, but with many details worked out and several additional subjects from a variety of sources.

Similarly, a reasonable understanding of general relativity is, in my opinion, necessary for a good understanding of string theory. For this my choices are Carroll's [12] and Zee's [13] books.

Lastly it is useful to have some background on more specific mathematical topics. For symmetry and group theory I recommend Zee once more [14]. For geometry and topology turn to Nakahara [15].

There are a number of articles and reviews that I have found useful as well. Polchinski has written a condensed version of his first volume [16]. It may be a condensed version, but it sometimes has explanations that clarify his Big Book. Tong's Notes on string theory [17] are very useful to read alongside Polchinski's book. These notes are very much based on the first half of Polchinski's first volume, but often contain more details or approach the subject from a slightly different angle. Ginsparg's Notes [18] on conformal field theory remain an absolute classic. It is also worth mentioning that there exists a solution manual [19] for about half the problems in Polchinski's book. I should also mention a couple of articles that have been very useful in understanding some specific parts of Polchinski's books. The article by Lüst and Skliros on handle operators [20], and in particular their section 2, has been of great help to understand Weyl transformations on Riemann surfaces, a topic discussed in chapter 3. The review article by Giveon, Porrati & Rabinovici on target space duality [21] has helped me understand better toroïdal compactification and its duality group in chapter 8. For a better understanding of the Born-Infeld term in the Lagrangian of a D-brane I have used the lectures of Szabo [22], and complemented this with the original article of Fradkin and Tseytlin [23].

Finally I should mention the Physics Stack Exchange at physics.stackexchange.com which is a great repository of questions and answers on physics and has active groups on quantum field theory, general relativity and string theory.

All the above references reflect, of course, my personal choices only. There are a variety of other sources that someone else may find better suited for her or his purpose.

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Chapter 1

A First Look at Strings

1.1 p 12: Eq. (1.2.15) The Variation of the Determinant of the Metric

Use

$$\ln\det M = \mathrm{tr}\ \ln M \qquad [1.1]$$

to write

$$\gamma^{-1}\delta\gamma = \delta\ln\gamma = \delta\mathrm{tr}\ \ln\gamma = \mathrm{tr}\ \delta\ln\gamma = \mathrm{tr}\ \gamma^{-1}\delta\gamma = \gamma^{ab}\delta\gamma_{ba} \qquad [1.2]$$

We have used the fact that $\left(\gamma^{-1}\right)_{ab} = \gamma^{ab}$. So $\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab}$. The second equation is obtained by using $\gamma^{ab}\gamma_{bc} = \delta_{ac}$ from which it follows that $\delta\gamma^{ab}\gamma_{bc} + \gamma^{ab}\delta\gamma_{bc} = 0$.

1.2 p 15: Eq. (1.2.32) The Change in the Curvature under a Weyl Rescaling

This is a formula that will come back several times, and it is quite rare to see it worked out in detail, so it is useful to do this here. We wish to show that under a local Weyl rescaling $g_{ab} \to g'_{ab} = e^{2\omega(\sigma)}g_{ab}$ the Ricci scalar satisfies

$$(g')^{1/2}R' = g^{1/2}(R - 2\nabla^2\omega) \qquad [1.3]$$

We have gone to Euclidean space and called the worldsheet metric g in stead of γ, just to save us some typing. One way to show this to write the Ricci scalar out in terms of the Riemann curvature, write that one out in terms of the connections and those in terms of the metric. We then transform the metric, make sure we don't get dizzy from all the terms, indices, and different contractions and hope this all works out. The other way is to be smart about it and ignore all terms we don't need, focussing on only what we do need.

Let us first recall some basic facts. The Ricci scalar is given by

$$\begin{aligned} R &= g^{ab} R_{ab} = g^{ab} R^c_{acb} \\ &= g^{ab} \left(\partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{cd} \Gamma^d_{ba} - \Gamma^c_{bd} \Gamma^d_{ca} \right) \end{aligned} \quad [1.4]$$

We have used the definition of the Riemann curvature

$$R^a_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} \quad [1.5]$$

The connection is given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) \quad [1.6]$$

When we replace the metric by $g'_{ab} = e^{2\omega(\sigma)} g_{ab}$ we have $g'^{ab} = e^{-2\omega(\sigma)} g^{ab}$ and the connection becomes

$$\begin{aligned} \Gamma'^a_{bc} &= \frac{1}{2} g'^{ad} \left(\partial_b g'_{cd} + \partial_c g'_{bd} - \partial_d g'_{bc} \right) \\ &= \frac{1}{2} e^{-2\omega} g^{ad} \left[\partial_b \left(e^{2\omega} g_{cd} \right) + \partial_c \left(e^{2\omega} g_{bd} \right) - \partial_d \left(e^{2\omega} g_{bc} \right) \right] \\ &= \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) + g^{ad} \left(g_{cd} \partial_b \omega + g_{bd} \partial_c \omega - g_{bc} \partial_d \omega \right) \\ &= \Gamma^a_{bc} + \Delta^a_{bc} \end{aligned} \quad [1.7]$$

where

$$\Delta^a_{bc} = g^{ad} \left(g_{cd} \partial_b \omega + g_{bd} \partial_c \omega - g_{bc} \partial_d \omega \right) \quad [1.8]$$

Let us now think, before we blindly start calculating. The Ricci scalar contains connections and their derivatives and these in turn contain derivatives of the Weyl factor ω. The $\sqrt{g}$ on both sides just makes sure that the $e^{2\omega}$ is overall cancelled. So, R' is an expression that will contain terms without ω's and terms with $\partial_n \omega, \partial_n \omega \partial_m \omega$ or $\partial_n \partial_m \omega$. The terms without any ω obviously combine to give $g^{1/2} R$ again, so it is the terms containing ω's that should carry our attention.

Now R is a scalar under diffeomorphism, as all its indices are nicely contracted. Therefore we should be able to write everything in terms of covariant derivatives of ω. A moment's thought reveals that there are only two possible combinations with at most a second order derivative, viz. $\nabla \omega \cdot \nabla \omega$ and $\nabla^2 \omega$. We should therefore be able to write

$$g'^{1/2} R' = g^{1/2} \left(R + a \nabla \omega \cdot \nabla \omega + b \nabla^2 \omega \right) \quad [1.9]$$

for some a and b that may depend on the metric and its derivatives, but not on ω. Let us now think about how we can fix these coefficients. We will do this for a general dimension D as we will need that formula later as well, and set $D = 2$ at the end.

Let first fix the coefficient b. We can write

$$\nabla^2\omega = \nabla_a\nabla^a\omega = \nabla_a\partial^a\omega = \partial_a\partial^a + \Gamma^a_{ab}\partial^b\omega = \partial^2\omega + \cdots \quad [1.10]$$

where the dots are for terms that do not contain any second order derivatives. We can thus just identify the part of R' with a second order derivative and that should give us the value of b. All the other terms then should fit in line, without us having to perform an actual calculation. Writing out R' in terms of the connections we find that

$$R'^a_{bcd} = \partial_c(\Gamma^a_{db} + \Delta^a_{db}) - \partial_d(\Gamma^a_{cb} + \Delta^a_{cb}) + (\Gamma^a_{ce} + \Delta^a_{ce})(\Gamma^e_{db} + \Delta^e_{db}) - (\Gamma^a_{de} + \Delta^a_{de})(\Gamma^e_{cb} + \Delta^e_{cd}) \quad [1.11]$$

Only the terms $\partial_c\Delta^a_{db}$ and $\partial_d\Delta^a_{cb}$ have a second derivative of ω. Therefore

$$\begin{aligned} R' &= g'^{ab}R'_{ab} = g'^{ab}R'^c_{acb} = g'^{ab}\left(\partial_c\Gamma'^c_{ba} - \partial_b\Gamma'^c_{ca} + \cdots\right) \\ &= e^{-2\omega}g^{ab}\left(\partial_c\Delta^c_{ba} - \partial_b\Delta^c_{ca} + \cdots\right) \end{aligned} \quad [1.12]$$

Now

$$\begin{aligned} g^{ab}\partial_c\Delta^c_{ba} &= g^{ab}\partial_c\left[g^{ce}(g_{ae}\partial_b\omega + g_{be}\partial_a\omega - g_{ba}\partial_e\omega)\right] \\ &= g^{ab}\left(\delta^c_a\partial_c\partial_b\omega + \delta^c_b\partial_c\partial_a\omega - g^{ce}g_{ab}\partial_c\partial_e\omega\right) \\ &= g^{bc}\partial_c\partial_b\omega + g^{ac}\partial_c\partial_a\omega - \delta^a_a g^{ce}\partial_c\partial_e\omega = (2-D)\partial^2\omega \end{aligned} \quad [1.13]$$

and similarly

$$\begin{aligned} g^{ab}\partial_b\Delta^c_{ca} &= g^{ab}\partial_b\left[g^{ce}(g_{ce}\partial_a\omega + g_{ae}\partial_c\omega - g_{ac}\partial_e\omega)\right] \\ &= g^{ab}\left(\delta^c_c\partial_a\partial_b\omega + \partial_a\partial_b\omega - \partial_a\partial_b\omega\right) = D\partial^2\omega \end{aligned} \quad [1.14]$$

Therefore

$$g'^{1/2}R' = e^{2\omega}g^{1/2}R' = e^{2\omega}g^{1/2}e^{-2\omega}(2 - D - D\partial^2\omega) + \cdots = -2(D-1)g^{1/2}2\partial^2\omega + \cdots \quad [1.15]$$

and we see that indeed $b = -2(D-1)$, and so for $D = 2$ we have indeed $b = -2$.

Let us next fix the coefficient a. Because ω is a scalar, we can write

$$a\nabla\omega\cdot\nabla\omega = a\partial^a\omega\partial_a\omega \quad [1.16]$$

We thus need to identify any terms that have a $\partial_a\omega\partial_b\omega$ in R'. These can only come from the terms that have a product of two connections:

$$\begin{aligned} R' &= g'^{ab}\left(\Gamma'^c_{ce}\Gamma'^e_{ba} - \Gamma'^c_{be}\Gamma'^e_{ca}\right) + \cdots \\ &= e^{-2\omega}g^{ab}\left(\Delta^c_{ce}\Delta^e_{ba} - \Delta^c_{be}\Delta^e_{ca}\right) + \cdots \end{aligned} \quad [1.17]$$

Now

$$\begin{aligned}
\Delta^c_{ce}\Delta^e_{ba} &= g^{cd}\big(g_{dc}\partial_e\omega + g_{de}\partial_c\omega - g_{ce}\partial_d\omega\big)g^{ef}\big(g_{fb}\partial_a\omega + g_{fa}\partial_b\omega - g_{ab}\partial_f\omega\big)\\
&= g^{cd}g^{ef}\big(g_{dc}g_{fb}\partial_e\omega\partial_a\omega + g_{dc}g_{fa}\partial_e\omega\partial_b\omega - g_{dc}g_{ab}\partial_e\omega\partial_f\omega\\
&\qquad + g_{de}g_{fb}\partial_c\omega\partial_a\omega + g_{de}g_{fa}\partial_c\omega\partial_b\omega - g_{de}g_{ab}\partial_c\omega\partial_f\omega\\
&\qquad - g_{ce}g_{fb}\partial_d\omega\partial_a\omega - g_{ce}g_{fa}\partial_d\omega\partial_b\omega + g_{dc}g_{ab}\partial_d\omega\partial_f\omega\big)\\
&= \delta^c_c\partial_a\omega\partial_b\omega + \delta^c_c\partial_a\omega\partial_b\omega - \delta^c_c g_{ab}\partial^d\omega\partial_d\omega + \partial_a\omega\partial_b\omega + \partial_a\omega\partial_b\omega - g_{ab}\partial^c\omega\partial_c\omega\\
&\quad - \partial_a\omega\partial_b\omega - \partial_a\omega\partial_b\omega + g_{ab}\partial^c\omega\partial_c\omega\\
&= 2d\partial_a\omega\partial_b\omega - d g_{ab}\partial^c\omega\partial_c\omega
\end{aligned} \tag{1.18}$$

Similarly

$$\begin{aligned}
\Delta^c_{be}\Delta^e_{ca} &= g^{cd}\big(g_{db}\partial_e\omega + g_{de}\partial_b\omega - g_{be}\partial_d\omega\big)g^{ef}\big(g_{fc}\partial_a\omega + g_{fa}\partial_c\omega - g_{ac}\partial_f\omega\big)\\
&= g^{cd}g^{ef}\big(g_{db}g_{fc}\partial_e\omega\partial_a\omega + g_{db}g_{fa}\partial_e\omega\partial_c\omega - g_{db}g_{ac}\partial_e\omega\partial_f\omega\\
&\qquad + g_{de}g_{fc}\partial_b\omega\partial_a\omega + g_{de}g_{fa}\partial_b\omega\partial_c\omega - g_{de}g_{ac}\partial_b\omega\partial_f\omega\\
&\qquad - g_{be}g_{fc}\partial_d\omega\partial_a\omega - g_{be}g_{fa}\partial_d\omega\partial_c\omega + g_{be}g_{ac}\partial_d\omega\partial_f\omega\big)\\
&= \partial_a\omega\partial_b\omega + \partial_a\omega\partial_b\omega - g_{ab}\partial^c\omega\partial_c\omega + \delta^c_c\partial_a\omega\partial_b\omega + \partial_a\omega\partial_b\omega - \partial_a\omega\partial_b\omega\\
&\quad - \partial_a\omega\partial_b\omega - g_{ab}\partial^c\omega\partial_c\omega + \partial_a\omega\partial_b\omega\\
&= (d+2)\partial_a\omega\partial_b\omega - 2g_{ab}\partial^c\omega\partial_c\omega
\end{aligned} \tag{1.19}$$

Therefore

$$\begin{aligned}
e^{-2\omega}g^{ab}(\Delta^c_{ce}\Delta^e_{ba} - \Delta^c_{be}\Delta^e_{ca}) &= e^{-2\omega}g^{ab}\big[2d\partial_a\omega\partial_b\omega - dg_{ab}\partial^c\omega\partial_c\omega\\
&\qquad - (D+2)\partial_a\omega\partial_b\omega + 2g_{ab}\partial^c\omega\partial_c\omega\big]\\
&= e^{-2\omega}g^{ab}(D-2)\big(\partial_a\omega\partial_b\omega - g_{ab}\partial^c\omega\partial_c\omega\big)\\
&= e^{-2\omega}(D-2)(1-D)\partial\omega\cdot\partial\omega
\end{aligned} \tag{1.20}$$

We thus find $a = -(D-1)(D-2)$ and this vanishes for $D = 2$, setting $a = 0$ in this case.

We have thus shown that for general D we have

$$(g')^{1/2}R' = g^{1/2}\big[R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial\omega\cdot\partial\omega\big] \tag{1.21}$$

For $D = 2$ we have

$$(g')^{1/2}R' = g^{1/2}(R - 2\nabla^2\omega) \tag{1.22}$$

which is (1.2.32).

1.3 p 15: Below Eq. (1.2.32) Invariance of χ under Weyl Rescaling

We first go to Euclidean space so we don't have to carry around minus signs. We wish to show that

$$\partial_a(\sqrt{g}\, v^a) = \sqrt{g}\nabla_a v^a \tag{1.23}$$

for an arbitrary four-vector v^a. Start with the LHS

$$\begin{aligned} LHS = \partial_a(\sqrt{g}\, v^a) = (\partial_a\sqrt{g})\, v^a + \sqrt{g}\partial_a v^a = \frac{1}{2\sqrt{g}}\sqrt{g}g^{bc}\partial_a g_{bc}v^a + \sqrt{g}\partial_a v^a \\ = \sqrt{g}\left(\frac{1}{2}g^{bc}\partial_a g_{bc}v^a + \partial_a v^a\right) \end{aligned} \tag{1.24}$$

For the RHS we find

$$\begin{aligned} RHS &= \sqrt{g}\,\nabla_a v^a = \sqrt{g}\left(\partial_a v^a + \Gamma^a_{ab}v^b\right) \\ &= \sqrt{g}\left[\partial_a v^a + \frac{1}{2}\left(g^{ac}\partial_a g_{cb} + g^{ac}\partial_b g_{ca} - g^{ac}\partial_c g_{ab}\right)v^b\right] \\ &= \sqrt{g}\left(\partial_a v^a + \frac{1}{2}g^{ac}\partial_b g_{ca}v^b\right) \end{aligned} \tag{1.25}$$

which is equal to the LHS.

Now the variation of χ in (1.2.31) after a Weyl rescaling is, and going back to Minkowski space,

$$\begin{aligned} \delta\chi &= \frac{1}{4\pi}\int_M d\tau d\sigma\,(-\gamma)^{1/2}(-2\nabla^2\omega) = -\frac{1}{2\pi}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\nabla_a(\nabla^a\omega) \\ &= -\frac{1}{2\pi}\int_M d\tau d\sigma\,\partial_a\big[(-\gamma)^{1/2}\nabla^a\omega\big] \end{aligned} \tag{1.26}$$

and is indeed a total derivative.

1.4 p 16: The Variation of the Einstein-Hilbert Action

We wish to compute the variation of the Einstein-Hilbert action

$$S_{EH} = \int d^2\sigma\,(-\gamma)^{1/2}\,R \tag{1.27}$$

under a change of metric. It is more convenient to consider a change δg^{ab} than δg_{ab}. The result should of course be equivalent. Using $R = g^{ab}R_{ab}$ we can write the variation as

$$\delta S_{EH} = (\delta S_{EH})_1 + (\delta S_{EH})_2 + (\delta S_{EH})_3 \tag{1.28}$$

with

$$
\begin{aligned}
(\delta S_{EH})_1 &= \int d^2\sigma\,(-\gamma)^{1/2}\,g^{ab}\delta R_{ab} \\
(\delta S_{EH})_2 &= \int d^2\sigma\,(-\gamma)^{1/2}\,(\delta g^{ab})R_{ab} \\
(\delta S_{EH})_3 &= \int d^2\sigma\,R\delta(-\gamma)^{1/2}
\end{aligned}
\qquad [1.29]
$$

Let us start with the first one. We first wish to calculate the variation of the Riemann curvature. We will first write it in terms of a variation of the connection $\delta\Gamma^a_{bc}$. We find

$$
\begin{aligned}
\delta R^a_{bcd} &= \delta\left[\partial_c\Gamma^a_{db} + \Gamma^a_{ce}\Gamma^e_{db} - (c\leftrightarrow d)\right] \\
&= \partial_c\delta\Gamma^a_{db} + \delta\Gamma^a_{ce}\Gamma^e_{db} + \Gamma^a_{ce}\delta\Gamma^e_{db} - (c\leftrightarrow d) \\
&= \partial_c\delta\Gamma^a_{db} + \delta\Gamma^a_{ce}\Gamma^e_{db} + \Gamma^a_{ce}\delta\Gamma^e_{db} - \partial_d\delta\Gamma^a_{bc} - \delta\Gamma^a_{de}\Gamma^e_{cb} - \Gamma^a_{de}\delta\Gamma^e_{cb} \\
&= \partial_c\delta\Gamma^a_{db} + \delta\Gamma^a_{ce}\Gamma^e_{db} - \delta\Gamma^a_{de}\Gamma^e_{cb} - \Gamma^e_{cd}\delta\Gamma^d_{be} - \left(\partial_d\delta\Gamma^a_{bc} - \Gamma^a_{ce}\delta\Gamma^e_{db} - \Gamma^e_{cd}\delta\Gamma^d_{be} + \Gamma^a_{de}\delta\Gamma^e_{cb}\right) \\
&= \nabla_c\delta\Gamma^a_{bd} - \nabla_d\delta\Gamma^a_{bc}
\end{aligned}
\qquad [1.30]
$$

In the fourth line we have added and subtracted $-\Gamma^e_{cd}\delta\Gamma^d_{be}$ and we have made use of the fact that the difference of two connections is a tensor, so that we can introduce the covariant derivatives of the tensors. Therefore

$$
\delta R_{bd} = \delta R^a_{bad} = \nabla_a\delta\Gamma^a_{bd} - \nabla_d\delta\Gamma^a_{ba} \qquad [1.31]
$$

and

$$
\begin{aligned}
(\delta S_{EH})_1 &= \int d^2\sigma\,(-\gamma)^{1/2}\,g^{bd}\,(\nabla_a\delta\Gamma^a_{bd} - \nabla_d\delta\Gamma^a_{ba}) \\
&= \int d^2\sigma\,(-\gamma)^{1/2}\,\left[\nabla_a(g^{bd}\delta\Gamma^a_{bd}) - \nabla_d(g^{bd}\delta\Gamma^a_{ba})\right]
\end{aligned}
\qquad [1.32]
$$

Where we have used the fact that the metric tensor has covariant derivative zero: $\nabla_a g^{bc} = 0$. We can now use $(-\gamma)^{1/2}\nabla_a v^a = \partial_a((-\gamma)^{1/2}\,v^a)$ to rewrite this as a total derivative, so that we see that this variation is equal to the boundary at infinity and hence zero[1]. Thus $(\delta S_{EH})_1 = 0$.

[1]We are ignoring potential global effects.

$(\delta S_{EH})_2$ is already of the form $\delta g^{ab} \times \text{something}$ so no further work is necessary. It remains to look at $(\delta S_{EH})_3$ .But here we can use (1.2.15):

$$\delta(-\gamma)^{1/2} = -\frac{1}{2}(-\gamma)^{-1/2}\delta\gamma = -\frac{1}{2}(-\gamma)^{-1/2} \times (-\gamma\gamma_{ab}\delta\gamma^{ab}) = -\frac{1}{2}(-\gamma)^{1/2}\gamma_{ab}\delta\gamma^{ab} \quad [1.33]$$

and thus

$$(\delta S_{EH})_3 = \int d^2\sigma\, (-\gamma)^{1/2} \left(-\frac{1}{2}\gamma_{ab}R\right) \delta\gamma^{ab} \quad [1.34]$$

We conclude that

$$\delta S_{EH} = \int d^2\sigma\, (-\gamma)^{1/2} \left(R_{ab} - \frac{1}{2}\gamma_{ab}R\right) \delta\gamma^{ab} \quad [1.35]$$

and so by varying the metric we do get the equations of motion

$$R_{ab} = \frac{1}{2}\gamma_{ab}R \quad [1.36]$$

1.5 p 16: Two-Dimensional Gravity has no Dynamics

It is very tedious to show that $R_{ab} = \frac{1}{2}\gamma_{ab}R$ in two dimensions. Joe's book claims that it follows from symmetry reasons, but it still requires detailed calculation. For example the expression the Ricci scalar is given by

$$\begin{aligned} R =& \Big[2g_{12}^2(\partial_2^2 g_{11} - 2\partial_1\partial_2 g_{12} + \partial_1^2 g_{22}) + g_{12}(-\partial_2 g_{11}(2\partial_2 g_{12} + \partial_1 g_{22}) \\ &+ \partial_1 g_{11}\partial_2 g_{22} + 2\partial_1 g_{12}(2\partial_2 g_{12} - \partial_1 g_{22})) + g_{11}(\partial_2 g_{11}\partial_2 g_{22} - 2\partial_1 g_{12}\partial_2 g_{22} + \partial_1 g_{22}^2) \\ &+ g_{22}(\partial_1 g_{11}(\partial_1 g_{22} - 2\partial_2 g_{12}) - 2g_{11}(\partial_2^2 g_{11} - 2\partial_1\partial_2 g_{12} + \partial_1^2 g_{22}) + \partial_2 g_{11}^2)\Big] / \\ &\Big[2(g_{12}^2 - g_{11}g_{22})^2\Big] \end{aligned} \quad [1.37]$$

It is actually easiest to show this via Mathematica. The corresponding code is shown in fig.1.1. It defines the connections, the Riemann curvature and Ricci scalar for an arbitrary two-dimensional metric. The test is that $tt[a, b] = 0$ for $a, b = 1, 2$.

```
In[721]:= Clear[G, dG, gu, g, dg, ddg, dgu, R, RR, m, mu, tt];
m = {{g11[x, y], g12[x, y]}, {g12[x, y], g22[x, y]}};
mu = Inverse[m];
g[a_, b_] := m[[a, b]]
gu[a_, b_] := mu[[a, b]]
dg[1, a_, b_] := D[g[a, b], x]
dg[2, a_, b_] := D[g[a, b], y]
dgu[1, a_, b_] := D[gu[a, b], x]
dgu[2, a_, b_] := D[gu[a, b], y]
ddg[1, 1, a_, b_] := D[D[g[a, b], x], x]
ddg[1, 2, a_, b_] := D[D[g[a, b], x], y]
ddg[2, 1, a_, b_] := D[D[g[a, b], y], x]
ddg[2, 2, a_, b_] := D[D[g[a, b], y], y]
G[a_, b_, c_] := (1/2) * Sum[gu[a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c]), {d, 2}]
dG[e_, a_, b_, c_] := Simplify[(1/2) * Sum[dgu[e, a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c])
+ gu[a, d] * (ddg[e, b, c, d] + ddg[e, c, b, d] - ddg[e, d, b, c]), {d, 2}]]
R[a_, b_, c_, d_] := Simplify[dG[c, a, d, b] - dG[d, a, c, b]
+ Sum[G[a, c, e] * G[e, d, b] - G[a, d, e] * G[e, c, b], {e, 2}]]
R[a_, b_] := Simplify[Sum[R[c, a, c, b], {c, 2}]]
RR = Simplify[Sum[ gu[a, b] * R[a, b], {a, 2}, {b, 2}]];
tt[a_, b_] := R[a, b] - (1/2) * g[a, b] * RR

In[740]:= {Simplify[tt[1, 1]], Simplify[tt[1, 2]], Simplify[tt[2, 1]], Simplify[tt[2, 2]]}
Out[740]= {0, 0, 0, 0}
```

Figure 1.1: Mathematica code for showing that two-dimensional gravity has no dynamics

1.6 p 17: Below Eq. (1.3.7) Determining p_+

We start from the mass-shell condition $-m^2 = p^2 = -2p^-p^+ + p^ip^i$. Thus $p^ip^i + m^2 = 2p^-p^+$. Use this in (1.3.6) to give

$$H = \frac{2p^-p^+}{2p^-} = p^- = -p^+ \qquad [1.38]$$

1.7 p 18: Eq. (1.3.9) Invariance of $f d\sigma$

We have $\sigma' = \sigma'(\sigma, \tau)$ and $\tau' = \tau$. Therefore

$$\begin{aligned}
\gamma_{\sigma\sigma} &= \frac{\partial \sigma'^c}{\partial \sigma}\frac{\partial \sigma'^d}{\partial \sigma}\gamma'_{cd} = \frac{\partial \sigma'}{\partial \sigma}\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\sigma\sigma} = \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'_{\sigma\sigma} \\
\gamma_{\tau\tau} &= \frac{\partial \sigma'^c}{\partial \tau}\frac{\partial \sigma'^d}{\partial \tau}\gamma'_{cd} = \frac{\partial \tau'}{\partial \tau}\frac{\partial \tau'}{\partial \tau}\gamma'_{\tau\tau} + 2\frac{\partial \tau'}{\partial \tau}\frac{\partial \sigma'}{\partial \tau}\gamma'_{\tau\sigma} + \frac{\partial \sigma'}{\partial \tau}\frac{\partial \sigma'}{\partial \tau}\gamma'_{\sigma\sigma} \\
&= \gamma'_{\tau\tau} + 2\frac{\partial \sigma'}{\partial \tau}\gamma'_{\tau\sigma} + \left(\frac{\partial \sigma'}{\partial \tau}\right)^2 \gamma'_{\sigma\sigma} \\
\gamma_{\tau\sigma} &= \frac{\partial \sigma'^c}{\partial \tau}\frac{\partial \sigma'^d}{\partial \sigma}\gamma'_{cd} = \frac{\partial \tau'}{\partial \tau}\frac{\partial \tau'}{\partial \sigma}\gamma'_{\tau\tau} + \frac{\partial \tau'}{\partial \tau}\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\tau\sigma} + \frac{\partial \sigma'}{\partial \tau}\frac{\partial \tau'}{\partial \sigma}\gamma'_{\sigma\tau} + \frac{\partial \sigma'}{\partial \tau}\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\sigma\sigma} \\
&= \frac{\partial \sigma'}{\partial \sigma}\gamma'_{\tau\sigma} + \frac{\partial \sigma'}{\partial \tau}\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\sigma\sigma} \qquad [1.39]
\end{aligned}$$

So

$$\begin{aligned}
\det\gamma &= \gamma_{\sigma\sigma}\gamma_{\tau\tau} - \gamma_{\tau\sigma}^2 \\
&= \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'_{\sigma\sigma}\left[\gamma'_{\tau\tau} + 2\frac{\partial \sigma'}{\partial \tau}\gamma'_{\tau\sigma} + \left(\frac{\partial \sigma'}{\partial \tau}\right)^2 \gamma'_{\sigma\sigma}\right] \\
&\quad - \left[\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\tau\sigma} + \frac{\partial \sigma'}{\partial \tau}\frac{\partial \sigma'}{\partial \sigma}\gamma'_{\sigma\sigma}\right]^2 \\
&= \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'_{\sigma\sigma}\gamma'_{\tau\tau} + 2\left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \frac{\partial \sigma'}{\partial \tau}\gamma'_{\sigma\sigma}\gamma'_{\tau\sigma} + \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2\left(\frac{\partial \sigma'}{\partial \tau}\right)^2 \gamma'^2_{\sigma\sigma} \\
&\quad - \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'^2_{\tau\sigma} - \left(\frac{\partial \sigma'}{\partial \tau}\right)^2\left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'^2_{\sigma\sigma} - 2\left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \frac{\partial \sigma'}{\partial \tau}\gamma'_{\tau\sigma}\gamma'_{\sigma\sigma} \\
&= \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \left(\gamma'_{\sigma\sigma}\gamma'_{\tau\tau} - \gamma'^2_{\tau\sigma}\right) = \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \det\gamma' \qquad [1.40]
\end{aligned}$$

We thus find, using the fact that $d\tau = d\tau' = 0$,

$$\begin{aligned}
f d\sigma &= \gamma_{\sigma\sigma}(-\det\gamma)^{-1/2} d\sigma \\
&= \left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \gamma'_{\sigma\sigma}\left[-\left(\frac{\partial \sigma'}{\partial \sigma}\right)^2 \det\gamma'\right]^{-1/2} \frac{\partial \sigma}{\partial \sigma'} d\sigma' = \gamma'_{\sigma\sigma}(-\det\gamma')^{-1/2} d\sigma' \qquad [1.41]
\end{aligned}$$

1.8 p 18: Below Eq. (1.3.9) Fixing the Gauge

I feel this may need a bit more explanation. We have shown that $f d\sigma$ is an invariant under reparametrisations of σ with τ kept fixed. So for every given τ we can use this to define an invariant length $dl = f d\sigma$ along the string. This is independent of the choice of σ as long as τ is fixed. So we can now select a specific worldsheet coordinate system. We don't change τ but we define σ as being the proportional to the invariant length $\int f d\sigma$ from one of the endpoints of the open string[2]. The proportionality constant is determined by requiring that at the other end-point $\sigma = \ell$. In this coordinate system f is independent of σ. $f = dl/d\sigma$ and dl is an invariant length hence it is independent of σ, but f can of course still depend on τ, i.e. $f = f(\tau)$. We now use the Weyl invariance to rescale the metric so that $\gamma = -1$. Now f is invariant under a Weyl rescaling as well, as both $\gamma_{\sigma\sigma}$ and $(-\gamma)^{1/2}$ transform in the same way, and so their ratio is invariant as well. Because f is invariant, this means that after the Weyl rescaling we still have $\partial_\sigma f = 0$. But this means that $0 = \partial_\sigma \left(\gamma_{\sigma\sigma}(-\gamma)^{-1/2})\right)$ and thus $\partial_\sigma \gamma_{\sigma\sigma} = 0$ as $-\gamma = 1$.

Let's recapitulate. We fix τ by setting it equal to x^+. We fix σ by defining it to be proportional to the invariant length. We fix γ using a Weyl rescaling. Combining these, we have shown that that we can satisfy $\partial_\sigma \gamma_{\sigma\sigma}$ and so this is an acceptable gauge choice.

1.9 p 18: Eq. (1.3.10) Invariance of $f d\sigma$

From $-1 = \gamma = \gamma_{\tau\tau}\gamma_{\sigma\sigma} - \gamma_{\tau\sigma}^2$ we get

$$\gamma_{\tau\tau} = \frac{\gamma_{\tau\sigma}^2 - 1}{\gamma_{\sigma\sigma}} \qquad [1.42]$$

It is easily checked that

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \qquad [1.43]$$

In our case $ac - b^2 = \gamma = -1$. Thus

$$\begin{pmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\tau\sigma} & \gamma^{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} \gamma_{\tau\tau} & \gamma_{\tau\sigma} \\ \gamma_{\tau\sigma} & \gamma_{\sigma\sigma} \end{pmatrix}^{-1} = \gamma^{-1} \begin{pmatrix} \gamma_{\sigma\sigma} & -\gamma_{\tau\sigma} \\ -\gamma_{\tau\sigma} & \gamma_{\tau\tau} \end{pmatrix} = \begin{pmatrix} -\gamma_{\sigma\sigma} & \gamma_{\tau\sigma} \\ \gamma_{\tau\sigma} & (1 - \gamma_{\tau\sigma}^2)/\gamma_{\sigma\sigma} \end{pmatrix} \qquad [1.44]$$

[2]For closed strings, there is no end-point, so we would have to chose a specific starting point on the closed string, see (1.4.1).

1.10 p 18: Eq. (1.3.11) The Lagrangian in the Light-Cone Gauge

It is important here that we talk about the Lagrangian and not the Lagrangian density. But let us start with the latter

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4\pi\alpha'}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= -\frac{1}{4\pi\alpha'}\Big(-\gamma^{ab}\partial_a X^+\partial_b X^- - \gamma^{ab}\partial_a X^-\partial_b X^+ + \gamma^{ab}\partial_a X^i\partial_b X^i\Big) \\ &= -\frac{1}{4\pi\alpha'}\Big(-\gamma^{\tau\tau}\partial_\tau X^+\partial_\tau X^- - \gamma^{\tau\sigma}\partial_\tau X^+\partial_\sigma X^- - \gamma^{\sigma\tau}\partial_\sigma X^+\partial_\tau X^- - \gamma^{\sigma\sigma}\partial_\sigma X^+\partial_\sigma X^- \\ &\qquad - \gamma^{\tau\tau}\partial_\tau X^-\partial_\tau X^+ - \gamma^{\tau\sigma}\partial_\tau X^-\partial_\sigma X^+ - \gamma^{\sigma\tau}\partial_\sigma X^-\partial_\tau X^+ - \gamma^{\sigma\sigma}\partial_\sigma X^-\partial_\sigma X^+ \\ &\qquad + \gamma^{\tau\tau}\partial_\tau X^i\partial_\tau X^i + 2\gamma^{\tau\sigma}\partial_\tau X^i\partial_\sigma X^i + \gamma^{\sigma\sigma}\partial_\sigma X^i\partial_\sigma X^i\Big)\end{aligned} \qquad [1.45]$$

We now use $X^+ = x^+ = \tau$ so that $\partial_\tau X^+ = 1$ and $\partial_\sigma X^+ = 0$ and use the explicit form of the inverse metric

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4\pi\alpha'}\Big(-2\gamma^{\tau\tau}\partial_\tau X^- - 2\gamma^{\tau\sigma}\partial_\sigma X^- + \gamma^{\tau\tau}\partial_\tau X^i\partial_\tau X^i + 2\gamma^{\tau\sigma}\partial_\tau X^i\partial_\sigma X^i + \gamma^{\sigma\sigma}\partial_\sigma X^i\partial_\sigma X^i\Big) \\ &= -\frac{1}{4\pi\alpha'}\Big(2\gamma_{\sigma\sigma}\partial_\tau X^- - 2\gamma_{\tau\sigma}\partial_\sigma X^- - \gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i + 2\gamma_{\tau\sigma}\partial_\tau X^i\partial_\sigma X^i \\ &\qquad + \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2)\partial_\sigma X^i\partial_\sigma X^i\Big)\end{aligned} \qquad [1.46]$$

We now write $X^-(\tau,\sigma) = x^-(\tau) + Y^-(\tau,\sigma)$ and go to the Lagrangian

$$\begin{aligned}L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\, \Big(&2\gamma_{\sigma\sigma}\partial_\tau x^- - 2\gamma_{\sigma\sigma}\partial_\tau Y^- - 2\gamma_{\tau\sigma}\partial_\sigma x^- - 2\gamma_{\tau\sigma}\partial_\sigma Y^- \\ &- \gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i + 2\gamma_{\tau\sigma}\partial_\tau X^i\partial_\sigma X^i + \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2)\partial_\sigma X^i\partial_\sigma X^i\Big)\end{aligned} \qquad [1.47]$$

By construction $\partial_\sigma x^- = 0$, but we also have since we have already established that in our gauge choice $\gamma_{\sigma\sigma}$ is independent of σ

$$\int_0^\ell d\sigma\,\gamma_{\sigma\sigma}\partial_\tau Y^- = \gamma_{\sigma\sigma}\partial_\tau\int_0^\ell d\sigma\, Y^- = 0 \qquad [1.48]$$

as Y^- has by construction mean value zero. Therefore, we find (1.3.11)

$$\begin{aligned}L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\,\Big[&\gamma_{\sigma\sigma}(2\partial_\tau x^- - \partial_\tau X^i\partial_\tau X^i) - 2\gamma_{\tau\sigma}(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) \\ &+ \gamma_{\sigma\sigma}^{-1}(1-\gamma_{\tau\sigma}^2)\partial_\sigma X^i\partial_\sigma X^i\Big]\end{aligned} \qquad [1.49]$$

1.11 p 19: Eq. (1.3.13) The Open String Boundary Conditions

(1.2.28) is $\partial^\sigma x^\mu(\tau,0) = \partial^\sigma x^\mu(\tau,\ell) = 0$. Writing this with derivatives with indices downstairs we get $\partial^\sigma x^\mu = g^{a\sigma}\partial_a X^\mu = g^{\sigma\sigma}\partial_\sigma X^\mu + g^{\sigma\tau}\partial_\tau X^\mu$ and so, inverting the metric $0 = -\gamma_{\tau\tau}\partial_\sigma X^\mu + \gamma_{\tau\sigma}\partial_\tau X^\mu$ at $\sigma = 0$ and $\sigma = \ell$.

1.12 p 22: Eq. (1.3.32) Regularising $\sum_n n$

This derivation is a delight to mathematicians.

$$\begin{aligned}
\sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} n e^{-\varepsilon a n} &= -\frac{d}{d\varepsilon a}\sum_{n=0}^{\infty} e^{-\varepsilon a n} = -\frac{d}{d\varepsilon a}\frac{1}{1-e^{\varepsilon a}} \\
&= -\frac{d}{d\varepsilon a}\frac{1}{\varepsilon a - \frac{1}{2!}(\varepsilon a)^2 + \frac{1}{3!}(\varepsilon a)^3 + \cdots} = -\frac{d}{d\varepsilon a}\frac{1}{\varepsilon a}\frac{1}{1-\frac{1}{2}\varepsilon a + \frac{1}{6}(\varepsilon a)^2 + \cdots} \\
&= -\frac{d}{d\varepsilon a}\frac{1}{\varepsilon a}\left[1 - \left(1 - \frac{1}{2}\varepsilon a + \frac{1}{6}(\varepsilon a)^2\right)) + \left(1 - \frac{1}{2}\varepsilon a + \frac{1}{6}(\varepsilon a)^2\right)^2 + \cdots\right] \\
&= -\frac{d}{d\varepsilon a}\left(\frac{1}{\varepsilon a} - \frac{1}{2} + \frac{1}{12}\varepsilon a + \cdots\right) = \frac{1}{(\varepsilon a)^2} - \frac{1}{12}
\end{aligned} \qquad [1.50]$$

There is a rather entertaining "proof" of the fact that $\sum_{n=1}^{\infty} n = -1/12$ that is due to Ramanujan. First, let us call the sum S, i.e. $S = 1+2+3+4+\cdots$. Now we subtract from this the sum $4S$ but not term by term, we subtract the terms $4S$ from each third term of S, i.e.

$$\begin{aligned}
S &= 1+2+3+4+5+6+\cdots \\
-4S &= \quad -4 \quad\; -8 \quad\; -12+\cdots \\
\tilde{S} = S - 4S &= 1-2+3-4+5-6+\cdots
\end{aligned} \qquad [1.51]$$

Now $\tilde{S}$ is just the alternating series so $\tilde{S} = 1/(1+x)^2$ at $x = 1$. So

$$S - 4S = -3S = \tilde{S} = \frac{1}{1+1}^{2} = \frac{1}{4} \Rightarrow S = -\frac{1}{12} \qquad [1.52]$$

But, of course, even for physicists this is taking it a bit far.

1.13 p 24: Eq. (1.3.43) The Regge Slope for Open Strings

We start by checking the Eigenvalue of S^{23} for the state $(\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle$ for a given m:

$$\begin{aligned} S^{23}(\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle &= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^2_{-n}\alpha^3_n - \alpha^3_{-n}\alpha^2_n\right)(\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle \\ &= -i\frac{1}{m}\left(\alpha^2_{-m}\alpha^3_m - \alpha^3_{-m}\alpha^2_m\right)(\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle \\ &= -i\frac{1}{m}\left(\alpha^2_{-m}\alpha^3_m i\alpha^3_{-m} - \alpha^3_{-m}\alpha^2_m\alpha^2_{-m}\right)|0;k\rangle \\ &= (\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle \end{aligned} \tag{1.53}$$

where we have used $[\alpha^i_m, \alpha^n_j] = m\delta^{ij}\delta_{m+n,0}$. So this state has spin one. Consider now

$$\begin{aligned} &S^{23}(\alpha^2_{-m} + i\alpha^3_{-m})^2\,|0;k\rangle = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^2_{-n}\alpha^3_n - \alpha^3_{-n}\alpha^2_n\right)(\alpha^2_{-m} + i\alpha^3_{-m})^2\,|0;k\rangle \\ &= -i\frac{1}{m}\left(\alpha^2_{-m}\alpha^3_m - \alpha^3_{-m}\alpha^2_m\right)(\alpha^2_{-m} + i\alpha^3_{-m})(\alpha^2_{-m} + i\alpha^3_{-m})\,|0;k\rangle \\ &= -i\frac{1}{m}\left(\alpha^2_{-m}\alpha^3_m - \alpha^3_{-m}\alpha^2_m\right)\left(\alpha^2_{-m}\alpha^2_{-m} + i\alpha^2_{-m}\alpha^3_{-m} + i\alpha^3_{-m}\alpha^2_{-m} - \alpha^3_{-m}\alpha^3_{-m}\right)|0;k\rangle \\ &= -i\frac{1}{m}\Big[\alpha^2_{-m}\alpha^3_m\left(i\alpha^2_{-m}\alpha^3_{-m} + i\alpha^3_{-m}\alpha^2_{-m} - \alpha^3_{-m}\alpha^3_{-m}\right) \\ &\qquad - \alpha^3_{-m}\alpha^2_m\left(\alpha^2_{-m}\alpha^2_{-m} + i\alpha^2_{-m}\alpha^3_{-m} + i\alpha^3_{-m}\alpha^2_{-m}\right)\Big]\,|0;k\rangle \\ &= -i\frac{1}{m}(im\alpha^2_{-m}\alpha^2_{-m} + im\alpha^2_{-m}\alpha^2_{-m} - m\alpha^2_{-m}\alpha^3_{-m} - \alpha^2_{-m}\alpha^3_{-m}\alpha^3_m\alpha^3_{-m} \\ &\qquad - m\alpha^3_{-m}\alpha^2_{-m} - \alpha^3_{-m}\alpha^2_{-m}\alpha^2_m\alpha^2_{-m} - im\alpha^3_{-m}\alpha^3_{-m} - im\alpha^3_{-m}\alpha^3_{-m})\,|0;k\rangle \\ &= 2(\alpha^2_{-m}\alpha^2_{-m} + i\alpha^2_{-m}\alpha^3_{-m} + i\alpha^3_{-m}\alpha^2_{-m} - \alpha^3_{-m}\alpha^3_{-m})\,|0;k\rangle \\ &= 2(\alpha^2_{-m} + i\alpha^3_{-m})^2\,|0;k\rangle \end{aligned} \tag{1.54}$$

Similarly we find

$$S^{23}(\alpha^2_{-m} + i\alpha^3_{-m})^N\,|0;k\rangle = N(\alpha^2_{-m} + i\alpha^3_{-m})^N\,|0;k\rangle \tag{1.55}$$

Any other state at level N will contain at least one less factor of $\alpha^2_{-m} + i\alpha^3_{-m}$ and as S^{23} only has non-zero Eigenvalue on that specific combination, it will give a spin lower than N. Using the mass-shell condition (1.3.36) in $D = 26$, i.e. $\alpha' m^2 = N - 1$ we thus find

$$S^{23} \leq N = \alpha' m^2 + 1 \tag{1.56}$$

which is (1.3.43)

1.14 p 29: Eq. (1.4.19) The Unoriented Strings

For the open string we have (1.3.22)

$$X^i = x^i + \frac{p^i}{p^+}\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\alpha_n^i}{n}\exp\left(-\frac{n\pi i c\tau}{\ell}\right)\cos\frac{n\pi\sigma}{\ell} \tag{1.57}$$

Under the transformation $\sigma \to \ell - \sigma$ we the operator X^i transforms as

$$\begin{aligned} X^i \to & \Omega X^i \Omega^{-1} \\ & = \Omega x^i \Omega^{-1} + \Omega\frac{p^i}{p^+}\Omega^{-1}\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\Omega\alpha_n^i\Omega^{-1}}{n}\exp\left(-\frac{n\pi i c\tau}{\ell}\right)\cos\frac{n\pi\sigma}{\ell} \end{aligned} \tag{1.58}$$

On the other hand we have $\cos n\pi\sigma/\ell \to \cos n\pi(\ell-\sigma)/\ell = (-1)^n \cos n\pi\sigma/\ell$ and so we see that

$$\Omega\alpha_n^i\Omega^{-1} = (-1)^n\alpha_n^i \tag{1.59}$$

For the closed string we have (1.4.4)

$$X^i = x^i + \frac{p^i}{p^+}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left\{\frac{\tilde\alpha_n^i}{n}\exp\left[-\frac{2n\pi i(\sigma+c\tau)}{\ell}\right] + \frac{\alpha_n^i}{n}\exp\left[+\frac{2n\pi i(\sigma-c\tau)}{\ell}\right]\right\} \tag{1.60}$$

and so as

$$\exp\left[-\frac{2n\pi i(\sigma+c\tau)}{\ell}\right] \to \exp\left[+\frac{2n\pi i(\sigma-c\tau)}{\ell}\right] \tag{1.61}$$

and

$$\exp\left[+\frac{2n\pi i(\sigma-c\tau)}{\ell}\right] \to \exp\left[-\frac{2n\pi i(\sigma+c\tau)}{\ell}\right] \tag{1.62}$$

we have

$$\begin{aligned} \Omega\alpha_n^i\Omega^{-1} &= \tilde\alpha_n^i \\ \Omega\tilde\alpha_n^i\Omega^{-1} &= \alpha_n^i \end{aligned} \tag{1.63}$$

Joe has these expressions not only for the X^i, but also for X^0 and X^1. You cannot deduce this in the light-cone gauge.

Chapter 2

Conformal Field Theory

2.1 p 33: Eq. (2.1.2) The Complex Coordinates

The inverse is

$$\sigma^1 = \frac{1}{2}(z + \bar{z}) \qquad , \qquad \sigma^2 = \frac{1}{2i}(z - \bar{z}) \tag{2.1}$$

2.2 p 33: Eq. (2.1.3) The Complex Derivatives

This isn't really a definition, but follows from Leibniz:

$$\begin{aligned} \partial = \partial_z = \partial_z \sigma^1 \partial_1 + \partial_z \sigma^2 \partial_2 = \frac{1}{2}(\partial_1 - i\partial_2) \\ \bar{\partial} = \partial_{\bar{z}} = \partial_{\bar{z}} \sigma^1 \partial_1 + \partial_{\bar{z}} \sigma^2 \partial_2 = \frac{1}{2}(\partial_1 + i\partial_2) \end{aligned} \tag{2.2}$$

The inverse is

$$\partial_1 = \partial + \bar{\partial} \qquad , \qquad \partial_2 = i(\partial - \bar{\partial}) \tag{2.3}$$

2.3 p 33: Eq. (2.1.6) The Complex Metric

Just for the sake of it, we will drive it in two ways First,

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = \frac{1}{4}(dz + d\bar{z})^2 - \frac{1}{4}(dz - d\bar{z})^2 = dzd\bar{z} \tag{2.4}$$

writing $ds^2 = g_{zz}dzdz + g_{z\bar{z}}dzd\bar{z} + g_{\bar{z}z}d\bar{z}dz + g_{\bar{z}\bar{z}}d\bar{z}d\bar{z}$ we find

$$g_{..} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \tag{2.5}$$

and its inverse

$$g^{\cdot\cdot} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad [2.6]$$

The dots refer top the location of the indices.
The second way is to use the transformation rule of the metric:

$$\begin{aligned} g_{zz} &= \frac{\partial\sigma^a}{\partial z}\frac{\partial\sigma^b}{\partial z}g_{ab} = \left(\frac{\partial\sigma^1}{\partial z}\right)^2 g_{11} + \left(\frac{\partial\sigma^2}{\partial z}\right)^2 g_{22} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2i}\right)^2 = 0 \\ g_{z\bar{z}} &= \frac{\partial\sigma^a}{\partial z}\frac{\partial\sigma^b}{\partial \bar{z}}g_{ab} = \frac{\partial\sigma^1}{\partial z}\frac{\partial\sigma^1}{\partial \bar{z}}g_{11} + \frac{\partial\sigma^2}{\partial z}\frac{\partial\sigma^2}{\partial \bar{z}}g_{22} \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2i}\right)\left(-\frac{1}{2i}\right) = \frac{1}{2} \end{aligned} \qquad [2.7]$$

etc.

2.4 p 33: Eq. (2.1.7) The Jacobian

We have

$$\begin{aligned} d^2z = dzd\bar{z} &= |\frac{\partial(z,\bar{z})}{\partial(\sigma^1,\sigma^2)}|d\sigma^1 d\sigma^2 = |\begin{pmatrix} \partial z/\partial\sigma^1 & \partial z/\partial\sigma^2 \\ \partial\bar{z}/\partial\sigma^1 & \partial\bar{z}/\partial\sigma^2 \end{pmatrix}|d\sigma^1 d\sigma^2 \\ &= |\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}|d\sigma^1 d\sigma^2 = 2d\sigma^1 d\sigma^2 \end{aligned} \qquad [2.8]$$

Note also that

$$g = \det\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = -\frac{1}{4} \qquad [2.9]$$

and so

$$(-g)^{1/2} = \frac{1}{2} \qquad [2.10]$$

2.5 p 36: Eq. (2.1.23) The Equation of Motion as Operator Equation

$$\begin{aligned} \partial_1\bar{\partial}_1 : X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) : &= \partial_1\bar{\partial}_1\left[X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2\right] \\ &= -\pi\alpha'\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu}\times 2\pi\delta^2(z-z',\bar{z}-\bar{z}') \\ &= 0 \end{aligned} \qquad [2.11]$$

2.6 p 36: Eq. (2.1.24) $\partial\bar{\partial}\ln|z|^2 = 2\pi\delta^2(z,\bar{z})$

As per the text, this is obvious when $z \neq 0$. To show the normalisation, let us go back to σ^a coordinates and integrate over the worldsheet. The RHS is

$$RHS = \int d^2\sigma\, 2\pi \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2) = \pi \qquad [2.12]$$

where we have used the delta function convention from p xviii. The LHS is

$$\begin{aligned} LHS &= \int d^2\sigma\, \frac{1}{2}(\partial_1 - i\partial_2)\frac{1}{2}(\partial_1 + i\partial_2)\ln\big(\sigma_1^2+\sigma_2^2\big) \\ &= \frac{1}{4}\int d^2\sigma\,(\partial_1^2+\partial_2^2)\ln\big(\sigma_1^2+\sigma_2^2\big) \\ &= \frac{1}{4}\int d^2\sigma\,\left(\partial_1\frac{2\sigma_1}{\sigma_1^2+\sigma_2^2} + \partial_2\frac{2\sigma_2}{\sigma_1^2+\sigma_2^2}\right) \\ &= \frac{1}{2}\int d^2\sigma\,\partial_a\frac{c_a}{\sigma_1^2+\sigma_2^2} = -\frac{1}{2}\oint_C \frac{\sigma_1 d\sigma_2 - \sigma_2 d\sigma_1}{\sigma_1^2+\sigma_2^2} \end{aligned} \qquad [2.13]$$

We have used the fact that we are in Euclidean space so that we don't have to bother about the location of the indices, and we have also used Stokes' theorem in the last equation. Let us now move to radial coordinates $\sigma_1 = r\cos\theta$ and $\sigma_2 = r\sin\theta$ so that

$$\sigma_1 d\sigma_2 - \sigma_2 d\sigma_1 = r\cos\theta(\sin\theta dr + r\cos\theta d\theta) - r\sin\theta(\cos\theta dr - r\sin\theta d\theta) = r^2 d\theta \qquad [2.14]$$

and we find where we have used the delta function convention from p xviii. The LHS is

$$LHS = \frac{1}{2}\oint_C \frac{r^2 d\theta}{r^2} = \frac{1}{2}\times\oint_C d\theta = \pi \qquad [2.15]$$

and so LHS = RHS, proving (2.1.24)

2.7 p 38: Eq. (2.2.4) A Taylor Expansion

The Taylor expansion in two variables is

$$A(x,y) = A(x_0,y_0) + \sum_{n-1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\partial_x^k\partial_y^{n-k}A(x_0,y_0) \qquad [2.16]$$

In our case $\partial_x = \partial$ and $\partial_y\bar{\partial}$. We moreover have $:\partial\bar{\partial}X(z,\bar{z})\cdots := 0$ as an operator equation and so the only terms contributing for a given n are those with $k=0$ and $k=n$. This immediately gives (2.2.4).

2.8 p 39: Eq. (2.2.5) and (2.2.8) Subtractions and Contractions

Note that this is not without any humour. Subtractions have a positive sign $+\frac{1}{2}\alpha'\eta^{\mu_i\mu_j}\ln|z_{ij}|^2$ and contractions have a negative sign $-\frac{1}{2}\alpha'\eta^{\mu_i\mu_j}\ln|z_{ij}|^2$

2.9 p 39: Below Eq. (2.2.6) Normal Ordered Products Satisfy the Equation of Motion

Let us first show it for three fields, after which we will argue it to be the case for any number of fields. Consider $\partial_1\bar{\partial}_1 :X^{\mu_1}(z_1,\bar{z}_1)X^{\mu_2}(z_2,\bar{z}_2)X^{\mu_3}(z_3,\bar{z}_3):$. We first write out the definition of normal ordering

$$\begin{aligned}\partial_1\bar{\partial}_1 :X^{\mu_1}(z_1,\bar{z}_1)X^{\mu_2}(z_2,\bar{z}_2)X^{\mu_3}(z_3,\bar{z}_3):= \partial_1\bar{\partial}_1\Big[&X^{\mu_1}(z_1,\bar{z}_1)X_2^{\mu}(z_2,\bar{z}_2)X_3^{\mu}(z_3,\bar{z}_3)\\ &+\frac{\alpha'}{2}\eta^{\mu_1\mu_2}\ln|z_{12}|^2 :X^{\mu_3}(z_3,\bar{z}_3): +\frac{\alpha'}{2}\eta^{\mu_1\mu_3}\ln|z_{13}|^2 :X^{\mu_2}(z_2,\bar{z}_2):\\ &+\frac{\alpha'}{2}\eta^{\mu_2\mu_3}\ln|z_{23}|^2 :X^{\mu_1}(z_1,\bar{z}_1): \Big]\end{aligned} \qquad [2.17]$$

We have kept the normal ordering symbols around the single fields as it will be convenient for when we generalise to more fields. We can now perform the derivatives. Using (2.1.19) acting on the first term gives us two contact terms of the form

$$-\pi\alpha'\eta^{\mu_1\mu_i}\delta^2(z_{1i},\bar{z}_{1i}) :X^{\mu_j}(z_j,\bar{z}_j): \qquad [2.18]$$

where (i,j) is either $(2,3)$ or $(3,2)$. These terms are cancelled by taking the derivatives of the second and the third term and using (2.1.24), i.e. $\partial\bar{\partial}\ln|z|^2 = 2\pi\delta^2(z,\bar{z})$. The derivative of the last term vanishes by the operator equation of motion $\partial\bar{\partial} :X^{\mu_1}(z_1,\bar{z}_1):= 0$.

We can now easily generalise this to n scalar fields

$$\partial_1\bar{\partial}_1 :X^{\mu_1}(z_1,\bar{z}_1)X^{\mu_2}(z_2,\bar{z}_2)\cdots X^{\mu_n}(z_n,\bar{z}_n): \qquad [2.19]$$

First we have the derivative on the not-normal order product. It will give $n-1$ contact terms. Then we look at the contractions, of which there are two types. Those involving X^{μ_1} and one of the fields X^{μ_i} for $i = 2,\cdots,n$ will cancel with the contact terms. The remaining contractions are between a X^{μ_i} and a X^{μ_j} for $i,j \neq 1$ and $i \neq j$. They result in a term proportional to

$$\begin{aligned}\partial_1\bar{\partial}_1 :&X^{\mu_1}(z_1,\bar{z}_1)\cdots X^{\mu_{i-1}}(z_{i-1},\bar{z}_{i-1})X^{\mu_{i+1}}(z_{i+1},\bar{z}_{i+1})\cdots\\ &X^{\mu_{j-1}}(z_{j-1},\bar{z}_{j-1})X^{\mu_{j+1}}(z_{j+1},\bar{z}_{j+1})\cdots X^{\mu_n}(z_n,\bar{z}_n):\end{aligned} \qquad [2.20]$$

But that is just a the same as our original expression but with $n-2$ fields i.o. n fields. We've shown this to be zero for $n = 1,2,3$ and so it follows that this term is zero by recursion and thus the n-point normal ordered product satisfy the equation of motion as well.

2.10 p 39: Eq. (2.2.10) The Product of Normal Ordered Operators

(2.2.10) has a factor of 2 compared to (2.2.8) because in (2.2.8) both derivatives can act on $:\mathcal{F}:$ and so we need to avoid the double-counting. In (2.2.10) each derivative can specifically act on one and only one of the operators.

2.11 p 40: Eq. (2.2.11) Calculating an OPE

Let us denote by $\overbracket{X^\mu(z_1)X}^\nu(z_2) = -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln|z_{12}|^2$ a contraction between two fields. We then have

$$\begin{aligned}
:\partial X^\mu\partial X_\mu:(z):\partial' X^\nu\partial' X_\nu:(z') = &:\partial X^\mu\partial X_\mu(z)\partial' X^\nu\partial' X_\nu(z'):\\
&+4\times\partial\overbracket{X^\mu(z)\partial' X}^\nu(z'):\partial X_\mu(z)\partial' X_\nu:(z')\\
&+2\times\partial\overbracket{X^\mu(z)\partial' X}^\nu(z')\partial\overbracket{X_\mu(z)\partial' X}_\nu(z')\\
=&\,\partial X^\mu\partial X_\mu(z)\partial' X^\nu\partial' X_\nu(z')\\
&-4\frac{1}{2}\alpha'\eta^{\mu\nu}\partial\partial'\ln|z-z'|^2:\partial X_\mu(z)\partial' X_\nu:(z')\\
&+2\left(\frac{1}{2}\alpha'\eta^{\mu\nu}\partial\partial'\ln|z-z'|^2\right)^2
\end{aligned}\qquad [2.21]$$

Now

$$\partial\partial'\ln|z-z'|^2 = \partial\left(-\frac{1}{z-z'}\right) = \frac{1}{(z-z')^2}\qquad [2.22]$$

and $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^\mu_\mu = D$, we get

$$\begin{aligned}
:\partial X^\mu\partial X_\mu:(z):\partial' X^\nu\partial' X_\nu:(z') &\sim \frac{D\alpha'/2}{(z-z')^4} - \frac{2\alpha'}{(z-z')^2}:\partial X^\mu(z)\partial' X_\mu:(z')\\
&\sim \frac{D\alpha'/2}{(z-z')^4} - \frac{2\alpha'}{(z-z')^2}:\partial' X^\mu\partial' X_\mu:(z') - \frac{2\alpha'}{z-z'}:\partial'^2 X^\mu\partial' X_\mu:(z')
\end{aligned}\qquad [2.23]$$

where $\sim$ denotes up to regular terms and we have expanded $:\partial X^\mu(z)\partial' X_\mu:(z')$ in a Taylor series in the last line.

2.12 p 41: Eq. (2.3.5) The Ward Identity

We use the fact that $\partial_a(g^{1/2}v^a(c)) = g^{1/2}\nabla_a v^a(c)$ as shown here

$$LHS = \partial_a(\sqrt{g}\,v^a) = (\partial_a\sqrt{g})\,v^a + \sqrt{g}\partial_a v^a = \frac{1}{2\sqrt{g}} g g^{bc}\partial_a g_{bc} v^a + \sqrt{g}\partial_a v^a$$
$$= \sqrt{g}\left(\frac{1}{2}g^{bc}\partial_a g_{bc} v^a + \partial_a v^a\right) \quad [2.24]$$

For the RHS we find

$$RHS = \sqrt{g}\,\nabla_a v^a = \sqrt{g}\left(\partial_a v^a + \Gamma^a_{ab} v^b\right)$$
$$= \sqrt{g}\left[\partial_a v^a + \frac{1}{2}\left(g^{ac}\partial_a g_{cb} + g^{ac}\partial_b g_{ca} - g^{ac}\partial_c g_{ab}\right) v^b\right]$$
$$= \sqrt{g}\left(\partial_a v^a + \frac{1}{2} g^{ac}\partial_b g_{ca} v^b\right) \quad [2.25]$$

The second and last term of the connection cancel after interchanging the dummy indices a and c. This is now equal to the LHS.

To derive (2.3.5) we start from (2.3.4), which implies

$$0 = \int d^d\sigma g^{1/2} j^a(\sigma)\partial_a\rho(\sigma) = -\int d^d\sigma \partial_a(g^{1/2}j^a(\sigma))\rho(\sigma)$$
$$= -\int d^d\sigma g^{1/2}\nabla_a j^a(\sigma)\rho(\sigma) \quad [2.26]$$

2.13 p 42: Eq. (2.3.11) The OPE with the Conserved Current Determines the Transformation Law

For a holomorphic current (2.3.10) becomes

$$\oint_C dz\, j(z)\mathcal{A}(z_0) = \frac{2\pi}{\varepsilon}\delta\mathcal{A}(z_0) \quad [2.27]$$

with C a contour counter-clockwise around z_0. We already see that the transformation rule of an operator $\mathcal{A}(z_0)$ under a symmetry is determined by its OPE of the corresponding symmetry current, i.e. $j(z)\mathcal{A}(z_0)$.

Now we have for a general function $f(z)$

$$\mathrm{Res}_{z\to z_0} f(z) = \frac{1}{2\pi i}\oint_C dz\, f(z) \quad [2.28]$$

with C a contour counter-clockwise around z_0. Therefore

$$\mathrm{Res}_{z\to z_0} j(z)\mathcal{A}(z_0) = \frac{1}{2\pi i}\oint_C dz\, j(z)\mathcal{A}(z_0) = \frac{1}{2\pi i}\frac{2\pi}{\varepsilon}\delta\mathcal{A}(z_0) = \frac{1}{i\varepsilon}\delta\mathcal{A}(z_0) \quad [2.29]$$

2.14 p 43: Eq. (2.3.14) Transformation of a Vertex Operator under a Space-Time Translation

We have

$$j^\mu(z) :e^{ik\cdot X(0)}: = \frac{i}{\alpha'}\partial X^\mu :\sum_{n=0}^{\infty}\frac{i^n}{n!}\big(k\cdot X(0)\big)^n:$$
$$\sim \frac{i}{\alpha'}\partial \overbrace{X^\mu(z) X^\nu}(0) k_\nu :\sum_{n=1}^{\infty}\frac{i^n}{n!}n\big(k\cdot X(0)\big)^{n-1}:$$
$$\sim \frac{i}{\alpha'}\left(-\frac{\alpha'}{2}\eta^{\mu\nu}\partial\ln|z|^2\right)k_\nu i :e^{ik\cdot X(0)}: \sim \frac{k^\mu}{2z} :e^{ik\cdot X(0)}: \qquad [2.30]$$

Let us now define $\mathcal{V}_k(z) =:e^{ik\cdot X(z)}:$, what we will later see is a vertex operator. We can then use (2.3.11) to find its transformation law under the space-time translation in the μ direction:

$$\delta^\mu \mathcal{V}_k(0) = i\varepsilon \mathrm{Res}_{z\to 0} j^\mu(z)\mathcal{V}_k(0) = i\varepsilon\frac{1}{2\pi i}\oint_C j^\mu(z)\mathcal{V}_k(0)$$
$$= \frac{i\varepsilon}{2\pi i}\oint_C \frac{k^\mu}{2z}\mathcal{V}_k(0) = \frac{i\varepsilon k^\mu}{2}\mathcal{V}_k(0) \qquad [2.31]$$

2.15 p 43: Eq. (2.3.15) The Energy-Momentum Tensor

Let us first check that the action is indeed invariant under a world-sheet translation:

$$\delta S = \delta\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial^a X^\mu \partial_a X_\mu = \frac{1}{4\pi\alpha'}\int d^2\sigma\, 2\partial^a X^\mu\partial_a\delta X_\mu$$
$$= \frac{1}{2\pi\alpha'}\int d^2\sigma\, \partial^a X^\mu \partial_a(-\varepsilon v^b\partial_b X_\mu) = -\frac{\varepsilon v^b}{2\pi\alpha'}\int d^2\sigma\, \partial^a X^\mu\partial_a\partial_b X_\mu$$
$$= \frac{\varepsilon v^b}{2\pi\alpha'}\int d^2\sigma\, \partial^a\partial_a X^\mu\partial_b X_\mu = 0 \qquad [2.32]$$

where in the last line we have used the equations of motion $\partial^2 X^\mu = 0$. Recall that the symmetry is only required the equations of motion are satisfied. Indeed the corresponding Noether charged is only conserved on shell.

Let us now derive the Noether current. How do we do that? Suppose you have a Lagrangian $\mathcal{L}[\phi]$ of some fields ϕ that is invariant under a continuous transformation $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi$. But as we can always add a total divergence to a Lagrangian - it just gives a boundary term in the action, invariance means that under this transformation the Lagrangian is unchanged up to a potential total divergence. I.e. The Lagrangian transforms as $\mathcal{L} \to \mathcal{L} + \alpha\partial_\mu\mathcal{J}^\mu$, so $\alpha\Delta\mathcal{L} = \alpha\partial_\mu\mathcal{J}^\mu$.

Now, consider the transformation of the Lagrangian for the change of its field:

$$\begin{aligned}\alpha\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi \\ &= \frac{\partial\mathcal{L}}{\partial\phi}\alpha\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\alpha\Delta\phi \\ &= \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi \end{aligned} \tag{2.33}$$

The second term vanishes by the equations of motion. We now equate both forms of the transformation of the Lagrangian

$$\alpha\partial_\mu\mathcal{J}^\mu = \alpha\Delta\mathcal{L} = \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) \tag{2.34}$$

and we see that

$$\partial_\mu\left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) - \mathcal{J}^\mu\right] = 0 \tag{2.35}$$

I.e. the Noether current

$$j^\mu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) - \mathcal{J}^\mu \tag{2.36}$$

is conserved, $\partial_\mu j^\mu = 0$.

Let us work this out for the worldsheet translation $\delta\sigma^a = \varepsilon v^a$ under which X^μ transforms as a worldsheet scalar: $\delta X^\mu = -\varepsilon v^a\partial_a X^\mu$. First we need to find how the Lagrangian transforms and from there the corresponding $\mathcal{J}$. This is quite easy. The Lagrangian is also a scalar under worldsheet transformation and so it must transform in a way similar to X^μ, i.e.

$$\delta\mathcal{L} = -\varepsilon v^a\partial_a\mathcal{L} = -\varepsilon v^a\partial_b(\delta_a^b\mathcal{L}) \Rightarrow \mathcal{J}^b = -\varepsilon v^a\delta_a^b\mathcal{L} \tag{2.37}$$

We also have, using the Lagrangian

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\partial^a X^\mu\partial_a X_\mu \tag{2.38}$$

that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_b X^\mu)}\Delta X^\mu = \frac{1}{2\pi\alpha'}\partial^b X_\mu(-\varepsilon v^a\partial_a X^\mu) = -\frac{1}{2\pi\alpha'}\varepsilon v^a\partial^b X_\mu\partial_a X^\mu \tag{2.39}$$

We therefore find

$$\begin{aligned} j^b &= -\frac{1}{2\pi\alpha'}\varepsilon v^a \partial^b X_\mu \partial_a X^\mu - (-\varepsilon v^a \delta_a^b \mathcal{L}) \\ &= -\frac{\varepsilon v^a}{2\pi\alpha'}\left(\partial^b X_\mu \partial_a X^\mu - \frac{1}{2}\delta_a^b \partial^c X^\mu \partial_c X_\mu\right) \end{aligned} \quad [2.40]$$

Bringing the index b down gives

$$j_b = -\frac{\varepsilon v^a}{2\pi\alpha'}\left(\partial_b X_\mu \partial_a X^\mu - \frac{1}{2}\delta_{ab}\partial^c X^\mu \partial_c X_\mu\right) \quad [2.41]$$

which gives us the energy-momentum tensor $j_a \propto v^b T_{ab}$ with

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X_\mu \partial_b X^\mu - \frac{1}{2}\delta_{ab}\partial^c X^\mu \partial_c X_\mu\right) \quad [2.42]$$

which is (2.3.15b).

2.16 p 43: Eq. (2.4.1) The Energy-Momentum Tensor is Traceless

Using the complex metric we have

$$g^{ab}T_{ab} = g^{zz}T_{zz} + 2g^{z\bar{z}}T_{z\bar{z}} + g^{\bar{z}\bar{z}}T_{\bar{z}\bar{z}} \quad [2.43]$$

The only non-vanishing components of the metric in complex coordinates are the off-diagonal ones. Tracelessness thus implies $T_{z\bar{z}} = 0$.

2.17 p 43: Eq. (2.4.2) The Energy-Momentum Tensor Splits into a Holomorphic and an Anti-holomorphic Part

Set $b = z$ in the conservation equation $0 = \partial^a T_{ab} = g^{ac}\partial_c T_{ab}$. Because of the tracelessness of the energy-momentum tensor the sum over a is only over z. As the metric is off-diagonal the sum over c is then only over $\bar{z}$ and the conservation equation therefore becomes $\bar{\partial}T_{zz} = 0$. The same of course holds for $\partial T_{\bar{z}\bar{z}} = 0$.

2.18 p 44: Eq. (2.4.6) The OPE with the Energy-Momentum Tensor

$$\begin{aligned} T(z)X^\mu(0) &= -\frac{1}{\alpha'} :\partial X^\nu \partial X_\nu: (z)X^\mu(0) = -\frac{1}{\alpha'}2\partial \overbrace{X^\nu(z) X^\mu}(0)\partial X_\nu(z) \\ &= -\frac{2}{\alpha'}\partial\left(-\frac{\eta^{\mu\nu}}{2\alpha'}\ln|z|^2\right)\partial X_\nu(z) \sim \frac{1}{z}\partial^\mu X(0) \end{aligned} \quad [2.44]$$

In the last line we have Taylor expanded $\partial^\mu X(z)$ and only kept the first term, as the higher order corrections give terms in the OPE that are regular.

2.19 p 44: Eq. (2.4.7) The Transformation of the Field X^μ

The Ward identity (2.3.11) tells us that the transformation of a field corresponding to the conserved current $T(z)$ is determined by the OPE of that current with that field. Thus

$$\begin{aligned}\delta X^\mu(w) &= i\varepsilon \mathrm{Res}_{z\to w} j^z(z)(X^\mu(w) = i\varepsilon \frac{1}{2\pi i}\oint_C iv(z)T(z)X^\mu(w) \\ &= -\frac{\varepsilon}{2\pi i}\oint_C \frac{v(z)}{z}\partial X^\mu(w) = -\varepsilon v(w)\partial X^\mu(w)\end{aligned} \qquad [2.45]$$

2.20 p 45: Fig 2.2. Examples of Conformal Transformations

It may be informative, or at least illustrative to give some examples of conformal transformations. The following are five pictures of how the contour lines of constant real part of z transform under different conformal transformations.

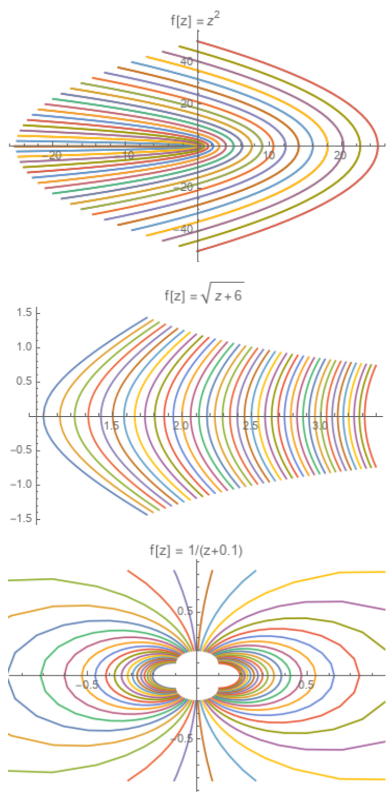


Figure 2.1: Conformal transformation $f(z) = z^2, \sqrt{z+6}$ and $1/(z+0.1)$. Contour lines for $\mathrm{Re}(z) = c^{\mathrm{te}}$

As an artistic aside, mapping such conformal transformations can actually give very pretty pictures. Below are just two more examples

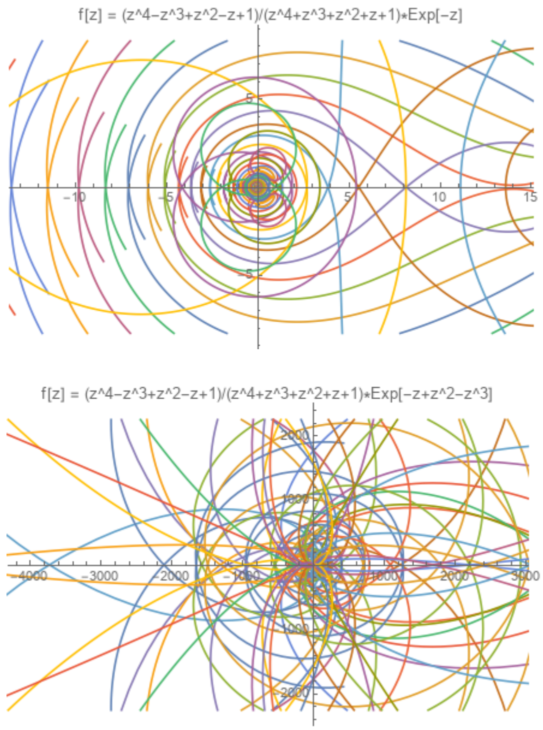


Figure 2.2: Conformal transformation $f(z) = \frac{z^4-z^3+z^2-z+1}{z^4+z^3+z^2+z+1}e^{-z}$ and $\frac{z^4-z^3+z^2-z+1}{z^4+z^3+z^2+z+1}e^{-z+z^2-z^3}$. Contour lines for $\mathrm{Re}(z) = c^{\mathrm{te}}$

2.21 p 46: Eq. (2.4.12) Conformal Transformation of an Operator, I

From the residues theorem (2.3.11) we have, expanding $v(z)$ in a Taylor series,

$$\begin{aligned}\delta\mathcal{A}(z_0) &= i\varepsilon\frac{1}{2\pi i}\oint_C j(z)\mathcal{A}(z_0) = i\varepsilon\frac{1}{2\pi i}\oint_C iv(z)T(z)\mathcal{A}(z_0)\\ &= -\frac{\varepsilon}{2\pi i}\oint_C\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{k!}\partial^k v(z_0)\sum_{n=0}^{\infty}\frac{\mathcal{A}^{(n)}(z_0)}{(z-z_0)^{n+1}}\\ &= -\frac{\varepsilon}{2\pi i}\sum_{k,n=0}^{\infty}\frac{1}{k!}\frac{\partial^k v(z_0)\mathcal{A}^{(n)}(z_0)}{(z-z_0)^{n-k+1}} = -\varepsilon\sum_{n=0}^{\infty}\frac{1}{n!}\partial^n v(z_0)\mathcal{A}^{(n)} \qquad [2.46]\end{aligned}$$

as the residue picks up the $k = n$ pole only. The same, of course, holds for the anti-holomorphic part.

2.22 p 46: Eq. (2.4.14) Conformal Transformation of an Operator, II

We wish to determine the coefficients $\mathcal{A}^{(n)}(w)$ in the OPE

$$T(z)\mathcal{A}(w) \sim \sum_{n=0}^{\infty} \frac{\mathcal{A}^{(n)}(w)}{(z-w)^{n+1}} \qquad [2.47]$$

for an operator that satisfies $\mathcal{A}'(z') = \zeta^{-h}\mathcal{A}(z)$ under a conformal transformation $z' = \zeta z$. Let us first go to an infinitesimal rescaling $\zeta = 1 + \varepsilon$. Then $z' = (1+\varepsilon)z = z + \varepsilon v(z)$ for $v(z) = z$. From (2.4.12), we have then that the only non-vanishing terms are

$$\begin{aligned}\delta\mathcal{A}(z) &= -\varepsilon \sum_{n=0}^{\infty} \frac{1}{n!}\partial^n v(z)\mathcal{A}^{(n)}(z) \\ &= -\varepsilon\left(\frac{1}{0!}\partial^0 v(z)\mathcal{A}^{(0)}(z) + \frac{1}{1!}\partial^1 v(z)\mathcal{A}^{(1)}(z)\right) \\ &= -\varepsilon(z\mathcal{A}^{(0)}(z) + \mathcal{A}^{(1)}(z)) \end{aligned} \qquad [2.48]$$

But we also have the transformation law

$$\mathcal{A}'(z') = \zeta^{-h}\mathcal{A}(z) = (1+\varepsilon)^{-h}\mathcal{A}(z) = (1-h\varepsilon)\mathcal{A}(z) \qquad [2.49]$$

But we also have, to first oder in ε

$$\mathcal{A}'(z') = \mathcal{A}'(z+\varepsilon z = \mathcal{A}'(z) + \varepsilon z\partial\mathcal{A}'(z) = \mathcal{A}'(z) + \varepsilon z\partial\mathcal{A}(z) \qquad [2.50]$$

Therefore

$$(1-h\varepsilon)\mathcal{A}(z) = \mathcal{A}'(z) + \varepsilon z\partial\mathcal{A} \qquad [2.51]$$

and thus

$$\delta\mathcal{A}(z) = \mathcal{A}'(z) - \mathcal{A}(z) = -\varepsilon z\partial\mathcal{A}(z) - h\varepsilon\mathcal{A}(z) \qquad [2.52]$$

Comparing both results for $\delta\mathcal{A}(z)$ we see that

$$\mathcal{A}^{(0)}(z) = \partial\mathcal{A}(z) \qquad \text{and} \qquad \mathcal{A}^{(1)}(z) = h\mathcal{A}(z) \qquad [2.53]$$

which shows (2.4.14).

Similarly, under a translation $z' = z+\varepsilon = z+\varepsilon v(z)$ for $v(z) = 1$, the only non-vanishing term in (2.4.12) is

$$\delta\mathcal{A}(z) = -\varepsilon\partial^0 v(z)\mathcal{A}^{(0)}(z) = -\varepsilon\mathcal{A}^{(0)}(z) \qquad [2.54]$$

Comparing this with $\delta\mathcal{A}(z) = -\varepsilon v(z)\partial\mathcal{A}(z) = -\varepsilon\partial\mathcal{A}(z)$ we recover

$$\mathcal{A}^{(0)}(z) = \partial\mathcal{A}(z) \qquad [2.55]$$

2.23 p 46: Eq. (2.4.16) Conformal Transformation of a Primary Field

This should now be straightforward. Take the infinitesimal transformation $z' = z + \varepsilon v(z)$. We then have for a primary field, on the one hand

$$\mathcal{O}'(z') = \mathcal{O}'(z + \varepsilon v(z)) = \mathcal{O}'(z) + \varepsilon v(z)\partial\mathcal{O}'(z) = \mathcal{O}'(z) + \varepsilon v(z)\partial\mathcal{O}(z) \qquad [2.56]$$

and on the other hand

$$\begin{aligned}\mathcal{O}'(z') &= (\partial z')^{-h}\mathcal{O}(z) = [\partial(z + \varepsilon v(z))]^{-h}\,\mathcal{O}(z) = (1 + \varepsilon\partial v(z))^{-h}\mathcal{O}(z)\\ &= (1 - h\varepsilon\partial v(z))\mathcal{O}(z)\end{aligned} \qquad [2.57]$$

Equating both expressions for $\mathcal{O}'(z')$ we get

$$\delta\mathcal{O}(z) = \mathcal{O}'(z) - \mathcal{O}(z) = -\varepsilon v(z)\partial\mathcal{O}(z) - h\varepsilon\partial v(z)\mathcal{O}(z) \qquad [2.58]$$

and so, because $v(z)$ is an arbitrary holomorphic function, comparing with (2.4.12)

$$\mathcal{O}^{(0)}(z) = \partial\mathcal{O}^{(0)}(z); \qquad \mathcal{O}^{(1)}(z) = h\mathcal{O}^{(0)}(z); \qquad \mathcal{O}^{(n\geq 2)}(z) = 0 \qquad [2.59]$$

and this corresponds to (2.4.16).

2.24 p 46: Eq. (2.4.17) Conformal Transformation of Typical Operators

$T(z)X^\mu(w)$ is given in (2.4.6). From this we get

$$T(z)\partial X^\mu(w) = \partial_w\left(\frac{\partial X^\mu(w)}{z - w}\right) = \frac{\partial X^\mu(w)}{(z - w)^2} + \frac{\partial(\partial X^\mu)(w)}{z - w} \qquad [2.60]$$

i.e. ∂X^μ is a $(1, 0)$ primary field. Taking one more derivative

$$\begin{aligned}T(z)\partial^2 X^\mu(w) &= \partial_w\left[\frac{\partial X^\mu(w)}{(z - w)^2} + \frac{\partial(\partial X^\mu)(w)}{z - w}\right]\\ &= \frac{2\partial X^\mu(w)}{(z - w)^3} + \frac{2\partial X^\mu(w)}{(z - w)^2} + \frac{\partial(\partial^2 X^\mu)(w)}{z - w}\end{aligned} \qquad [2.61]$$

so this is a $(2,0)$ operator, but it is not primary because it has a term in $(z-w)^{-3}$. Finally

$$\begin{aligned}
T(z) :e^{ik\cdot X(w)}:= &-\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: (z) \sum_{n=0}^{\infty} \frac{i^n}{n!} :(k\cdot X)^n: (w) \\
= &-\frac{1}{\alpha'}\left[\sum_{n=0}^{\infty} \frac{2ni^n}{n!} k_\nu \partial \overbrace{X^\mu(z) X}^{\nu}(w) :(k\cdot X)^{n-1}: (w)\right. \\
&+\left.\left[\sum_{n=0}^{\infty} \frac{2n(n-1)i^n}{n!} k_\nu k_\sigma \partial \overbrace{X^\mu(z) X}^{\nu}(w) \partial \overbrace{X_\mu(z) X}^{\sigma}(w) :(k\cdot X)^{n-2}: (w)\right]\right. \\
= &-\frac{1}{\alpha'}\left[2k_\nu\left(-\frac{\eta^{\mu\nu}\alpha'}{2(z-w)}\right) i \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} :\partial X^\mu(z)(k\cdot X)^{n-1}(w):\right. \\
&\left.+ k_\nu k_\sigma \left(-\frac{\eta^{\mu\nu}\alpha'}{2(z-w)}\right)\left(-\frac{\delta^\sigma_\mu \alpha'}{2(z-w)}\right) i^2 \sum_{n=2}^{\infty} \frac{i^{n-2}}{(n-2)!} :(k\cdot X)^{n-2}(w):\right] \\
= &\frac{ik_\mu :\partial^\mu X(z) e^{ik\cdot X(w)}:}{z-w} + \frac{\alpha' k^\mu k_\mu :e^{ik\cdot X(w)}:}{4(z-w)^2} \\
\sim &\frac{\frac{\alpha' k^2}{4} :e^{ik\cdot X(w)}:}{(z-w)^2} + \frac{\partial :e^{ik\cdot X(w)}:}{z-w}
\end{aligned} \qquad [2.62]$$

In the last line we have Taylor expanded $\partial X^\mu(z)(k\cdot X)^{n-1}(w)$. Thus, $:e^{ik\cdot X(w)}:$ is a primary field with weight $\alpha' k^2/4$.

2.25 p 48: Eq. (2.4.23) Conformal Transformation of the Energy-Momentum Tensor

The OPE of the energy-momentum tensor with itself is

$$T(z)T(w) \sim \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \qquad [2.63]$$

Using (4.4.11) we immediately find from this that, writing

$$T(z)T(w) \sim \sum_{n=0}^{\infty} \frac{T^{(n)}(w)}{(z-w)^{n+1}} \qquad [2.64]$$

the only non-vanishing $T^{(n)}(z)$ are

$$T^{(3)}(z) = D/2; \qquad T^{(1)}(z) = 2T(z); \qquad T^{(0)}(z) = \partial T(z) \qquad [2.65]$$

Using this in (2.4.12) we find

$$\begin{aligned}\delta T(z) =& -\varepsilon\left[\frac{1}{3!}\partial^3 v(z)T^{(3)}(z)+\frac{1}{1!}\partial^1 v(z)T^{(1)}(z)+\frac{1}{0!}\partial^0 v(z)T^{(0)}(z)\right]\\ =& -\varepsilon\left[\frac{D}{12}\partial^3 v(z)+2\partial v(z)T(z)+v(z)\partial T(z)\right]\end{aligned}\qquad [2.66]$$

This is (2.4.23), but for some reason Joe has put the ε in the $v(z)$ here.

It may be instructive to calculate this as well directly from the contour integration.

$$\begin{aligned}\delta T(w) &= i\varepsilon\frac{1}{2\pi i}\oint_C iv(z)T(z)T(w)\\ &= i\varepsilon\frac{1}{2\pi i}\oint_C iv(z)\left[\frac{D/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}\right]\end{aligned}\qquad [2.67]$$

Using

$$\begin{aligned}\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^n} &= \frac{1}{2\pi i}\oint_C f(z)\frac{(-1)^{n-1}}{(n-1)!}\partial^{n-1}\frac{1}{z-z_0}\\ &= \frac{1}{(n-1)!}\frac{1}{2\pi i}\oint_C\frac{\partial^{n-1}f(z)}{z-z_0} = \frac{1}{(n-1)!}\partial^{n-1}f(z_0)\end{aligned}\qquad [2.68]$$

this gives

$$\delta T(w) = -\varepsilon\left[\frac{D}{12}\partial^3 v(w)+2\partial v(w)T(w)+v(w)\partial T(w)\right]\qquad [2.69]$$

which is, of course, the same transformation rule.

2.26 p 48: Eq. (2.4.27) The Schwarzian Derivative

Let us check the infinitesimal form with $f(z) = z' = z+\varepsilon v(z)$:

$$\{z',z\} = \frac{2\partial^3 z'\partial z' - 3(\partial^2 z')^2}{2(\partial z')^2} = \frac{2\varepsilon\partial^3 v(1+\varepsilon\partial v)-3(\varepsilon\partial^2 v)^2}{2(1+\varepsilon\partial v)^2} = \varepsilon\partial^3 v + o(\varepsilon^2)\qquad [2.70]$$

Moreover

$$\begin{aligned}(\partial z')^2 T'(z') &= (1+\varepsilon\partial v)^2\left[T'(z)+\varepsilon v\partial T'(z)\right]\\ &= T'(z)+\varepsilon v\partial T(z)+2\varepsilon\partial v T(z)+o(\varepsilon^2)\end{aligned}\qquad [2.71]$$

Therefore

$$T'(z)+\varepsilon v\partial T(z)+2\varepsilon\partial v T(z) = T(z)-\frac{c}{12}\varepsilon\partial^3 v\qquad [2.72]$$

and thus we recover

$$\delta T(z) = T'(z) - T(z) = -\frac{c}{12}\varepsilon\partial^3 v - 2\varepsilon\partial v T(z) - \varepsilon v\partial T(z) \qquad [2.73]$$

We will leave it as an exercise that the Schwarzian derivative satisfies the correct composition rule. Doing it here, does not really add any value.

2.27 p 49: Eq. (2.5.2) The Linear Dilaton Central Charge

We just focus on the additional contribution to central charge. It comes from the contraction of the double derivatives

$$\begin{aligned} V_\mu V_\nu \partial^2 \overbrace{X^\mu(z)\partial^2 X}^{}{}^\mu(0) &= V_\mu V_\nu \partial_z^2\partial_w^2\left(-\frac{\alpha'\eta^{\mu\nu}}{2}\ln|z|^2\right) = -\frac{\alpha'}{2}\left(-\frac{6}{(z-w)^4}\right) \\ &= \frac{6\alpha' V^2/2}{(z-w)^4} \end{aligned} \qquad [2.74]$$

2.28 p 49: Eq. (2.5.3) The Linear Dilaton Transformation

We now have

$$\begin{aligned} T(z)X^\mu(0) &= \left(-\frac{1}{\alpha'}\partial X^\nu\partial X_\nu(z) + V_\nu\partial^2 X^\nu\right)X^\mu(0) \\ &= -\frac{2}{\alpha'}\partial \overbrace{X^\nu(z)X}^{}{}^\mu(0)\partial X_\nu(z) + V_\nu\partial^2\overbrace{X^\nu(z)X}^{}{}^\mu(0) \\ &\sim -\frac{2}{\alpha'}\left(-\frac{\alpha'\eta^{\mu\nu}}{2}\partial\ln|z|^2\right)\partial X_\nu(z) + V_\nu\left(-\frac{\alpha'\eta^{\mu\nu}}{2}\partial^2\ln|z|^2\right) \\ &\sim \frac{\alpha' V^\mu/2}{z^2} + \frac{\partial X^\mu(0)}{z} \end{aligned} \qquad [2.75]$$

This gives

$$X^{\mu(0)} = \partial X^\mu; \qquad X^{\mu(1)} = \alpha' V^\mu/2 \qquad [2.76]$$

and thus

$$\delta X^\mu(z) = -\varepsilon\left(v(z)\partial X^\mu(z) - \frac{\alpha' V^\mu}{2}\partial v(z)\right) \qquad [2.77]$$

Here and in the remainder we will not explicitly write the normal ordering symbols. We will always assume they are present, unless explicitly stated. Note the the OPE $T(z)X^\mu(0)$ does have a second order pole but the numerator is not of the form $hX^\mu(0)$, so under this energy-momentum tensor it is not a primary field.

2.29 p 50: Eq. (2.5.4) The bc Action is Conformally Invariant

Under a conformal transformation the fields b and c transforms as

$$\delta b = -\varepsilon(v\partial b + \lambda\partial v b); \qquad \delta c = -\varepsilon(v\partial c + (1-\lambda)\partial v c) \qquad [2.78]$$

for a $v = v(z)$. We therefore have

$$\delta(b\bar{\partial}c) = (\delta b)\bar{\partial}c + b\bar{\partial}\delta c = -\varepsilon\left[(v\partial b + \lambda\partial v b)\bar{\partial}c + b\bar{\partial}(v\partial c + (1-\lambda)\partial v c)\right] \qquad [2.79]$$

Let us collect the terms independent of λ, and ignoring $-\varepsilon$:

$$\begin{aligned} v\partial b\bar{\partial}c + b\bar{\partial}(v\partial c) + b\bar{\partial}(\partial v c) &= v\partial b\bar{\partial}c + bv\bar{\partial}\partial c + b\partial v\bar{\partial}c \\ &= v\partial b\bar{\partial}c + bv\bar{\partial}\partial c - \partial b v\bar{\partial}c - bv\partial\bar{\partial}c = 0 \end{aligned} \qquad [2.80]$$

by the fact that $\bar{\partial}v(z) = 0$. Likewise for the terms in λ we find

$$\partial v b\bar{\partial}c - b\bar{\partial}(\partial v c) = \partial v b\bar{\partial}c - b(\bar{\partial}\partial)vc - b\partial v\bar{\partial}c = 0 \qquad [2.81]$$

again using $\bar{\partial}v(z) = 0$.

2.30 p 50: Eq. (2.5.11) The Ghost Energy-Momentum Tensor

Let us check that the ghost fields are primary fields with the correct weights. The ghost energy-momentum tensor can be written as

$$T(z) = (1-\lambda)\partial bc(z) - \lambda b\partial c(z) \qquad [2.82]$$

Once more we don't write the normal ordering symbols. Thus, using $b(z)c(0) = c(z)b(0) \sim 1/z$,

$$\begin{aligned} T(z)b(0) &= (1-\lambda)\partial bc(z)b(0) - \lambda b\partial c(z)b(0) \\ &= \frac{(1-\lambda)\partial b(z)}{z} + \frac{\lambda b(z)}{z^2} \sim \frac{\lambda b(0)}{z^2} + \frac{\partial b(0)}{z} \end{aligned} \qquad [2.83]$$

where we have Taylor expanded $b(z)$ in the last line. Similarly

$$\begin{aligned} T(z)c(0) &= (1-\lambda)\partial bc(z)c(0) - \lambda b\partial c(z)c(0) \\ &= -(1-\lambda)c\partial b(z)c(0) + \lambda\partial cb(z)c(0) \\ &= \frac{(1-\lambda)c(z)}{z^2} + \frac{\lambda\partial c(z)}{z^2} \sim \frac{(1-\lambda)c(0)}{z^2} + \frac{\partial c(0)}{z} \end{aligned} \qquad [2.84]$$

and so $b(z)$ and $c(z)$ are indeed primary fields with weights λ and $1-\lambda$ respectively.

2.31 p 51: Eq. (2.5.12) The Ghost Central Charge

We'll just work out the central charge of the ghost energy-momentum tensor. I needs to come from the double contractions in $T(z)T(0)$:

$$\begin{aligned} T(z)T(0) &= [(1-\lambda)\partial bc(z) - \lambda b\partial c(z)]\,[(1-\lambda)\partial bc(0) - \lambda b\partial c(0)] \\ &= (1-\lambda)^2 \partial bc(z)\partial bc(0) + \lambda^2 b\partial c(z) b\partial c(0) \\ &\quad - \lambda(1-\lambda)\,[\partial bc(z) b\partial c(0) + b\partial c(z)\partial bc(0)] \\ &\sim (1-\lambda)^2 \partial b(z)c(0)c(z)\partial b(0) + \lambda^2 b(z)\partial c(0)\partial c(z)b(0) \\ &\quad - \lambda(1-\lambda)\left[\partial b(z)\partial c(0)c(z)b(0) + b(z)c(0)\partial c(z)\partial b(0)\right] + \cdots \\ &= (1-\lambda)^2\left(-\frac{1}{z^2}\right)\left(+\frac{1}{z^2}\right) + \lambda^2\left(+\frac{1}{z^2}\right)\left(-\frac{1}{z^2}\right) \\ &\quad - \lambda(1-\lambda)\left[\left(-\frac{2}{z^3}\right)\left(\frac{1}{z}\right) + \left(\frac{1}{z^2}\right)\left(-\frac{2}{z^3}\right)\right] \\ &= \frac{-(1-\lambda)^2 - \lambda^2 + 4\lambda(1-\lambda)}{z^4} = \frac{-6\lambda^2 + 6\lambda - 1}{z^4} \end{aligned} \qquad [2.85]$$

and so the central charge is given bu

$$c = -12\lambda^2 + 12\lambda - 2 = -3(2\lambda - 1)^2 + 1 \qquad [2.86]$$

2.32 p 51: Eq. (2.5.14) The Ghost Charge Current

This is obviously a symmetry of the action:

$$\delta(b\bar{\partial}c) = (\delta b)\bar{\partial}c + b(\bar{\partial}\delta c) = -i\varepsilon b\bar{\partial}c + i\varepsilon b\bar{\partial}c = 0 \qquad [2.87]$$

This variation is zero and not a total divergence $\partial_\mu \mathcal{J}^\mu$, so we have $\mathcal{J}^\mu = 0$ and the Noether current [**??**] is given by

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) - \mathcal{J}^\mu \Rightarrow \frac{\partial\mathcal{L}}{\partial(\bar{\partial}c)}\Delta c = i\varepsilon bc \Rightarrow j(z) = -bc(z) \qquad [2.88]$$

2.33 p 51: Eq. (2.5.15) The Conformal Transformation of the Ghost Charge, I

$$
\begin{aligned}
T(z)j(0) &= [(1-\lambda)\partial bc(z) - \lambda b\partial c(z)]\,(-bc(0)) \\
&= -(1-\lambda)\Big[\overbracket{\partial b(z)c(0)} :c(z)b(0): + \overbracket{c(z)b(0)} :\partial b(z)c(0): + \overbracket{\partial b(z)c(0)}\,\overbracket{c(z)b(0)}\Big] \\
&\quad + \lambda\Big[\overbracket{b(z)c(0)} :\partial c(z)b(0): + \overbracket{\partial c(z)b(0)} :b(z)c(0): + \overbracket{b(z)c(0)}\,\overbracket{\partial c(z)b(0)}\Big] \\
&\sim -(1-\lambda)\left[-\frac{cb(0)+z\partial cb(0)}{z^2} + \frac{\partial bc(0)}{z} - \frac{1}{z^2}\frac{1}{z}\right] \\
&\quad + \lambda\left[\frac{\partial cb(0)}{z} - \frac{bc(0)+z\partial bc(0)}{z^2} - \frac{1}{z}\frac{1}{z^2}\right] \\
&\sim \frac{(1-\lambda)-\lambda}{z^3} + \frac{(1-\lambda)cb(0)-\lambda bc(0)}{z^2} \\
&\quad + \frac{(1-\lambda)(\partial cb(0)-\partial bc(0)) + \lambda(\partial cb(0)-\partial bc(0))}{z} \\
&\sim \frac{1-2\lambda}{z^3} + \frac{-bc(0)}{z^2} + \frac{\partial(-bc(0))}{z} \sim \frac{1-2\lambda}{z^3} + \frac{j(0)}{z^2} + \frac{\partial j(0)}{z}
\end{aligned}
\qquad [2.89]
$$

2.34 p 51: Eq. (2.5.16) The Conformal Transformation of the Ghost Charge, II

From the OPE $T(z)j(0)$ we read off

$$
j^{(2)} = 1-2\lambda; \qquad j^{(1)} = j; \qquad j^{(0)} = \partial j \qquad [2.90]
$$

from which we readily find using (2.4.12)

$$
\delta j(z) = -\varepsilon\left[\frac{1-2\lambda}{2}\partial^2 v + j\partial v + v\partial j\right] \qquad [2.91]
$$

Joe's book just has the ε included in the v.

2.35 p 51: Eq. (2.5.17) The Conformal Transformation of the Ghost Charge, III

The finite form of the conformal transformation follows from a calculation identical to that of (2.4.26)

2.36 p 52: Eq. (2.5.24) The Central Charge of the $\beta\gamma$ System

We will not bother working this out as it should be straightforward by now. However we will point out already here how the critical dimensions arise in a path integral quantisation of the bosonic and the superstring.

For the bosonic string we have D fields X^μ, with central charge $c_X = D$, and we then have a bc ghost system with $\lambda = 2$, i.e. a central charge $c_{bc} = -3(2\lambda - 1)^2 + 1 = -26$. Together the bosonic string has central charge

$$c = c_X + c_{bc} = D - 26 \qquad [2.92]$$

and this central charge vanishes for $D = 26$.

For the superstring we add D fermions ψ with central charge $c_\psi = D/2$ and a $\beta\gamma$ ghost system with $\lambda = 3/2$ and central charge $c_{\beta\gamma} = 3(2\lambda - 1)^2 - 1 = 11$. The total central charge of the superstring is thus

$$c = c_X + c_\psi + c_{bc} + c_{\beta\gamma} = D + \frac{D}{2} - 26 + 11 = \frac{3D}{2} - 15 \qquad [2.93]$$

and this central charge vanishes for $D = 10$.

Of course, all this will have to be explained later, including the fact why the central charge needs to vanish. Spoiler: this is to keep the theory anomaly free.

2.37 p 53: Eq. (2.6.4) The Complex Coordinates

For the complex coordinates $z = \exp\left(i\sigma^2 + i\sigma^1\right)$ we see that $\sigma^2 = -\infty$ which corresponds to worldsheet time at minus infinity corresponds to the point $z = 0$. The fact that this is a single point, irrespective of the value of σ^1 will be important later when the state-operator correspondence is discussed. It basically means that every asymptotic state at $\sigma^2 = -\infty$ can be mapped to the action of an operator at the origin of the complex plane.

2.38 p 53: Eq. (2.6.7) The Fourier Expansion

We can write (2.6.7.a) as

$$T_{ww}(w) = -\sum_{n=-\infty}^{\infty} e^{inw} T_n \qquad [2.94]$$

making manifest that it is just a Fourier expansion.

2.39 p 54: Eq. (2.6.8) The Relation Between L_m and T_m

First we write T_{zz} in terms of w:

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_n}{z^{m+2}} = \sum_{m=-\infty}^{\infty} L_m e^{+i(m+2)w} \qquad [2.95]$$

We now use (2.6.9) to write T_{zz} in terms of T_{ww} and use $\partial_w z = \partial_w e^{-iw} = -ie^{-iw} = -iz$

$$\begin{aligned} T_{zz}(z) &= (\partial_w z)^{-2} \left(T_{ww}(w) - \frac{c}{24}\right) = (-iz)^{-2} \left(T_{ww}(w) - \frac{c}{24}\right) \\ &= -\, e^{-2iw} \left(- \sum_{m=-\infty}^{\infty} e^{imw} T_m - \frac{c}{24}\right) \\ &= \sum_{m=-\infty}^{\infty} e^{i(m+2)w} \left(T_m + \frac{c}{24}\delta_{m,0}\right) \end{aligned} \qquad [2.96]$$

Comparing the two expressions for $T_z z(z)$ gives

$$L_m = T_m + \frac{c}{24}\delta_{m,0} \qquad [2.97]$$

2.40 p 54: Eq. (2.6.9) The Relation Between T_{zz} and T_{ww}

T_{zz} and T_{ww} are related by a conformal transformation $z = e^{-iw}$, so we can use (2.4.26)

$$(\partial_w z)^2 T_{zz}(z) = T_{ww}(w) - \frac{c}{12}\{z, w\} \qquad [2.98]$$

The Schwarzian derivative is, using $\partial_w z = \partial_w e^{-iw} = -ie^{-iw} = -iz$ etc

$$\{z, w\} = \frac{2\partial_w^3 z \partial_w z - 3(\partial_w^2 z)^2}{2(\partial_w z)^2} = \frac{2(iz)(-iz) - 3(-z)^2}{2(-iz)^2} = \frac{2-3}{-2} = \frac{1}{2} \qquad [2.99]$$

Therefore

$$T_{ww}(w) = (\partial_w z)^2 T_{zz}(z) + \frac{c}{24} \qquad [2.100]$$

2.41 p 54: Eq. (2.6.10) The Hamiltonian

We first wish to write T_{22} in terms of $T(w)$. By definition (2.3.15)

$$\begin{aligned} T_{22} &= -\frac{1}{\alpha'}\left[\partial_2 X^\mu \partial_2 X_\mu - \frac{1}{2}(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial^2 X_\mu)\right] \\ &= \frac{1}{2\alpha'}\left(\partial_1 X^\mu \partial_1 X_\mu - \partial_2 X^\mu \partial_2 X_\mu\right) \end{aligned} \qquad [2.101]$$

Now $w = \sigma^1 + i\sigma^2$ and $\bar{w} = \sigma^1 - i\sigma^2$ and therefore

$$\begin{aligned}\partial_1 &= \partial_1 w\partial + \partial_1\bar{w}\bar{\partial} = \partial + \bar{\partial}\\ \partial_2 &= \partial_2 w\partial + \partial_2\bar{w}\bar{\partial} = i(\partial - \bar{\partial})\end{aligned} \quad [2.102]$$

where $\partial = \partial_w$ and $\bar{\partial} = \partial_{\bar{w}}$ Thus

$$\begin{aligned}T_{22} &= \frac{1}{2\alpha'}\left[(\partial+\bar{\partial})X^\mu(\partial+\bar{\partial})X_\mu - i(\partial-\bar{\partial})X^\mu i(\partial-\bar{\partial})X_\mu\right]\\ &= \frac{1}{2\alpha'}\left(2\partial X^\mu\partial X_\mu + 2\bar{\partial}X^\mu\bar{\partial}X_\mu\right)\\ &= -\left(T_{ww}(w) + T_{\bar{w}\bar{w}}(\bar{w})\right)\end{aligned} \quad [2.103]$$

Let us focus on the $T_{ww}(w)$ part. The contribution from $T_{\bar{w}\bar{w}}(\bar{w})$ will then be straightforward. The integration of σ^1 between 0 and 2π corresponds in the $z = e^{-iw}$ plane to a contour integration around the origin, see fig. 2.3.b if that is not clear. But because of the formula $z = e^{-iw}$ that integration is clockwise, and we pick up an extra sign to bring it to the standard counter-clockwise form. The measure also changes from $d\sigma^1 = dw = idz/z$. So

$$\begin{aligned}H &= \int_0^{2\pi}\frac{d\sigma^1}{2\pi}T_{22} = \int_0^{2\pi}\frac{dw}{2\pi}(-T_{ww}(w)) = \oint\frac{idz}{2\pi z}\left[(\partial_w z)^2 T_z z(z) + \frac{c}{24}\right]\\ &= -\oint\frac{dz}{2\pi iz}\left[-z^2T_{zz}(z) + \frac{c}{24}\right] = L_0 - \frac{c}{24}\end{aligned} \quad [2.104]$$

Adding the $T_{\bar{w}\bar{w}}(\bar{w})$ part gives

$$H = L_0 + \bar{L}_0 - \frac{c+\tilde{c}}{24} \quad [2.105]$$

2.42 p 55: Fig 2.3 The Contracted Contour Integration

Start from **(a)** in the figure below with the three (counter-clockwise) contours C_1, C_2 and C_3.

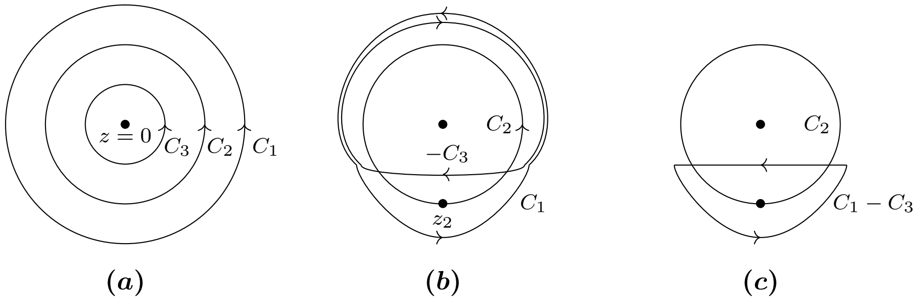


Figure 2.3: Deforming Contours. By deforming C_3 and C_1 is follows that $C_1 - C_3$ is equivalent to a contour around z_2

Flip C_3 to clockwise and deform it over C_2 as drawn in **(b)** to become $-C_3$. This contour deformation does not cross any singularities so the result is the same before and after deformation. Also deform C_1 as drawn in **(b)**. Now consider the sum of C_1 and $-C_3$. On the top half the contours C_1 and $-C_3$ cancel one another and what we are left with is a contour encircling the point z_2 as drawn in **(c)**.

2.43 p 55: Eq. (2.6.14) Switching Between OPEs and Commutation Relations

We start from (2.6.12) but change the indices of the charges to letters as this is less confusing. It is the radial ordering of the contours that determines the ordering of the corresponding operators. So $Q_a(C_1)Q_b(C_2) - Q_a(C_3)Q_b(C_2)$ means as operators on the Hilbert space $\hat{Q}_a\hat{Q}_b - \hat{Q}_b\hat{Q}_a = [\hat{Q}_a, \hat{Q}_b]$ as C_1 is the most outward contour, i.e. the largest time, and C_3 is the most inward contour, i.e. the smallest time. Now we also have

$$\begin{aligned} Q_a(C_1)Q_b(C_2) - Q_a(C_3)Q_b(C_2) &= Q_b(C_2)\big[Q_a(C_1) - Q_a(C_3)\big] \\ &= Q_b(C_2)Q_a(C_1 - C_3) \end{aligned} \qquad [2.106]$$

by the contour deformation of fig. 2.4. But, on the one hand the transformation of of an operator is given by the commutation relation with the corresponding charge, $\delta Q = [Q, A]$ and on the other hand we have from (2.3.11) that $\delta A(z) = \frac{1}{2\pi i}\oint dw\, j(w)A(z)$ with $j(z)$ the conserved current. Using a transformation under Q_a on an operator Q_b for a point on C_2 and the definition of the charges (2.6.11) we get

$$\delta Q_b\{C_2\} = [Q_a, Q_b]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_a(z_1) j_b(z_2) \qquad [2.107]$$

But the latter part just picks up the residue of the OPE $j_a(z_1)j_b(z_2)$ when $z_1 \to z_2$. Thus

$$\delta Q_b\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2} j_a(z_1)j_b(z_2) \tag{2.108}$$

and so we find

$$[\hat{Q}_a, \hat{Q}_b] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2} j_a(z_1)j_b(z_2) \tag{2.109}$$

Which is the relation that allows us to pass from OPEs to the commutation relations of conserved charges.

2.44 p 56: Eq. (2.6.19) The Virasoro Algebra

Let us apply the link between the commutation relations of the conserved charges and the OPE of the corresponding current Eq. (2.6.14) to derive one of the most important formula of string theory, viz. the Virasoro Algebra satisfied by the L_m. Form the definition (2.6.6), i.e. $L_m = \oint \frac{dz}{2\pi i} z^{m+1}T(z)$ and (2.6.14) we get, using repeatedly partial integration,

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2}\, z_1^{m+1}T(z_1)z_2^{n+1}T(z_2) \\
&= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2}\, z_1^{m+1}z_2^{n+1}\left[\frac{c/2}{z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}}\right] \\
&= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2}\, z_1^{m+1}z_2^{n+1}\left[-\frac{c}{12}\partial_1^3\frac{1}{z_{12}} - 2T(z_2)\partial_1\frac{1}{z_{12}} + \partial T(z_2)\frac{1}{z_{12}}\right] \\
&= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2}\, \left[\frac{c}{12}\partial_1^3 z_1^{m+1}z_2^{n+1} + 2\partial_1 z_1^{m+1}z_2^{n+1}T(z_2) + z_1^{m+1}z_2^{n+1}\partial T(z_2)\right]\frac{1}{z_{12}} \\
&= \oint \frac{dz_2}{2\pi i} \left[\frac{c}{12}(m+1)m(m-1)z_2^{m+n+1} + 2(m+1)z^{m+n+1}T(z_2) - \partial_2 z^{m+n+2}T(z_2)\right] \\
&= \oint \frac{dz_2}{2\pi i} \left[\frac{c}{12}(m^3-m)z_2^{m+n+1} + \big[2(m+1)-(m+n+2)\big]z^{m+n+1}T(z_2)\right] \\
&= \oint \frac{dz_2}{2\pi i} \left[\frac{c}{12}(m^3-m)z_2^{m+n+1} + (m-n)z^{m+n+1}T(z_2)\right] \\
&= \frac{c}{12}(m^3-m)\delta_{m+n,0} + (m-n)L_{m+n}
\end{aligned} \tag{2.110}$$

2.45 p 56: Eq. (2.6.24) The Transformation of Primary Fields

$$\begin{aligned}
[L_m, \mathcal{O}_n] &= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2} z_1^{m+1} T(z_1) z_2^{n+h-1}\mathcal{O}(z_2) \\
&= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2} z_1^{m+1} z_2^{n+h-1} \left[\frac{h\mathcal{O}(z_2)}{z_{12}^2} + \frac{\partial\mathcal{O}(z_2)}{z_{12}}\right] \\
&= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1\to z_2} z_1^{m+1} z_2^{n+h-1} \left[-h\mathcal{O}(z_2)\partial_1 + \partial\mathcal{O}(z_2)\right] \frac{1}{z_{12}} \\
&= \oint \frac{dz_2}{2\pi i} \left[\partial_1 z_1^{m+1} z_2^{n+h-1} h\mathcal{O}(z_2) + z_1^{m+1} z_2^{n+h-1}\partial\mathcal{O}(z_2)\right] \frac{1}{z_{12}} \\
&= \oint \frac{dz_2}{2\pi i} \left[h(m+1) z_2^{m+n+h-1}\mathcal{O}(z_2) - (m+n+h) z_2^{m+n+h-1}\mathcal{O}(z_2)\right] \\
&= \oint \frac{dz_2}{2\pi i} \left[(h-1)m - n\right] z_2^{m+n+h-1}\mathcal{O}(z_2) \\
&= \left[(h-1)m - n\right] \mathcal{O}_{m+n}
\end{aligned} \qquad [2.111]$$

2.46 p 56: Eq. (2.6.25) The Open String Boundary

As $w = \sigma^1 + i\sigma^2$ the condition $0 \le \operatorname{Re} w \le \pi$ is the same as $\sigma^1 \in [0, \pi]$. Now

$$z = -e^{-iw} = -e^{-i(\sigma^1 + i\sigma^2)} = -e^{\sigma^2 - i\sigma^1} = -e^{\sigma^2}\cos\sigma^1 + ie^{\sigma^2}\sin\sigma^1 \qquad [2.112]$$

Now $\sigma^1 \in [0, \pi]$ clearly implies that $\operatorname{Im} z \ge 0$. And vice-versa $\operatorname{Im} z \ge 0$ is only possible for $\sigma^1 \in [0, \pi] (\operatorname{mod} 2\pi)$.

2.47 p 58: Eq. (2.7.2) The Single Valuedness of X^μ

We have

$$\begin{aligned}
\alpha_0^\mu - \tilde{\alpha}_0^\mu &= \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} \partial X^\mu + \sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{\partial} X^\mu \\
&= \sqrt{\frac{2}{\alpha'}} \frac{1}{2\pi} \oint \left[dz \partial X^\mu + d\bar{z} \bar{\partial} X^\mu\right] \\
&= \sqrt{\frac{2}{\alpha'}} \frac{1}{2\pi} i \int_R d^2 z \left[\partial\bar{\partial} X^\mu - \bar{\partial}\partial X^\mu\right] = 0
\end{aligned} \qquad [2.113]$$

where we have used the divergence theorem (2.1.9).

2.48 p 58: Eq. (2.7.3) The Space-Time Momentum

The Noether current for the space-time translation is given in (2.3.13). The conserved charge of a current j^a is obtained by integrating over the entire space. Here, for closed strings, this is an integration over σ^1 between 0 and 2π in z coordinates this is the same as integrating over a circle in complex z plane. So the conserved charge p^μ corresponding to the space-time translational invariance is just proportional to $\oint dz\, j^\mu$. The anti-holomorphic part adds a $\oint d\bar{z}\, \tilde{\jmath}^\mu$ but with a minus sign because the contour has to be switched to counter-clockwise. Thus, with $j^\mu = (i/\alpha')\partial X^\mu$ and $\tilde{\jmath}^\mu = (i/\alpha')\bar{\partial} X^\mu$

$$\begin{aligned} p^\mu &= \frac{1}{2\pi i} \oint (dz\, j^\mu - d\bar{z}\, \tilde{\jmath}^\mu) \\ &= \frac{1}{2\pi i} \oint dz\, \frac{i}{\alpha'} \left(dz\, \partial X^\mu - d\bar{z} \bar{\partial} X^\mu\right) = \frac{1}{\alpha'} \left(\oint \frac{dz}{2\pi} \partial X^\mu - \oint \frac{d\bar{z}}{2\pi} \bar{\partial} X^\mu \right) \\ &= \frac{1}{\alpha'} \sqrt{\frac{\alpha'}{2}} (\alpha_0^\mu + \tilde{\alpha}_0^\mu) = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu \end{aligned} \qquad [2.114]$$

2.49 p 58: Eq. (2.7.4) Integrating ∂X^μ

We write

$$\partial X^\mu = -i\sqrt{\frac{\alpha'}{2}} \frac{\alpha_0^\mu}{z} - i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} = -i\frac{\alpha'}{2} p^\mu - i\sqrt{\frac{\alpha'}{2}} p^\mu - \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} \qquad [2.115]$$

Integrating gives

$$X^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} p^\mu \ln z + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \frac{\alpha_m^\mu}{z^m} + f(\bar{z}) \qquad [2.116]$$

for any arbitrary function $f(\bar{z})$. Similarly we have

$$X^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} p^\mu \ln \bar{z} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m} + \tilde{f}(z) \qquad [2.117]$$

Combining both expressions we find (2.7.4) where x^μ is just a constant.

2.50 p 59: Eq. (2.7.7) Normal Ordering for L_0

We have, first ignoring any normal ordering issues,

$$\begin{aligned}
L_0 = &= \oint \frac{dz}{2\pi i z} z^2 T(z) = -\frac{1}{\alpha'} \oint \frac{dz}{2\pi i} z \partial X^\mu \partial X_\mu \\
&= -\frac{1}{\alpha'} \oint \frac{dz}{2\pi i} z \left(-i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} \right) \left(-i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}} \right) \\
&= \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \oint \frac{\alpha_m^\mu \alpha_n^\mu}{z^{m+n+1}} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m^\mu \alpha_{\mu -m} \\
&= \frac{1}{2} \alpha_0^\mu \alpha_{\mu 0} + \frac{1}{2} \sum_{m=1}^{\infty} \alpha_m^\mu \alpha_{\mu -m} + \frac{1}{2} \sum_{m=1}^{\infty} \alpha_{-m}^\mu \alpha_{\mu m} \\
&= \frac{1}{4} \alpha' p^\mu + \sum_{m=1}^{\infty} \alpha_{-m}^\mu \alpha_{\mu m} + a^X
\end{aligned} \quad [2.118]$$

The a^X is a constant we add – possibly non- zero, but it will turn out to be zero – of interchanging the creation and annihilation operators in $\sum_{m=1}^{\infty} \alpha_m^\mu \alpha_{\mu -m}$. On p 22 we saw that in the light-cone gauge this amounted to the zero-point energies and was proportional to $\sum_{n=1}^{\infty} n$ which we regularised to $-1/12$. Here clearly the $\sum_{n=1}^{\infty} n$ from the commutation relations $[\alpha_m^\mu, \alpha_n^\nu] = m\, \eta^{\mu\nu} \delta_{m+n}$, but we need a better treatment than what we have seen before.

2.51 p 59: Eq. (2.7.9) $a^X = 0$

First we have from (2.7.7)

$$L_0 \left|0;0\right\rangle = \left(\frac{1}{4} \alpha' p^\mu + \sum_{m=1}^{\infty} \alpha_{-m}^\mu \alpha_{\mu m} + a^X \right) \left|0;0\right\rangle = a^X \left|0;0\right\rangle \quad [2.119]$$

By the fact that $p^\mu \left|0;0\right\rangle = 0$ and $\alpha_{\mu m} \left|0;0\right\rangle = 0$ for $m > 0$. But by the Virasoro algebra $L_0 = [L_1 L_{-1} - L_{-1} L_1]$ and we have

$$\begin{aligned}
L_1 \left|0;0\right\rangle \propto \sum_{n=-\infty}^{\infty} \alpha_{1-n}^\mu \alpha_{\mu n} \left|0;0\right\rangle &= \sum_{n=-\infty}^{-1} \alpha_{1-n}^\mu \alpha_{\mu n} \left|0;0\right\rangle \\
&= \sum_{n=-\infty}^{-1} \alpha_{\mu n} \alpha_{1-n}^\mu \left|0;0\right\rangle = 0
\end{aligned} \quad [2.120]$$

by the fact, once more, that $p^\mu |0;0\rangle = \alpha_{\mu m} |0;0\rangle = 0$ for $m > 0$. Similarly

$$\begin{aligned} L_{-1} |0;0\rangle \propto \sum_{n=-\infty}^{\infty} \alpha^\mu_{-1-n}\alpha_{\mu n} |0;0\rangle &= \sum_{n=-\infty}^{-1} \alpha^\mu_{-1-n}\alpha_{\mu n} |0;0\rangle \\ &= \sum_{n=-\infty}^{-1} \alpha_{\mu n}\alpha^\mu_{-1-n} |0;0\rangle = 0 \end{aligned} \quad [2.121]$$

Therefore $L_0 |0;0\rangle = [L_1 L_{-1} - L_{-1}L_1] |0;0\rangle = 0$ and so also $a^X = 0$.

Note that we have seen that $|0;0\rangle$ is invariant under $\{L_0, L_1, L_{-1}\}$. This means that $|0;0\rangle$ is invariant under the $SL(2,\mathbb{R})$ subalgebra of the Virasoro algebra.

2.52 p 60: Eq. (2.7.11) The Creation-Annihilation Normal Ordering

For $|z| > |z'|$, i.e. the worldsheet time coordinate σ^2 of z is at a later time than that of z' we have

$$\begin{aligned} X^\mu(z,\bar z)X^\nu(z',\bar z') = & \left[x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha^\mu_m}{z^m} + \frac{\tilde\alpha^\mu_m}{\bar z^m}\right)\right] \\ & \times \left[x^\nu - i\frac{\alpha'}{2}p^\nu \ln|z'|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\left(\frac{\alpha^\nu_n}{z'^n} + \frac{\tilde\alpha^\nu_n}{\bar z'^n}\right)\right] \end{aligned} \quad [2.122]$$

We want to bring all annihilation operators α^σ_k and $\tilde\alpha^\sigma_k$ for $k > 0$ to the right of the creation operators that have $k < 0$. So we need to keep track of the non-vanishing commutation relations in the products. Let us first look at x and p as they don't commute. Their products give

$$-i\frac{\alpha'}{2}\left(x^\mu p^\nu \ln|z'|^2 + p^\mu x^\nu \ln|z|^2\right) \quad [2.123]$$

We have defined p as a lowering operator and x as a raising operator, so we need the former on the right. The above product thus becomes, using $[x^\mu, p^\nu] = i\eta^{\mu\nu}$,

$$\begin{aligned} & -i\frac{\alpha'}{2}\left(x^\mu p^\nu \ln|z'|^2 + (x^\nu p^\mu - i\eta^{\mu\nu})\ln|z|^2\right) \\ = & -i\frac{\alpha'}{2}\left((x^\mu p^\nu \ln|z'|^2 + x^\nu p^\mu \ln|z|^2\right) - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2 \end{aligned} \quad [2.124]$$

We can thus write

$$\begin{aligned}&\left(x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2\right)\left(x^\nu - i\frac{\alpha'}{2}p^\nu \ln|z'|^2\right)\\ &= \mathop{\circ}\limits_{\circ}\left(x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2\right)\left(x^\nu - i\frac{\alpha'}{2}p^\nu \ln|z'|^2\right)\mathop{\circ}\limits_{\circ} - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2\end{aligned} \tag{2.125}$$

The other part that has non-vanishing commutation relations are the sums. We can focus on the "holomorphic" part α^μ only as they commute with the $\tilde{\alpha}^\nu$. The latter can just be added. Focussing on that part we find

$$\left(\sum_{m=-\infty}^{-1}\frac{1}{m}\frac{\alpha_m^\mu}{z^m} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{\alpha_m^\mu}{z^m}\right)\left(\sum_{n=-\infty}^{-1}\frac{1}{n}\frac{\alpha_n^\mu}{z'^n} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{\alpha_n^\mu}{z'^n}\right) \tag{2.126}$$

The only part of this that is not already in creation-annihilation normal ordering is the combination of the sums $\sum_{m=1}^{\infty}$ and $\sum_{n=-\infty}^{-1}$. This can be written as

$$\begin{aligned}&\sum_{m=1}^{\infty}\frac{1}{m}\frac{\alpha_m^\mu}{z^m}\sum_{n=-\infty}^{-1}\frac{1}{n}\frac{\alpha_n^\nu}{z'^n} = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{-1}\frac{1}{mn}\frac{1}{z^m z'^n}(\alpha_n^\nu\alpha_m^\mu + m\delta_{m+n,0}\eta^{\mu\nu})\\ &= \mathop{\circ}\limits_{\circ}\sum_{m=1}^{\infty}\frac{1}{m}\frac{\alpha_m^\mu}{z^m}\sum_{n=-\infty}^{-1}\frac{1}{n}\frac{\alpha_n^\nu}{z'^n}\mathop{\circ}\limits_{\circ} - \eta^{\mu\nu}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{z'}{z}\right)^m\end{aligned} \tag{2.127}$$

There is a similar contribution for the anti-holomorphic part. We can now bring all pieces together and find

$$\begin{aligned}X^\mu(z,\bar{z})X^\nu(z',\bar{z}') = &\ \mathop{\circ}\limits_{\circ}\left[x^\mu - i\frac{\alpha'}{2}p^\mu\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right)\right]\\ &\times\left[x^\nu - i\frac{\alpha'}{2}p^\nu\ln|z'|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}\left(\frac{\alpha_n^\nu}{z'^n} + \frac{\tilde{\alpha}_n^\nu}{\bar{z}'^n}\right)\right]\mathop{\circ}\limits_{\circ}\\ &- \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z|^2 - \left(i\sqrt{\frac{\alpha'}{2}}\right)^2\eta^{\mu\nu}\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{z'}{z}\right)^m + \left(\frac{\bar{z}'}{\bar{z}}\right)^m\right]\\ = &\ \mathop{\circ}\limits_{\circ}X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\mathop{\circ}\limits_{\circ} + \frac{\alpha'}{2}\eta^{\mu\nu}\left\{-\ln|z|^2 + \sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{z'}{z}\right)^m + \left(\frac{\bar{z}'}{\bar{z}}\right)^m\right]\right\}\end{aligned} \tag{2.128}$$

Now, we can write for $|z| > |z'|$

$$\begin{aligned}\ln|z-z'|^2 &= \ln z\left(1-\frac{z'}{z}\right)\bar{z}\left(1-\frac{\bar{z}'}{\bar{z}}\right)\\ &= \ln|z|^2 + \ln\left(1-\frac{z'}{z}\right) + \ln\left(1-\frac{\bar{z}'}{\bar{z}}\right)\end{aligned} \quad [2.129]$$

Using $\ln(1-x) = -\sum_{n=1}^{\infty} x^2/n$ we get

$$\ln|z-z'|^2 = \ln|z|^2 - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{z'}{z}\right)^m - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\bar{z}'}{\bar{z}}\right)^m \quad [2.130]$$

and thus we obtain

$$X^\mu(z,\bar{z})X^\nu(z',\bar{z}') = \,\circ\!\!\!\circ X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\circ\!\!\!\circ - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 \quad [2.131]$$

which is (2.7.11).

2.53 p 60: a^X from the Normal Ordering

From $\circ\!\!\!\circ X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\circ\!\!\!\circ = :X^\mu(z,\bar{z})X^\nu(z',\bar{z}'):$ it follows that

$$\sum_{k=-\infty}^{\infty}\frac{L_k}{z^{k+2}} = T(z) = -\frac{1}{\alpha'}:\partial X^\mu \partial X_\mu:(z) = -\frac{1}{\alpha'}\circ\!\!\!\circ\partial X^\mu\partial X_\mu\circ\!\!\!\circ(z) \quad [2.132]$$

Thus, using (2.7.1)

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{L_k}{z^{k+2}} &= -\frac{1}{\alpha'}\circ\!\!\!\circ\left(-i\sqrt{\frac{\alpha'}{2}}\right)^2\sum_{m=-\infty}^{\infty}\frac{\alpha_m^\mu}{z^{m+1}}\sum_{n=-\infty}^{\infty}\frac{\alpha_{\mu n}}{z^{n+1}}\circ\!\!\!\circ\\ &= \frac{1}{2}\sum_{m,n=-\infty}^{\infty}\frac{\circ\!\!\!\circ\alpha_m^\mu\alpha_{\mu n}\circ\!\!\!\circ}{z^{m+n+2}} = \frac{1}{2}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\circ\!\!\!\circ\alpha_m^\mu\alpha_{\mu k-m}\circ\!\!\!\circ}{z^{k+2}}\end{aligned} \quad [2.133]$$

Therefore

$$L_k = \frac{1}{2}\sum_{m=-\infty}^{\infty}\circ\!\!\!\circ\alpha_m^\mu\alpha_{\mu k-m}\circ\!\!\!\circ \quad [2.134]$$

and in particular

$$L_0 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\circ\!\!\!\circ\alpha_m^\mu\alpha_{\mu -m}\circ\!\!\!\circ \quad [2.135]$$

which gives (2.7.7) with $a^X = 0$.

2.54 p 61: Eq. (2.7.15) The Virasoro Generators for the Linear Dilaton CFT

$$L_m = \oint \frac{dz}{2\pi i z} z^{m+2} T(z) = \oint \frac{dz}{2\pi i z} z^{m+2} \left(-\frac{1}{\alpha'} \partial X^\mu \partial X_\mu + V_\mu \partial^2 X^\mu \right) \qquad [2.136]$$

The first part is just the same as for the standard X^μ CFT and gives $L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} {}^{\circ}_{\circ}\alpha^\mu_n \alpha_{\mu m-n}{}^{\circ}_{\circ}$. The second part gives

$$\begin{aligned} V_\mu \oint \frac{dz}{2\pi i z} z^{m+2} \partial \left(-i\sqrt{\frac{\alpha'}{2}} \right) \sum_{k=-\infty}^{\infty} \frac{\alpha^\mu_k}{z^{k+1}} &= i\sqrt{\frac{\alpha'}{2}} \sum_{k=-\infty}^{\infty} (k+1) V_\mu \alpha^\mu_k \oint \frac{dz}{2\pi i} \frac{1}{z^{m-k+1}} \\ &= i\sqrt{\frac{\alpha'}{2}} (m+1) V_\mu \alpha^\mu_m \end{aligned} \qquad [2.137]$$

Combining the two parts gives (2.7.15)

2.55 p 61: Eq. (2.7.17) The bc Ghost Commutators

The ghost components are given by

$$b_m = \oint \frac{dz}{2\pi i} z^{m+\lambda-1} b(z) \qquad \text{and} \qquad c_m = \oint \frac{dz}{2\pi i} z^{m-\lambda} c(z) \qquad [2.138]$$

Therefore

$$\begin{aligned} \{b_m, c_n\} &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^{m+\lambda-1} b(z_1) z_2^{n-\lambda} c(z_2) \\ &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^{m+\lambda-1} z_2^{n-\lambda} \frac{1}{z_1 - z_2} \\ &= \oint \frac{dz_2}{2\pi i} z_2^{m+n-1} = \delta_{m+n,0} \end{aligned} \qquad [2.139]$$

2.56 p 61: Eq. (2.7.18) The bc Vacuum States

Because $\{b_0, c_0\} = 1$ a state cannot be both annihilated by b_0 and c_0. Indeed, if there were such a state, say $|0\rangle$ then it would satisfy $b_0 |0\rangle = c_0 |0\rangle = 0$ and hence also $(b_0 c_0 + c_0 b_0)|0\rangle = 0$. But this is also $\{b_0, c_0\}|0\rangle = |0\rangle$. This means that $|0\rangle = 0$, i.e. there exists no such state.

So let us call $|\downarrow\rangle$ the state annihilated by b_m and c_m for $m > 0$ and by b_0. Let us then call $c_0 |\downarrow\rangle = |\uparrow\rangle$. We then have $c_0 |\uparrow\rangle = c_0 c_0 |\downarrow\rangle = 0$ as $c_0^2 = 0$. Let us now act with b_0 on $|\uparrow\rangle$. What does it give us? Well $b_0 |\uparrow\rangle = b_0 c_0 |\downarrow\rangle = (1 - c_0 b_0)|\downarrow\rangle = |\downarrow\rangle$. All this gives the relations (2.7.18).

2.57 p 61: Eq. (2.7.19) The bc Virasoro Generators

$$\begin{aligned}
L_k &= \oint \frac{dz}{2\pi i} z^{k+1} T(z) = \oint \frac{dz}{2\pi i} z^{k+1} \left[(1-\lambda)\partial b c - \lambda b \partial c\right] \\
&= \oint \frac{dz}{2\pi i} z^{k+1} \sum_{m,n=-\infty}^{\infty} b_m c_n \left[(1-\lambda)\left(\partial \frac{1}{z^{m+\lambda}}\right) \frac{1}{z^{n+1-\lambda}} - \lambda \frac{1}{z^{m+\lambda}} \left(\partial \frac{1}{z^{n+1-\lambda}}\right)\right] \\
&= \sum_{m,n=-\infty}^{\infty} b_m c_n \left[-(1-\lambda)(m+\lambda) + \lambda(n+1-\lambda)\right] \oint \frac{dz}{2\pi i} z^{k+1-m-\lambda-n-1+\lambda-1} \\
&= \sum_{m,n=-\infty}^{\infty} \left[\lambda(m+n) - m\right] b_m c_n \oint \frac{dz}{2\pi i} z^{k-m-n-1} \\
&= \sum_{m,n=-\infty}^{\infty} \left[\lambda(m+n) - m\right] b_m c_n \delta_{m+n-k,0} = \sum_{m=-\infty}^{\infty} \left[\lambda k - m\right] b_m c_{k-m}
\end{aligned} \tag{2.140}$$

We now bring this into creation-annihilation normal ordering. This only is an issue for $k = 0$ and thus introduces a constant times $\delta_{k,0}$:

$$L_k = \sum_{m=-\infty}^{\infty} \left[\lambda k - m\right] {}^\circ_\circ b_m c_{k-m} {}^\circ_\circ + \delta_{k,0} a^{\mathrm{g}} \tag{2.141}$$

Thus for L_0 we have

$$\begin{aligned}
L_0 &= -\sum_{m=-\infty}^{\infty} m b_m c_{-m} = -\left[\sum_{m=-\infty}^{-1} m b_m c_{-m} + \sum_{m=1}^{\infty} m b_m c_{-m}\right] \\
&= -\sum_{m=-\infty}^{\infty} m b_m c_{-m} = -\left[\sum_{m=-\infty}^{-1} m b_m c_{-m} + \sum_{m=1}^{\infty} m(-c_{-m} b_m + 1)\right] \\
&= -\sum_{m=-\infty}^{\infty} m {}^\circ_\circ b_m c_{-m} {}^\circ_\circ - \sum_{m=1}^{\infty} m
\end{aligned} \tag{2.142}$$

We would be tempted to use the heuristic $\sum_{m=1}^{\infty} m = -1/12$ here again, but this happens not to be correct this time.

2.58 p 61: Eq. (2.7.21) The bc Normal Ordering Constant a^{g}

On the one hand we have

$$L_0 \left|\downarrow\right\rangle = \left(-\sum_{n=-\infty}^{\infty} n {}^\circ_\circ b_n c_{-n} {}^\circ_\circ + a^{\mathrm{g}}\right) \left|\downarrow\right\rangle = a^{\mathrm{g}} \left|\downarrow\right\rangle \tag{2.143}$$

and on the other hand we have $2L_0 = [L_1, L_{-1}]$. We first compute

$$\begin{aligned} L_{-1}\left|\downarrow\right\rangle &= \sum_{n=-\infty}^{\infty}(-\lambda-n)\,\mathopen{\circ}b_nc_{-1-n}\mathclose{\circ}\left|\downarrow\right\rangle \\ &= -\left(\sum_{n=-\infty}^{-2}(\lambda+n)b_nc_{-1-n} + (\lambda-1)b_{-1}c_0 + \sum_{n=0}^{\infty}(\lambda+n)b_nc_{-1-n}\right)\left|\downarrow\right\rangle \\ &= -(\lambda-1)b_{-1}c_0\left|\downarrow\right\rangle = -(\lambda-1)b_{-1}\left|\uparrow\right\rangle \end{aligned} \quad [2.144]$$

We have used the fact that $c_m\left|\downarrow\right\rangle = 0$ for $m > 0$, that $b_n\left|\downarrow\right\rangle = 0$ for $n \geq 0$ and that $b_0\left|\downarrow\right\rangle = \left|\uparrow\right\rangle$. Continuing we find

$$\begin{aligned} L_1L_{-1}\left|\downarrow\right\rangle &= \sum_{n=-\infty}^{\infty}(\lambda-n)\,\mathopen{\circ}b_nc_{1-n}\mathclose{\circ}\left(-(\lambda-1)b_{-1}\right)\left|\uparrow\right\rangle \\ &= -(\lambda-1)\left[\sum_{n=-\infty}^{-1}(\lambda-n)b_nc_{1-n} + \lambda b_0c_1 - \sum_{n=1}^{\infty}(\lambda-n)c_{1-n}b_n\right]b_{-1}\left|\uparrow\right\rangle \\ &= -(\lambda-1)\left[0 + \lambda b_0c_1b_{-1} + 0\right]\left|\uparrow\right\rangle = -(\lambda-1)\lambda b_0(-b_{-1}c_1 + 1)\left|\uparrow\right\rangle \\ &= -\lambda(\lambda-1)b_0\left|\uparrow\right\rangle = -\lambda(\lambda-1)\left|\downarrow\right\rangle \end{aligned} \quad [2.145]$$

We also have

$$\begin{aligned} L_1\left|\downarrow\right\rangle &= \sum_{n=-\infty}^{\infty}(\lambda-n)\,\mathopen{\circ}b_nc_{1-n}\mathclose{\circ}\left|\downarrow\right\rangle \\ &= \left(\sum_{n=-\infty}^{0}(\lambda-n)b_nc_{1-n} - \sum_{n=1}^{\infty}(\lambda-n)c_{1-n}b_n\right)\left|\downarrow\right\rangle \\ &= \;= 0 \end{aligned} \quad [2.146]$$

We conclude that

$$2a^{\mathrm{g}}\left|\downarrow\right\rangle = [L_1, L_{-1}]\left|\downarrow\right\rangle = -\lambda(\lambda-1)\left|\downarrow\right\rangle \quad [2.147]$$

and thus

$$a^{\mathrm{g}} = \frac{1}{2}\lambda(1-\lambda) \quad [2.148]$$

2.59 p 62: a^{g} from Normal Ordering

We start by comparing $:b(z)c(z':)$ and ${}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ}$

$$
\begin{aligned}
{}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{n+1-\lambda}} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{0}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{n+1-\lambda}} + \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{n+1-\lambda}} \\
&= \sum_{m=-\infty}^{-1}\sum_{n=-\infty}^{0}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{m+1-\lambda}} + \sum_{m=0}^{\infty}\sum_{n=-\infty}^{0}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{n+1-\lambda}} \\
&\quad + \sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{m+1-\lambda}} + + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{{}^{\circ}_{\circ}b_m c_n{}^{\circ}_{\circ}}{z^{m+\lambda}z'^{n+1-\lambda}}
\end{aligned}
\tag{2.149}
$$

To bring this in creation-annihilation normal ordering, we only have to change the order of the ghost operators for the second term

$$
{}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} = \cdots - \sum_{m=0}^{\infty}\sum_{n=-\infty}^{0}\frac{c_n b_m}{z^{m+\lambda}z'^{n+1-\lambda}} + \cdots
\tag{2.150}
$$

Here $\cdots$ represent the first, third and fourth term. Thus

$$
\begin{aligned}
{}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} &= \cdots - \sum_{m=0}^{\infty}\sum_{n=-\infty}^{0}\frac{(-c_n b_m + \delta_{m+n,0})}{z^{m+\lambda}z'^{n+1-\lambda}} + \cdots + \\
&= \cdots + \sum_{m=0}^{\infty}\sum_{n=-\infty}^{0}\frac{b_m c_n}{z^{m+\lambda}z'^{m+1-\lambda}} - \sum_{m=0}^{\infty}\sum_{n=-\infty}^{0}\frac{\delta_{m+n,0}}{z^{m+\lambda}z'^{n+1-\lambda}} + \cdots
\end{aligned}
\tag{2.151}
$$

Therefore

$$
\begin{aligned}
{}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} &= b(z)c(z') - \sum_{m=0}^{\infty}\frac{1}{z^{m+\lambda}z'^{-n+1-\lambda}} = b(z)c(z') - \frac{1}{z^{\lambda}z'^{1-\lambda}}\sum_{m=0}^{\infty}\left(\frac{z'}{z}\right)^{m} \\
&= b(z)c(z') - \frac{1}{z^{\lambda}z'^{1-\lambda}}\frac{1}{1-z'/z} = b(z)c(z') - \left(\frac{z}{z'}\right)^{1-\lambda}\frac{1}{z-z'}
\end{aligned}
\tag{2.152}
$$

Now from (2.5.7) we have $:b(z)c(z') := b(z)c(z') - 1/(z-z')$ and thus

$$
\begin{aligned}
:b(z)c(z') :&= {}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} + \left(\frac{z}{z'}\right)^{1-\lambda}\frac{1}{z-z'} - \frac{1}{z-z'} \\
&= {}^{\circ}_{\circ}b(z)c(z'){}^{\circ}_{\circ} + \frac{(z/z')^{1-\lambda} - 1}{z-z'}
\end{aligned}
\tag{2.153}
$$

Note, en passant that

$$
\begin{aligned}
: bc : (z) = \lim_{z' \to z} : b(z)c(z') &:= \lim_{z' \to z} {}^{\circ}_{\circ} b(z)c(z') {}^{\circ}_{\circ} + \lim_{z' \to z} \frac{(z/z')^{1-\lambda} - 1}{z - z'} \\
&= {}^{\circ}_{\circ} bc {}^{\circ}_{\circ}(z) + \frac{1-\lambda}{z}
\end{aligned}
\tag{2.154}
$$

From this we find

$$
: \partial b(z)c(z') := {}^{\circ}_{\circ} \partial b(z)c(z') {}^{\circ}_{\circ} + \frac{(1-\lambda)(z/z')^{1-\lambda}}{z(z-z')} - \frac{(z/z')^{1-\lambda} - 1}{(z-z')^2}
\tag{2.155}
$$

and thus taking $z' \to z$

$$
: \partial bc : (z) = {}^{\circ}_{\circ} \partial bc {}^{\circ}_{\circ}(z) - \frac{(1-\lambda)\lambda}{2z^2}
\tag{2.156}
$$

Similarly we find

$$
: b\partial c : (z) = {}^{\circ}_{\circ} b\partial c {}^{\circ}_{\circ}(z) - \frac{(1-\lambda)(2-\lambda)}{2z^2}
\tag{2.157}
$$

Therefore

$$
\begin{aligned}
: T(z) &:= (1-\lambda) : \partial bc : - \lambda : b\partial c : \\
&= (1-\lambda){}^{\circ}_{\circ} \partial bc {}^{\circ}_{\circ}(z) - \frac{(1-\lambda)^2\lambda}{2z^2} - \lambda {}^{\circ}_{\circ} b\partial c {}^{\circ}_{\circ}(z) + \frac{(1-\lambda)(2-\lambda)\lambda}{2z^2} \\
&= {}^{\circ}_{\circ} T(z) {}^{\circ}_{\circ} - \frac{\lambda(1-\lambda)\left[(1-\lambda) - (2-\lambda)\right]}{2z^2} = {}^{\circ}_{\circ} T(z) {}^{\circ}_{\circ} + \frac{\lambda(1-\lambda)}{2z^2}
\end{aligned}
\tag{2.158}
$$

From this we immediately see that

$$
L_m = \sum_{n=-\infty}^{\infty} (m\lambda - n) {}^{\circ}_{\circ} b_n c_{m-n} {}^{\circ}_{\circ} + \frac{\lambda(1-\lambda)}{2} \delta_{m,0}
\tag{2.159}
$$

and thus indeed $a^{\mathrm{g}} = \frac{1}{2}\lambda(1-\lambda)$.

2.60 p 62: Eq. (2.7.22) The Ghost Number Operator

$$
N^{\mathrm{g}} = \frac{1}{2\pi i} \int_0^{2\pi} d\sigma^1 j(\sigma) = \frac{1}{2\pi i} \int_0^{2\pi} dw j(w)
\tag{2.160}
$$

From (2.5.17) we have

$$
(\partial_w z) j_z(z) = j_w(w) + \frac{2\lambda - 1}{2} \frac{\partial_w^2 z}{\partial_w z}
\tag{2.161}
$$

with $z = e^{-iw}$ we have $\partial_w z = -iz$ and $\partial_w^2 z = -z$ so that

$$-izj_z(z) = j_w(w) + \frac{2\lambda - 1}{2}\frac{-z}{-iz} \Rightarrow j_w(w) = -izj(z) + i\frac{2\lambda - 1}{2} \qquad [2.162]$$

Using this and $dw = idz/z$ we have

$$\begin{aligned} N^{\mathrm{g}} &= \frac{1}{2\pi i}\oint \frac{idz}{z} i\left(-zj(z) + \frac{2\lambda - 1}{2}\right) \\ &= -\frac{1}{2\pi i}\oint dz\,\left(-j(z) + \frac{2\lambda - 1}{2z}\right) \end{aligned} \qquad [2.163]$$

We now use [2.154]

$$N^{\mathrm{g}} = -\frac{1}{2\pi i}\oint dz\,\left(: bc : (z) + \frac{2\lambda - 1}{2}\right) = -\frac{1}{2\pi i}\oint dz\,\left({}^\circ_\circ bc {}^\circ_\circ(z) + \frac{1-\lambda}{z} + \frac{2\lambda - 1}{2z}\right) \qquad [2.164]$$

Let us first quickly do the last two terms. They give

$$-\frac{1}{2\pi i}\oint dz \frac{1}{2z} = -\frac{1}{2} \qquad [2.165]$$

The first terms gives

$$\begin{aligned} &-\frac{1}{2\pi i}\oint \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{{}^\circ_\circ b_m c_n {}^\circ_\circ}{z^{m+\lambda+n+1-\lambda}} = -\frac{1}{2\pi i}\oint \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{{}^\circ_\circ b_m c_n {}^\circ_\circ}{z^{m++n+1}} \\ &= -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} {}^\circ_\circ b_m c_n {}^\circ_\circ \delta_{m+n,0} = -\sum_{m=-\infty}^{\infty} {}^\circ_\circ b_m c_{-m} {}^\circ_\circ \\ &= -\sum_{m=-\infty}^{-1} {}^\circ_\circ b_m c_{-m} {}^\circ_\circ - {}^\circ_\circ b_0 c_0 {}^\circ_\circ - \sum_{m=1}^{\infty} {}^\circ_\circ b_m c_{-m} {}^\circ_\circ \\ &= -\sum_{m=-\infty}^{-1} b_m c_{-m} + c_0 b_0 + \sum_{m=1}^{\infty} c_{-m} b_m = \sum_{m=1}^{\infty}(c_{-m}b_m - b_{-m}c_m) + c_0 b_0 \end{aligned} \qquad [2.166]$$

Combining both contributions we find

$$N^{\mathrm{g}} = \sum_{m=1}^{\infty}(c_{-m}b_m - b_{-m}c_m) + c_0 b_0 - \frac{1}{2} \qquad [2.167]$$

2.61 p 62: Eq. (2.7.23) The Ghost Number of the Ghost Fields

Let us first check $[N^{\mathrm{g}}, b_m]$:

$$
\begin{aligned}
[N^{\mathrm{g}}, b_m] &= [\sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}, b_m] \\
&= \sum_{n=1}^{\infty}[c_{-n}b_n, b_m] - \sum_{n=1}^{\infty}[b_{-n}c_n, b_m] + [c_0 b_0, b_m]
\end{aligned} \tag{2.168}
$$

Now

$$
[c_\ell b_n, b_m] = c_\ell b_n b_m - b_m c_\ell b_n = -c_\ell b_m b_n - b_m c_\ell b_n = -\{c_\ell, b_m\} b_n = -\delta_{\ell+m,0} b_n \tag{2.169}
$$

and

$$
[b_\ell c_n, b_m] = b_\ell c_n b_m - b_m b_\ell c_n = b_\ell c_n b_m + b_\ell b_m c_n = b_\ell \{c_n, b_m\} = \delta_{m+n,0} b_\ell \tag{2.170}
$$

Thus

$$
[N^{\mathrm{g}}, b_m] = \sum_{n=1}^{\infty}(-\delta_{m-n,0} b_n - \delta_{m+n,0} b_{-n}) - \delta_{m,0} b_0 \tag{2.171}
$$

Let us first take $m > 0$. In that case, only the first term survives as $\delta_{m+n,0}$ has no solution for $m > 0$ and $n \geq 1$ and also $\delta_{m>0,0} = 0$. Thus $[N^{\mathrm{g}}, b_{m>0}] = -b_m$. Similarly, for $m < 0$ only the second term survives and we find $[N^{\mathrm{g}}, b_{m<0}] = -b_m$. Finally for $m = 0$ only the third term survives and we find $[N^{\mathrm{g}}, b_0] = -b_0$. In summary we have

$$
[N^{\mathrm{g}}, b_m] = -b_m \tag{2.172}
$$

Consider next $[N^{\mathrm{g}}, c_m]$:

$$
\begin{aligned}
[N^{\mathrm{g}}, c_m] &= [\sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}, c_m] \\
&= \sum_{n=1}^{\infty}[c_{-n}b_n, c_m] - \sum_{n=1}^{\infty}[b_{-n}c_n, c_m] + [c_0 b_0, c_m]
\end{aligned} \tag{2.173}
$$

Now

$$
[c_\ell b_n, c_m] = c_\ell b_n c_m - c_m c_\ell b_n = c_\ell b_n c_m + c_\ell c_m b_n = c_\ell \{b_n, b_m\} = \delta_{m+n,0} c_\ell \tag{2.174}
$$

and

$$
[b_\ell c_n, c_m] = b_\ell c_n c_m - c_m b_\ell c_n = -b_\ell c_m c_n - c_m b_\ell c_n = -\{b_\ell, c_m\} c_n = -\delta_{\ell+m,0} c_n \tag{2.175}
$$

Thus

$$[N^{\rm g}, c_m] = \sum_{n=1}^{\infty}(\delta_{m+n,0}c_{-n} + \delta_{m-n,0}c_n) + \delta_{m,0}c_0 \qquad [2.176]$$

Let us first take $m > 0$. In that case, only the second term survives as $\delta_{m+n,0}$ has no solution for $m > 0$ and $n \geq 1$ and also $\delta_{m>0,0} = 0$. Thus $[N^{\rm g}, c_{m>0}] = c_m$. Similarly, for $m < 0$ only the first term survives and we find $[N^{\rm g}, c_{m<0}] = c_m$. Finally for $m = 0$ only the third term survives and we find $[N^{\rm g}, c_0] = -c_0$. In summary we have

$$[N^{\rm g}, c_m] = c_m \qquad [2.177]$$

2.62 p 62: Eq. (2.7.24) The Ghost Number of the Vacuum

$$N^{\rm g}\left|\downarrow\right\rangle = \left(\sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0b_0 - \frac{1}{2}\right)\left|\downarrow\right\rangle = -\frac{1}{2}\left|\downarrow\right\rangle \qquad [2.178]$$

and

$$\begin{aligned} N^{\rm g}\left|\uparrow\right\rangle &= \left(\sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0b_0 - \frac{1}{2}\right)\left|\uparrow\right\rangle = c_0\left|\downarrow\right\rangle - \frac{1}{2}\left|\uparrow\right\rangle \\ &= \left|\uparrow\right\rangle - \frac{1}{2}\left|\uparrow\right\rangle = \frac{1}{2}\left|\uparrow\right\rangle \end{aligned} \qquad [2.179]$$

2.63 p 63: Eq. (2.8.1) From the Semi-Infinite Cylinder to the Unit Disk

Recall we that have $w = \sigma^1 + i\sigma^2$. We are considering the semi-infinite cylinder with $\operatorname{Im} w = \sigma^2 \leq 0$. The z coordinate is defined as $z = e^{-iw} = e^{-i\sigma^1+\sigma^2}$ so that constant world-sheet time σ^2 corresponds to circles around the origin in the z-plane. Now $\sigma^2 \to \infty$ corresponds to $z \to 0$ and $\sigma^2 = 0$ corresponds to $z = e^{-i\sigma^1}$, i.e. the unit circle. An initial state defined at $\sigma^2 = \operatorname{Im} w \to -\infty$ thus corresponds to an operator acting on the origin of the complex plane.

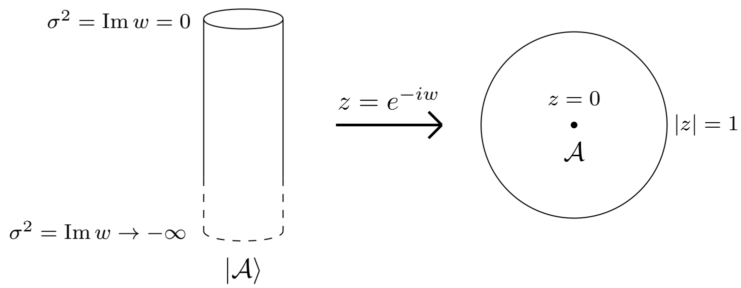


Figure 2.4: Mapping the semi-infinite cylinder to the unit disk. The points $\mathrm{Im}\, w = \sigma^2 \to -\infty$ correspond to the origin in the z-plane. The points $\mathrm{Im}\, w = \sigma^2 \to= 0$ correspond to the points on the unit disk in the z-plane

2.64 p 63: The State-Operator Isomorphism in 2d-CFTs

Let us clarify a bit why this is special and how it differs from an ordinary QFT. In an ordinary QFT we can of course obtain an asymptotic state by acting with an operator on the vacuum of the theory, i.e. $|\Phi\rangle = \lim_{t\to-\infty} \Phi\,|0\rangle$. In a 2d-CFT the inverse is also true because the states evolve radially from the origin. Any state in the Hilbert space can be evolved back with the Hamiltonian to the origin and that state can be created by the action of an operator at the origin. Thus any state in the Hilbert space of a CFT corresponds to a given operator as well. That is the isomorphism between states and operators in a 2d-CFT.

For a general QFT it is not true that we can evolve any state of the Hilbert space to a single point. We can of course evolve it to $t \to -\infty$, but there is still a $(D-1)$-dimensional hypersurface. So it is not clear where the operator acts exactly. In a QFT if two wave-packets evolve back in time in opposite directions, they don't end up in a single point, so this becomes inherently non-local. In a CFT both those wave packets will end up in the origin.

Another way to look at this is that in a CFT $t \to -\infty$ is exactly one point, the origin. We can thus compute correlation functions between fields at that point and other fields. In an ordinary QFT this is not possible.

Because of the isomorphism between states and operators in a CFT we can use either approach to make calculations, depending on which was is easier.

2.65 p 63: Eq (2.8.2) The Unit Operator and the Ground State

I personally find that the subtlety of this argument is not very clear in Joe's book, so it is worthwhile repeating it here. We consider the unit operator $\mathbf{1}$. It has trivial OPE with $\partial X^\mu(z)$, i.e. $\partial X^\mu(z)\mathbf{1} = \partial X^\mu(z)$ and the same for $\bar\partial X^\mu$. We know that the operator $\mathbf{1}$ corresponds to a certain state in the Hilbert space because of the state-operator isomorphism. Let us call that state $|?\rangle$ and work out what it is. Let us work out the action of α_m^μ for $m \geq 0$ on that state. By the isomorphism we know that, using (2.7.2a),

$$\alpha_m^\mu |?\rangle \equiv \oint \frac{dz}{2\pi i}\sqrt{\frac{2}{\alpha'}}z^m \partial X^\mu(z)\mathbf{1} \qquad [2.180]$$

The contour is a circle around the origin within the unit disk. The unit operator **1** acts on the origin as vertex operator. Within the contour ∂X^μ is holomorphic $\bar\partial(\partial X^\mu) = 0$. Indeed, as always $\bar\partial\partial X^\mu$ should be viewed as an operator equation within an expectation value

$$\langle \bar\partial\partial X^\mu \cdots \rangle = \text{contact terms} \qquad [2.181]$$

where only non-zero terms come from possible contact terms. Review the discussion on pages 35 and 36, leading to (2.1.20) if this is not clear. But as, by construction, the only operator within the contour is the unit operator at the origin, there are no contact terms, and so indeed $\bar\partial(\partial X^\mu) = 0$ and ∂X is holomorphic and thus has no divergences within the contour[1]. In other words, we know that $|?\rangle$ satisfies

$$\alpha_m^\mu |?\rangle = 0 \qquad \text{for} \qquad m \geq 0 \qquad [2.182]$$

But that is exactly the definition of the string ground state $|0;0\rangle$, we thus have the equivalence

$$\mathbf{1} \equiv |0;0\rangle \qquad [2.183]$$

2.66 p 64: Eq (2.8.4) The Isomorphism for General States

We can repeat the analysis of (2.8.2) when the charge on the contour is α_{-m}^μ with $m > 0$, i.e. it corresponds to a creation operator. We still have that ∂X^μ is holomorphic as within the contour, there is only the unit operator, acting at the origin. So we have from (2.7.2a)

$$\alpha_{-m}^\mu = \sqrt{\frac{2}{\alpha'}}\oint \frac{1}{2\pi}z^{-m}\partial X^\mu(z) \qquad [2.184]$$

[1]This has confused me for a long time. There seems to be a contradiction with the fact that we expand ∂X^μ in a Laurent series $\partial X^\mu(z) = -i\sqrt{\alpha'/2}\sum_{m=-\infty}^{\infty}\alpha_m^\mu/z^{m+1}$ and the fact that in our case at hand we find that ∂X^μ is holomorphic and thus has no divergence and a Taylor expansion would suffice. There is of course no contradiction, what we are showing is that ∂X^μ is holomorphic within the contour and so that in that region necessarily $\alpha_m^\mu \equiv 0$ for $m \geq 0$. All of this is, of course, to be viewed as operator equations.

Now $\partial X^\mu(z)$ is holomorphic within the contour, so it doesn't have a pole. But z^{-m} does have a pole. In order to extract the simple pole we write

$$z^{-m} = \frac{(-1)^{m-1}}{(m-1)!}\partial^{m-1}z^{-1} \qquad [2.185]$$

and we perform a partial integration

$$\begin{aligned}\alpha^\mu_{-m} &= \sqrt{\frac{2}{\alpha'}}\oint \frac{1}{2\pi}\frac{1}{(m-1)!}\frac{1}{z}\partial^{m-1}\partial X^\mu(z)\\ &= \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\oint\frac{1}{2\pi i}\frac{\partial^m X^\mu(z)}{z}\\ &= \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\partial^m X^\mu(0)\end{aligned} \qquad [2.186]$$

This gives us the isomorphism

$$\alpha^\mu_{-m}\left|0,0\right\rangle \equiv \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\partial^m X^\mu(0) \qquad \text{for} \qquad m>0 \qquad [2.187]$$

2.67 p 64: Eq (2.8.6) The Isomorphism for General States with an Operator Acting at the Origin, I

Consider for $m > 0$

$$\alpha^\mu_{-m} : \mathcal{A}(0) := \sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi}\frac{1}{z^m} : \partial X^\mu :: \mathcal{A}(0) : \qquad [2.188]$$

We now use (2..2.9)

$$: \mathcal{F} :: \mathcal{G} : = : \mathcal{F}\mathcal{G} : + \sum \text{cross-contractions} \qquad [2.189]$$

$$\begin{aligned}\alpha^\mu_{-m} : \mathcal{A}(0) :&= \sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi}\frac{1}{z^m} : \partial X^\mu \mathcal{A}(0) : + \sum \text{cross-contractions}\\ &= : \alpha^\mu_{-m}\mathcal{A}(0) : + \sum \text{cross-contractions}\end{aligned} \qquad [2.190]$$

We thus need to show that the cross-contractions don't contribute. Let us consider such a cross contraction. It would come from a factor $\partial^k X^\nu(0)$ with $k \geq 0$ in $: \mathcal{A}(0) :$. So we write $: \mathcal{A}(0) :=: \partial^k X^\nu(0)\tilde{\mathcal{A}}(0) :$. The cross-contraction therefore is proportional to

$$\partial \overline{X^\mu(z)\partial^k X}^\nu(0) : \tilde{\mathcal{A}}(0) := \frac{k!}{z^{k+1}} : \tilde{\mathcal{A}}(0) : \qquad [2.191]$$

and so the contribution from this cross-contraction is proportional to

$$\oint \frac{dz}{2\pi}\frac{1}{z^m}\frac{1}{z^{k+1}} : \tilde{\mathcal{A}}(0) := \oint \frac{dz}{2\pi}\frac{1}{z^{m+k+1}} : \tilde{\mathcal{A}}(0) : \qquad [2.192]$$

But with $m > 0$ and $k \geq 0$ this has at a double pole $1/z^2$ or a higher pole. We cannot use the partial integration trick[2] to extract a simple pole as $: \tilde{\mathcal{A}}(0) :$ does not depend on z. As a result this integral is zero as we set out to show.

2.68 p 65: Eq (2.8.7) The Isomorphism for General States with an Operator Acting at the Origin, II

We have for $m > 0$

$$\begin{aligned}\alpha^\mu_{-m}\left|\mathcal{A}\right\rangle &= \alpha^\mu_{-m} : \mathcal{A}(0) : \left|0\right\rangle =: \alpha^\mu_{-m}\mathcal{A}(0) : \left|0\right\rangle \\ &= \sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi} z^{-m} : \partial X^\mu(z)\mathcal{A}(0) : \left|0\right\rangle : \end{aligned} \qquad [2.193]$$

The normal order product $: \partial X^\mu(z)\mathcal{A}(0) : \left|0\right\rangle :$ ensures that this factor has no singularities as $z \to 0$, and so we can apply exactly the same reasoning as for the derivation of (2.8.4). We rewrite z^{-m}, use partial integration and get (2.8.7).

2.69 p 65: Eq (2.8.10) The Ghost Operators Acting on the Ground State

From (2.8.2) we know that the state $\left|0\right\rangle$ corresponds to the unit operator. We now have, for $\lambda = 2$

$$b_m\left|0\right\rangle \equiv \oint \frac{dz}{2\pi i} z^{m+1} b(z)\mathbf{1} = \oint \frac{dz}{2\pi i} z^{m+1} b(z) = 0 \quad \text{for} \quad m+1 \geq 0 \qquad [2.194]$$

This is because $\langle\bar{\partial}\cdots\rangle$ = contact terms, but there is no operator in the contour and so no contact terms. Therefore $b(z)$ is holomorphic inside the contour and has therefore no divergences. This is exactly the same argument we used for the holomorphicity of $\partial X^\mu(z)$ in (2.8.2).

Likewise we have

$$c_m\left|0\right\rangle \equiv \oint \frac{dz}{2\pi i} z^{m-2} c(z)\mathbf{1} = \oint \frac{dz}{2\pi i} z^{m-2} x(z) = 0 \quad \text{for} \quad m-2 \geq 0 \qquad [2.195]$$

because $c(z)$ is holomorphic inside the contour.

[2] We can in fact use it, but then we find something of the form $\oint (dz/z)\partial^{m+k} : \tilde{\mathcal{A}}(0) :$ which is zero because $: \tilde{\mathcal{A}}(0) :$ does not depend on z. This is, of course, just another way to say that the integral vanishes.

2.70 p 65: Eq (2.8.11) The Ground State and the Ghost Ground State

It is easily checked that $|0\rangle = b_{-1}|\downarrow\rangle$ satisfies (2.8.10). Indeed, for $m \geq -1$

$$b_m|0\rangle = b_m b_{-1}|\downarrow\rangle = -b_{-1}b_m|\downarrow\rangle = 0 \qquad [2.196]$$

This follows from (2.7.18a), i.e. $b_m|\downarrow\rangle = 0$ for $m \geq 0$ and $m = -1$ it follows from $b_{-1}^2 = 0$.
Similarly for $m \geq 2$

$$c_m|0\rangle = c_m b_{-1}|\downarrow\rangle = -b_{-1}c_m|\downarrow\rangle = 0 \qquad [2.197]$$

from (2.7.18c) i.e. $c_m|\downarrow\rangle = 0$ for $m > 0$.

2.71 p 65: The Ghost Number of the Ground State

$$\begin{aligned}N^{\mathrm{g}}|0\rangle = N^{\mathrm{g}}b_{-1}|\downarrow\rangle &= \left(\sum_{m=1}^{\infty}(c_{-m}b_m - b_{-m}c_m) + c_0b_0 - \frac{1}{2}\right)b_{-1}|\downarrow\rangle \\ &= \left(-b_{-1}c_1 - \frac{1}{2}\right)b_{-1}|\downarrow\rangle = -b_{-1}(-b_{-1}c_1 + 1)|\downarrow\rangle - \frac{1}{2}b_{-1}|\downarrow\rangle \\ &= -\frac{3}{2}b_{-1}|\downarrow\rangle = -\frac{3}{2}|0\rangle \qquad [2.198]\end{aligned}$$

We also have $N^{\mathrm{g}}|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$ from (2.7.24).
What operator does the ghost ground state $|\downarrow\rangle$ correspond to? We first note that

$$|\downarrow\rangle = c_1b_{-1}|\downarrow\rangle = c_1|0\rangle \qquad [2.199]$$

As $|0\rangle$ corresponds to the unit operator **1** we have

$$|\downarrow\rangle \equiv \oint \frac{dz}{2\pi i}z^{1-2}c(z)\mathbf{1} = \oint \frac{dz}{2\pi i}z^{-1}c(z) = c(z) \qquad [2.200]$$

and we see that, under the state-operator isomorphism, the ghost number $-1/2$ state $|\downarrow\rangle$ indeed corresponds to the ghost number 1 operator $c(z)$.

2.72 p 65: Eq (2.8.16) The Complex Coordinates for the Open String

In $w = \sigma^1 + i\sigma^2$ coordinates the open string $\sigma^1 = \mathrm{Re}\,w \in [0,\pi]$ and $\sigma^2 = \mathrm{Im}\,w \in]-\infty, 0]$. We map this using the conformal transformation $z = -e^{-iw}$ to the upper half unit disk. Indeed the points $sigma^2 = -\infty$ corresponds to the origin $z = 0$; the points $\sigma^2 = 0$ correspond to the unit circle $z = -e^{-i\sigma^1}$ with $\sigma^1 \in [0,\pi]$. For $(\sigma^1, \sigma^2) = (0,0)$ we have

$z = -1$, for $(\sigma^1, \sigma^2) = (\pi/2, 0)$ we have $z = i$ and for $(\sigma^1, \sigma^2) = (0, \pi)$ we have $z = 1$ and so the semi-infinite strip in the w-plane is mapped to the upper-half half-disk in the z-plane.

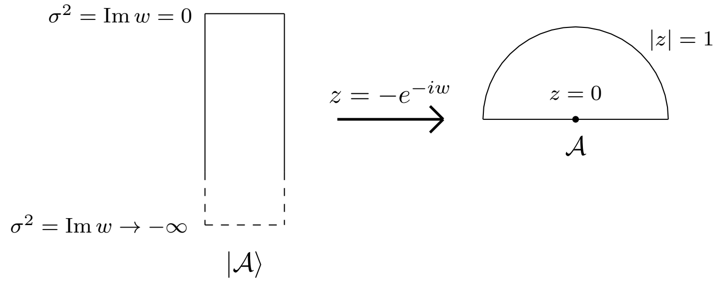


Figure 2.5: Mapping the semi-infinite strip to the upper-half unit disk. The points $\operatorname{Im} w = \sigma^2 \to -\infty$ correspond to the origin in the z-plane. The points $\operatorname{Im} w = \sigma^2 \to= 0$ correspond to the points on the unit disk in the upper-half z-plane

An operator on the boundary in the z-plane has $\operatorname{Im} z = 0$. From

$$z = -e^{-iw} = -e^{\sigma^2 - i\sigma^1} = -e^{\sigma^2}(\cos\sigma^1 - i\sin\sigma^1) \qquad [2.201]$$

we see that a point $\operatorname{Im} z = 0$ corresponds to $\sin\sigma^1 = 0$ or hence $\sigma^1 = 0$ or π. A point on the boundary of the upper-half unit disk thus corresponds to one of the end-points of the open string.

2.73 p 66: Eq (2.8.17) The State-Operator Mapping: from Operator to State

I developed a better understanding of the state-operator mapping from the path integral approach by reading David Tong's lectures on string theory and so I find it worthwhile to summarise what he has to say about it here.

Let us start with ordinary quantum mechanics. A wavefunction $\psi(x) = \langle\psi|x\rangle$ describes the probability to find a particle at x at a given time[3]. To describe a propagation of a particle that was initially at time τ_i and position x_i to a later time τ_f and position x_f, we use the propagator

$$G(x_f, x_i) = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x\, e^{iS} \qquad [2.202]$$

[3]The probability is of course $|\psi(x)|^2$, but that is a petty detail in this discussion. We will also ignore all normalisation factors in this discussion

If our system starts in a state $\psi_i(x_i)$ at time τ_i then it will evolve to $\psi_f(x_f)$ at time τ_f according to

$$\psi_f(x_f;\tau_f) = \int dx_i\, G(x_f,x_i)\psi_i(x_i;\tau_i) = \int dx_i \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x e^{iS}\psi_i(x_i;\tau_i) \qquad [2.203]$$

From that we learn that

1. The wavefunction ψ_f at x_f follows from the path integral restricting to paths that have $x(\tau_f) = \tau_f$;
2. The path integral is weighted by the initial state $\psi_i(x_i;\tau_i)$ and we also integrate over all initial positions x_i.

Let us now translate this to quantum field theory. The coordinates x are replaced by the fields ϕ and so a wave function $\psi(x)$ becomes a wavefunctional $\Psi[\phi(\sigma)]$. Here σ are the coordinates on the semi-infinite cylinder describing a closed string. Starting with a wavefunctional $\Psi_i[\phi_i(\sigma)]$ at time τ_i, we can use [2.203] to write down how it will evolve

$$\Psi_f[\phi_f(\sigma),\tau_f] = \int \mathcal{D}\phi_i \int_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} \mathcal{D}\phi\, e^{-S[\phi]}\Psi_i[\phi_i(\sigma),\tau_i] \qquad [2.204]$$

where we have gone to Euclidean space for convenience. Let us now go from the semi-infinite cylinder to the complex plane. States are defined on circles of constant radius, say $|z| = r$ and evolution happens via the dilatation operator $L_0 + \bar{L}_0$. Take an initial state that is defined on r_i. Eq [2.204] tells us to integrate over all field configurations with boundaries $\phi(\tau_i) = \phi(r_i) = \phi_i$ and $\phi(\tau_f) = \phi(r_f) = \phi_f$. These are configurations on the edges of an annulus with inner radius r_i and outer radius r_f, see the figure below. We also need to integrate over all boundary conditions at time τ_i, i.e. over all $\mathcal{D}\phi_i$. The state $\Psi_f[\phi_f(\sigma),\tau_f]$ with a given boundary condition ϕ_b at time τ_f, or equivalently radius r_f, is thus obtained by evolving all possible states at time τ_i, or equivalently radius r_i, to time τ_f and corresponding boundary condition. This is, of course, exactly how the path integral approach in QM works. But it is worth repeating.

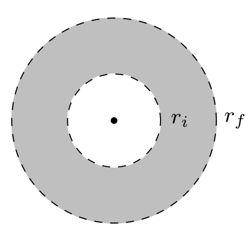


Figure 2.6: From Operator to State. Integration over an annulus between r_i at τ_i and r_f at τ_f

We thus get

$$\Psi_f[\phi_f(\sigma), r_f] = \int \mathcal{D}\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \mathcal{D}\phi \, e^{-S[\phi]} \Psi_i[\phi_i(\sigma), r_i] \qquad [2.205]$$

Once again we see that the wavefunctional at radius r_f is given by a path integral weighted by the wavefunctional at an "earlier" radius r_i with boundary conditions ϕ_i and ϕ_f and that being integrated over all possible initial field configurations ϕ_i.

Let us now consider the infinite past, i.e. $\sigma^2 \to -\infty$, or equivalently $z = 0$. As $r_i = 0$ we must now integrate over the entire disk with $|z| \le r_f$, rather than over an annulus. The weighting of the path integral is now changed by something acting at the point $z = 0$. That is exactly what we mean by a local operator. This means that if we have a local operator $\mathcal{A}(z)$ we can define a wave functional

$$\Psi_{\mathcal{A}}[\phi_f; r] = \int^{\phi(r)=\phi_f} \mathcal{D}\phi \, e^{-S[\phi]} \mathcal{A}(0) \qquad [2.206]$$

We are integrating over all field configurations within the disc, including all fields at the origin $z = 0$, which is analogous to integrating over the boundary of the inner circle $\int \mathcal{D}\phi_i$. A wavefunctional is nothing but the Schrödinger picture of a state, so this is the state-operator isomorphism explained in terms of path integrals.

One may wonder why we use a path integral here, and not in the earlier derivation of this isomorphism. But it only looks like we didn't use the path integral in the previous derivation. Indeed, in that derivation we always used operator equations (e.g. to argue that $\partial X^\mu(z)$ was holomorphic within the contour). But operator equations are equations valid in expectation values and expectation values can be calculated via a path integral. We have gone full circle.

2.74 p 67: Eq (2.8.18) The State-Operator Mapping: from State to Operator

Joe is very brief on this, so let's explain it a little bit more slowly. As you will see it isn't as magical as it sounds. So we consider an annular region where z is between r and 1 as in the figure below.

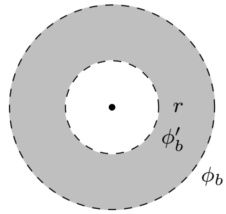


Figure 2.7: From State to Operator. Integration over an annulus between r and 1 with respective boundary conditions ϕ_b' and ϕ_b

On the inner circle at radius r we have boundary conditions ϕ_b', on the outer circle with radius 1 we have boundary conditions ϕ_b. Now note first that if we have a state on the inner circle evolving over time this is described in quantum theory by e^{-Ht} where H is the Hamiltonian and t is the parameter. For our worldsheet the Hamiltonian is given by (2.6.10) i.e.

$$H = L_0 + \bar{L}_0 + \frac{c + \tilde{c}}{24} \qquad [2.207]$$

and $t \equiv \sigma^2$. As $z = e^{-iw} = e^{\sigma^2 - i\sigma^1}$ we see that $|z| = r = e^{\sigma^2}$. The upshot of all of this is that time evolution is described by $r^{-L_0 - \bar{L}_0}$, where we have assumed that we are working in the critical dimension, $c = \tilde{c} = 0$.

Let us now turn to Eq (2.8.18)

$$\int [\mathcal{D}\phi_b'] \, [\mathcal{D}\phi_i]_{\phi_b' \text{to} \, \phi_b} e^{-S[\phi_i]} r^{L_0 + \bar{L}_0} \Psi[\phi_b'] \qquad [2.208]$$

This is a path integral over the annulus from $|z| = r$ with boundary conditions ϕ_b' to $|z| = 1$ with boundary conditions ϕ_b, with the action contribution $e^{-S[\phi_i]}$ weighted by a factor $r^{L_0 + \bar{L}_0} \Psi[\phi_b']$. We moreover integrate over all possible field configurations ϕ_b' of the inner circle. Now the path integral over the annulus $\int [\mathcal{D}\phi_b'] \, [\mathcal{D}\phi_i]_{\phi_b' \text{to} \, \phi_b} \exp(-S[\phi_i])$ just takes the operator it is acting on, viz. $r^{L_0 + \bar{L}_0} \Psi[\phi_b']$, and brings it to the outer circle, where the boundary condition is ϕ_b. But this evolution can also be described, as we have just seen, by the evolution operator, $r^{-L_0 - \bar{L}_0}$. Thus the effect of the path integral can be written as acting on the operator

$$r^{-L_0 - \bar{L}_0} \left(r^{L_0 + \bar{L}_0} \Psi[\phi_b'] \right) \to \Psi[\phi_b] \qquad [2.209]$$

We have evolved from the inner circle to the outer circle and so the wavefunctional is now taken with the outer boundary conditions. We thus find that this expression gives us $\Psi[\phi_b]$, i.e.

$$\Psi[\phi_b] = \int [\mathcal{D}\phi_b'] \, [\mathcal{D}\phi_i]_{\phi_b' \text{to} \, \phi_b} e^{-S[\phi_i]} r^{L_0 + \bar{L}_0} \Psi[\phi_b'] \qquad [2.210]$$

If we now take the limit of $r \to 0$, then the annulus becomes a disk. In this limit the path integral over the inner circle $\int [\mathcal{D}\phi_b']$ can be seen as the definition of some local operator at $z = 0$. As we have seen the path integral with this local operator then results in the state $\Psi[\phi_b]$ and this is how construct the operator when we are given a state.

2.75 p 67-68: The State-Operator Mapping for the Scalar Field X^μ: The Ground State

I find it more useful for this case, to describe the entire example rather than to focus on individual equations. We take a single free real scalar field X. The boundary condition of X is given by determining the value of the field on the unit circle, which we write as a Fourier expansion in the polar angle θ

$$X_b(\theta) = \sum_{n=-\infty}^{\infty} X_n e^{in\theta} \qquad [2.211]$$

Requiring the field to be real leads to $X_n^* = X_{-n}$. The boundary conditions is thus fully determined by the X_n and so is the wavefunctional on the boundary $\Psi[X_b] = \Psi[\{X_n\}]$.

Let us identify the state corresponding with the unit operator $\mathbf{1}$ with that given boundary. By (2.8.17) this is given by

$$\begin{aligned}\Psi_{\mathbf{1}}[X_b] &= \int [\mathcal{D}X_i]_{X_b} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i \bar{\partial} X_i\right) \mathbf{1} \\ &= \int [\mathcal{D}X_i]_{X_b} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i \bar{\partial} X_i\right) \qquad [2.212]\end{aligned}$$

Note that i is not and index here, it just means that it is the value of the field X in the interior of the unit disk. X_i should also not be confused with the X_n. This is a Gaussian integral, but with unusual boundary conditions. We can turn this into a Gaussian integral with standard boundary conditions by splitting the X_i as follows

$$X_i = X_{\text{cl}} + X_i' \qquad [2.213]$$

where X_{cl} is defined as

$$X_{\text{cl}} = X_0 + \sum_{n=1}^{\infty} (z^n X_n + \bar{z}^n X_{-n}) \qquad [2.214]$$

with $z = e^{in\theta}$. The reason split X_i with this definition of X_{cl} is because (1) X_{cl} satisfies the equation of motion $\bar{\partial}\partial X_{\text{cl}} = 0$ and (2) $X_i' = 0$ on the boundary. The former should be obvious and the latter is easily seen form rewriting [2.211] in terms of z:

$$\begin{aligned} X_b(\theta) &= \sum_{n=-\infty}^{-1} X_n e^{in\theta} + X_0 + \sum_{n=1}^{\infty} X_n e^{in\theta} \\ &= \sum_{n=1}^{\infty} X_{-n} e^{-in\theta} + X_0 + \sum_{n=1}^{\infty} X_n e^{in\theta} \\ &= X_0 + \sum_{n=1}^{\infty} (z^n X_n + \bar{z}^n X_{-n}) = X_{\text{cl}} \end{aligned} \qquad [2.215]$$

Thus

$$(X_i)_b = X_{\text{cl}} + (X_i')_b \Leftrightarrow X_{\text{cl}} = X_{\text{cl}} + (X_i')_b \quad \Rightarrow \quad (X_i')_b = 0 \qquad [2.216]$$

We can now write the action as

$$\begin{aligned} S[X_i] &= \frac{1}{2\pi\alpha'} \int d^2 z \partial (X_{\text{cl}} + X_i') \bar{\partial} (X_{\text{cl}} + X_i') \\ &= \frac{1}{2\pi\alpha'} \int d^2 z \left(\partial X_{\text{cl}} \bar{\partial} X_{\text{cl}} + \partial X_i' \bar{\partial} X_i' \right) \end{aligned} \qquad [2.217]$$

The cross terms vanish by construction that X_{cl} satisfies the equation of motion, after partial integration. Let us first work out

$$\begin{aligned} \int d^2 z \partial X_{\text{cl}} \bar{\partial} X_{\text{cl}} &= \int d^2 z \sum_{m=1}^{\infty} m z^{m-1} X_m \sum_{n=1}^{\infty} n \bar{z}^{n-1} X_{-n} \\ &= \sum_{m,n=1}^{\infty} mn X_m X_{-n} \int d^2 z \, z^{m-1} \bar{z}^{n-1} \end{aligned} \qquad [2.218]$$

The integral is easily evaluated in polar coordinates. Recall that the measure is $d^2 z = 2dxdy = 2rdrd\theta$. Thus

$$\begin{aligned} \int d^2 z \, z^{m-1} \bar{z}^{n-1} &= 2 \int_0^{2\pi} d\theta \int_0^1 r dr \left(r e^{i\theta} \right)^{m-1} \left(r e^{-i\theta} \right)^{n-1} \\ &= 2 \int_0^{2\pi} e^{i(m-n)\theta} d\theta \int_0^1 r^{m+n-1} dr \end{aligned} \qquad [2.219]$$

for $m \neq n$ the θ integration vanishes, whilst for $m = n$ it is just 2π. The r integration is just $1/(m+n)$. Thus

$$\int d^2 z \, z^{m-1} \bar{z}^{n-1} = \delta_{m-n,0} \frac{2\pi}{m+n} \qquad [2.220]$$

and so

$$\frac{1}{2\pi\alpha'}\int d^2z\partial X_{\rm cl}\bar{\partial}X_{\rm cl} = \frac{1}{2\pi\alpha'}\sum_{m,n=1}^{\infty} mnX_mX_{-n}\delta_{m-n,0}\frac{2\pi}{m+n}$$
$$= \frac{1}{\alpha'}\sum_{m=1}^{\infty} mX_mX_{-m} \qquad [2.221]$$

We can thus write [2.212] as

$$\Psi_{\mathbf{1}}[X_b] = e^{-S_{\rm cl}}\int [\mathcal{D}X_i']_{X_b=0}\ \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\partial X_i'\bar{\partial}X_i'\right) \qquad [2.222]$$

where

$$S_{\rm cl} = \frac{1}{\alpha'}\sum_{m=1}^{\infty} mX_mX_{-m} \qquad [2.223]$$

So we have rewritten the path integral in terms of the X_i' coordinates that vanish on the boundary of the unit disk. The impact of the boundary conditions now resides completely in $S_{\rm cl}$ that we can take out of the path integral. The remaining path integral is just a constant, independent of the boundary conditions, so we can write

$$\Psi_{\mathbf{1}}[X_b] \propto \exp\left(-\frac{1}{\alpha'}\sum_{m=1}^{\infty} mX_mX_{-m}\right) \qquad [2.224]$$

Now, if we have acted correctly then this state corresponds to the ground state of the theory, as the operator we started with was the unit operator. To check that it is indeed the ground state we need to check that $\Psi_{\mathbf{1}}[X_b]$ is annihilated by α_n for $n \geq 0$. Recall that we are working in the Schrödinger picture so we need anmmutation relations $[a, a^\dagger] = 1$ and when acting on wave functions $\psi(x)$ they are represented by $a = (ix + p)/\sqrt{2}$ and $a^\dagger = (ix - p)/\sqrt{2}$ as one easily checks. Here $p = i\partial_x$ when acting on wavefunctions. We now claim that in our case, in the Schrödinger picture we have

$$\alpha_n = -\frac{in}{\sqrt{2\alpha'}}X_{-n} - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_n}$$
$$\tilde{\alpha}_n = -\frac{in}{\sqrt{2\alpha'}}X_n - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_{-n}} \qquad [2.225]$$

and that these satisfy the commutation relations

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0} \qquad \text{and} \qquad [\alpha_m, \tilde{\alpha}_n] = 0 \qquad [2.226]$$

when acting on wavefunctional $\Phi[\{X_k\}]$. This is easily checked. First we note that with $m \neq -n$ we immediately see that $[\alpha_m, \alpha_n] = 0$. This is because in that case $\partial X_{-n}/\partial X_m = \partial X_{-m}/\partial X_n = 0$. So we need to check $[\alpha_m, \alpha_{-m}] = m$:

$$\begin{aligned}
[\alpha_m, \alpha_{-m}]\Phi[\{X_k\}] &= [-\frac{im}{\sqrt{2\alpha'}}X_{-m} - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_m}, \frac{im}{\sqrt{2\alpha'}}X_m - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_{-m}}]\Phi[\{X_k\}] \\
&= \frac{m}{2}\left[-X_{-m}\frac{\partial}{\partial X_{-m}} + \frac{\partial}{\partial X_m}X_m - X_m\frac{\partial}{\partial X_m} + \frac{\partial}{\partial X_{-m}}X_{-m}\right]\Phi[\{X_k\}] \\
&= \frac{m}{2}\left[-X_{-m}\frac{\partial}{\partial X_{-m}} + 1 + X_m\frac{\partial}{\partial X_m} - X_m\frac{\partial}{\partial X_m} + 1 + X_{-m}\frac{\partial}{\partial X_{-m}}\right]\Phi[\{X_k\}] \\
&= m\Phi[\{X_k\}]
\end{aligned} \quad [2.227]$$

We will leave it as an exercise to the reader to show that $[\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0}$ and $[\alpha_m, \tilde{\alpha}_n] = 0$. It now remains to show that $\Psi_{\mathbf{1}}[X_b]$ is the ground state, i.e. that it satisfies

$$\alpha_m \Psi_{\mathbf{1}}[X_b] = \tilde{\alpha}_m \Psi_{\mathbf{1}}[X_b] = 0 \qquad \text{for} \qquad m \geq 0 \quad [2.228]$$

Let us work this out. Take $m \geq 0$

$$\alpha_m \Psi_{\mathbf{1}}[X_b] = \left(-\frac{im}{\sqrt{2\alpha'}}X_{-m} - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_m}\right)\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right) \quad [2.229]$$

Focus on the second term:

$$\begin{aligned}
&-i\sqrt{\frac{\alpha'}{2}}\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right)\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} n\delta_{m,n}X_{-n}\right) \\
&= \frac{im}{\sqrt{2\alpha'}}X_{-m}\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right)
\end{aligned} \quad [2.230]$$

and this exactly cancels the first term. It should be obvious from this that we also have $\tilde{\alpha}_m\Psi_{\mathbf{1}}[X_b] = 0$ for $m \geq 0$. So we can indeed identify

$$\Psi_{\mathbf{1}}[X_b] = \exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right) \equiv |0,0\rangle \quad [2.231]$$

as the ground state.

2.76 p 68: Eq (2.8.28) The State-Operator Mapping for the Scalar Field X^μ: The State for the Operator $\partial^k X^\mu$

Let us now work out the state corresponding to the operator $\partial^k X$ for a single real scalar field. We start from (2.8.17), or in these notes [2.206]. Replicating what we had for the unit operator in [2.212] we have

$$\Psi_{\partial^k X}[X_b] = \int [\mathcal{D}X_i]_{X_b} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i \bar{\partial} X_i\right) \partial^k X_i(0) \qquad [2.232]$$

We again use the split [2.213]

$$X_i = X_{\text{cl}} + X_i' \qquad [2.233]$$

where X_{cl} is defined in [2.234], i.e.

$$X_{\text{cl}} = X_0 + \sum_{n=1}^{\infty}(z^n X_n + \bar{z}^n X_{-n}) \qquad [2.234]$$

We urge the reader to review the derivation of the state for the unit operator if necessary. The whole derivation is similar but for the extra operator $\partial^k X_i = \partial^k X_{\text{cl}} + \partial^k X_i'$. Now, one easily sees that

$$\partial^k X_{\text{cl}}(0) = \partial^k \left[X_0 + \sum_{n=1}^{\infty}(z^n X_n + \bar{z}^n X_{-n})\right]\Bigg|_{z=0} = k! X_k \qquad [2.235]$$

If this is not clear, just work out a few examples, $\partial X_{\text{cl}}(0), \partial^2 X_{\text{cl}}(0)$, etc. We can now just use [2.222] to write

$$\Psi_{\partial^k X}[X_b] = e^{-S_{\text{cl}}} \int [\mathcal{D}X_i']_{X_b=0} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i' \bar{\partial} X_i'\right) \partial^k (X_{\text{cl}} + X_i')(0) \qquad [2.236]$$

with again

$$S_{\text{cl}} = \frac{1}{\alpha'}\sum_{m=1}^{\infty} m X_m X_{-m} \qquad [2.237]$$

Thus

$$\begin{aligned}\Psi_{\partial^k X}[X_b] = k! X_k e^{-S_{\text{cl}}} \int [\mathcal{D}X_i']_{X_b=0} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i' \bar{\partial} X_i'\right) \\ + e^{-S_{\text{cl}}} \int [\mathcal{D}X_i']_{X_b=0} \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z \partial X_i' \bar{\partial} X_i'\right) \partial^k X_i'(0)\end{aligned} \qquad [2.238]$$

For some reason that escapes me, the second term will not contribute. What we can say is that the path integral is independent of the boundary and hence proportional to the ground state. We have

$$|\partial^k X\rangle = \Psi_{\partial^k X}[X_b] = k! X_k \Psi_{\mathbf{1}} + \beta \Psi_{\mathbf{1}} \qquad [2.239]$$

We want to show that this is the first excited state $\alpha_{-k}|0\rangle$. Let us work this out in the Schrödinger picture:

$$\begin{aligned}\alpha_{-k}|0\rangle &= \left(\frac{ik}{\sqrt{2\alpha'}}X_k - i\sqrt{\frac{\alpha'}{2}}\frac{\partial}{\partial X_{-k}}\right)\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right)\\ &= \frac{ik}{\sqrt{2\alpha'}}X_k \exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right)\\ &\quad - i\sqrt{\frac{\alpha'}{2}}\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right)\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_n\delta_{-k,-n}\right)\\ &= ik\sqrt{\frac{2}{\alpha'}}X_k\exp\left(-\frac{1}{\alpha'}\sum_{n=1}^{\infty} nX_nX_{-n}\right) = ik\sqrt{\frac{2}{\alpha'}}X_k\Psi_{\mathbf{1}}\end{aligned} \qquad [2.240]$$

and so, if we assume that $\beta = 0$ we have

$$|\partial^k X\rangle = k!X_k\Psi_{\mathbf{1}} = k!\frac{1}{ik}\sqrt{\frac{\alpha'}{2}}\alpha_{-k}|0\rangle = -i(k-1)!\sqrt{\frac{\alpha'}{2}}\alpha_{-k}|0\rangle \qquad [2.241]$$

which is (2.8.28).

Because it isn't clear to me a priori that we can set $\beta = 0$, let us show that $\alpha_{-k}|0\rangle \sim |\partial^k X\rangle$ in another way. Let us act on this state with the annihilation operator α_n for $n \geq 0$. On the one hand we know that

$$\alpha_n\alpha_{-k}|0\rangle = (\alpha_{-k}\alpha_n + \delta_{n-k})|0\rangle = \delta_{n-k}|0\rangle \qquad [2.242]$$

Let us reproduce this from the path integral approach

$$\alpha_n\Psi_{\partial^k X}[X_b] \sim \int [\mathcal{D}X_i]_{X_b}\, e^{-S[X]} \oint \frac{dw}{2\pi i} w^n \partial X(w)\, \partial^k X_i(0) \qquad [2.243]$$

Let us not be confused here. As all of this is valid as operator equations, it needs to be valid within a path integral weighted by the exponential of (minus) the action. So the factor containing the exponential should not be inside the contour integral, and we should not take its OPE with $\partial X(w)$. Now $\partial X(w)\,\partial^k X_i(0) \propto w^{-k-1}$ as is easily checked. We thus find

$$\begin{aligned}\alpha_n\Psi_{\partial^k X}[X_b] &\propto \int [\mathcal{D}X_i]_{X_b}\, e^{-S[X]} \oint \frac{dw}{2\pi i} w^n w^{-k-1}\\ &= \int [\mathcal{D}X_i]_{X_b}\, e^{-S[X]} \oint \frac{dw}{2\pi i} \frac{1}{w^{k-n+1}}\end{aligned} \qquad [2.244]$$

and this is zero unless $k = n$ so that we recover indeed $\alpha_n\Psi_{\partial^k X}[X_b] \propto \delta_{n-k}\Psi_{\mathbf{1}}[X_b]$. This thus also means that $\beta = 0$.

2.77 p 70: Eq (2.9.3) The OPE of Three Operators

The OPE of the product of three operators

$$\mathcal{A}_i(0)\mathcal{A}_j(1)\mathcal{A}_k(z) \qquad [2.245]$$

Will have potential singularities as $z \to 0$ and as $z \to 1$ via their OPEs. These OPEs have radius of convergence respectively $|z|, 1$ and $|1-z| < z$, see the figure

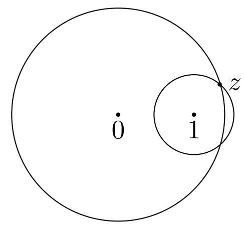


Figure 2.8: Radius of Convergence for the OPEs $\mathcal{A}_i(0)\mathcal{A}_j(1)\mathcal{A}_k(z)$

We can perform the OPEs in two ways. First we can perform the $\mathcal{A}_j(1)\mathcal{A}_k(z)$ OPE and then perform the OPE that result with $\mathcal{A}_i(0)$. This gives

$$\begin{aligned}
\mathcal{A}_i(0)\mathcal{A}_j(1)\mathcal{A}_k(z) &= \mathcal{A}_i(0)\sum_\ell (1-z)^{h_\ell - h_j - h_k} c^\ell_{jk}\mathcal{A}_\ell(1-z) \\
&= \sum_\ell \sum_m (1-z)^{h_\ell - h_j - h_k}(1-z)^{h_m - h_i - h_\ell} c^\ell_{jk} c^m_{i\ell}\ \mathcal{A}_m(1-z) \\
&= \sum_{\ell,m} (1-z)^{h_m - h_i - h_j - h_k} c^\ell_{jk} c^m_{i\ell}\mathcal{A}_m(1-z) \qquad [2.246]
\end{aligned}$$

Or we can first perform the $\mathcal{A}_i(0)\mathcal{A}_k(z)$ OPE and then perform the OPE that result with $\mathcal{A}_j(1)$, giving

$$\begin{aligned}
\mathcal{A}_i(0)\mathcal{A}_j(1)\mathcal{A}_k(z) &= \mathcal{A}_j(1)\sum_\ell z^{h_\ell - h_i - h_k} c^\ell_{ik}\mathcal{A}_\ell(z) \\
&= \sum_{\ell,m} z^{h_\ell - h_i - h_k}(1-z)^{h_m - h_j - h_\ell} c^\ell_{ik} c^m_{j\ell}\mathcal{A}_m(1-z) \qquad [2.247]
\end{aligned}$$

These two expressions need to be equal. As the $\mathcal{A}_n$ are chosen to be a complete basis for the local operators, this implies that

$$\sum_\ell (1-z)^{h_m - h_i - h_j - h_k} c^\ell_{jk} c^m_{i\ell} = \sum_\ell z^{h_\ell - h_i - h_k}(1-z)^{h_m - h_j - h_\ell} c^\ell_{ik} c^m_{j\ell} \qquad [2.248]$$

As an aside, we can of course repeat the same analysis for the product of four local operators. In that case it is convenient to work with a four-point function defined as

$$G^{\ell k}_{nm}(x) = z^{2h_k} \langle \mathcal{A}_k(\infty)\mathcal{A}_\ell(1)\mathcal{A}_n(x)\mathcal{A}_m(0)\rangle \qquad [2.249]$$

Applying the same methodology i.e. requiring that the result via the s, t, and u channels has to be the same gives a set of equations that are schematically of the form

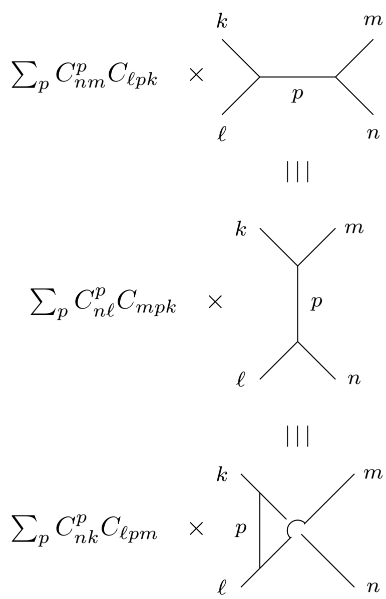


Figure 2.9: Conformal Bootstrap Equations from Four-Point Functions)

These equations are known as the conformal bootstrap equations and can be used to derive the conformal transformation properties of all the operators and this is tantamount to solving the conformal field theory completely.

2.78 p 72: Eq (2.9.14) Non-Highest Weight States in Unitary CFTs

A non-highest weights states can by definition be obtained by the action of a combination of L_{-m} with $m > 0$ on a highest weights state. Let us call that highest weight state $|\mathcal{O}\rangle$ with highest weight $h_{\mathcal{O}}$. The non-highest weight state is then of the form $L_{-k_1}L_{-k_2}\cdots L_{-k_n}|\mathcal{O}\rangle$.

Note that some of the L_{-k_i} can be equal to one another. The weight of that state is then

$$\begin{aligned} L_0 L_{-k_1} L_{-k_2} \cdots L_{-k_n} |\mathcal{O}\rangle &= (L_{-k_1} L_0 + k_1 L_{-k_1}) L_{-k_2} \cdots L_{-k_n} |\mathcal{O}\rangle \\ &= [k_1 L_{-k_1} L_{-k_2} \cdots L_{-k_n} + L_{-k_1}(L_{-k_2} L_0 + k_2 L_{-k_2}) \cdots L_{-k_n}] |\mathcal{O}\rangle \\ &= [(k_1 + k_2 + \cdots k_n) L_{-k_1} L_{-k_2} \cdots L_{-k_n} + L_{-k_1} L_{-k_2} \cdots L_{-k_n} L_0] |\mathcal{O}\rangle \\ &= (k_1 + k_2 + \cdots k_n + h_{\mathcal{O}}) L_{-k_1} L_{-k_2} \cdots L_{-k_n} |\mathcal{O}\rangle \end{aligned} \qquad [2.250]$$

and so has weight $h_{\mathcal{O}} + k_1 + k_2 + \cdots k_n$ which is positive and larger that $h_{\mathcal{O}}$ as all the k_i are positive.

2.79 p 72: Eq (2.9.15) $h_{\mathcal{O}} = 0$ Operators

From (2.9.14) we have that if $h_{\mathcal{O}} = 0$ then $L_{-1} \cdot \mathcal{O} = 0$. But (2.9.7) tells us that $L_{-1} \cdot \mathcal{O} = \partial \mathcal{O}$. and so $\partial \mathcal{O} = 0$.

2.80 p 73: The Normal Ordering Constants from the State-Operator Mapping

For any theory we have for the state corresponding to the unit operator by definition $L_0 |0\rangle = 0$. For the X^μ theory the state $|0\rangle$ is also the ground state. Now using (2.7.7) we have

$$L_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty} \alpha^\mu_{-n} \alpha_{\mu\, n} + a^X \qquad [2.251]$$

and so $L_0 |0\rangle$ immediately implies $a^X = 0$.

For the bc system we have $|0\rangle = b_{-1} |{\downarrow}\rangle$. From $L_0 |0\rangle = 0$ and (2.7.19) we thus find

$$\begin{aligned} 0 &= \left(- \sum_{n=-\infty}^{\infty} n \,{}^\circ_\circ b_n c_{-n} {}^\circ_\circ + a^{\mathrm{g}} \right) b_{-1} |{\downarrow}\rangle \\ &= \left[- \sum_{n=-\infty}^{-2} n b_n c_{-n} - (-1) b_{-1} c_1 + 0 + \sum_{n=1}^{\infty} n c_{-n} b_n + a^{\mathrm{g}} \right] b_{-1} |{\downarrow}\rangle \\ &= (-c_1 b_{-1} + 1 + a^{\mathrm{g}}) b_{-1} |{\downarrow}\rangle = (1 + a^{\mathrm{g}}) |{\downarrow}\rangle \end{aligned} \qquad [2.252]$$

and so $a^{\mathrm{g}} = 0$.

Chapter 3

The Polyakov Path Integral

Open Questions

I have a number of unanswered points for this chapter. They are briefly mentioned here and more detail is given under the respective headings. Any help in resolving them can be sent to hepnotes@hotmail.com and is more than welcome.

- ♣ (3.4.22) In expanding $\ln Z[\delta + h]/Z[\delta]$ to second order in h we are considering the term in $(hT)^2$, but we are ignoring the term in $h^2\delta^2 S/\delta g^2|_{g=\delta}$. Why can we do that? It may be related to the fact that this term gives a contact term (two delta functions) and that this does not contribute. But I am still confused why that would be the case.
- ♣ (3.6.18) I have not found a (relatively) simple proof of the equation $[\nabla^2 X^\mu e^{ik\cdot X}]_r = (i\alpha'\gamma/4)R[e^{ik\cdot X}]_r$. I have provided a detailed, but certainly worthwhile, exposition on this subject of around 20 pages that ends with very strong circumstantial evidence that is equation is correct. But any more direct proof is certainly welcome.

3.1 p 79: Fig 3.4 Open String Processes

The figure below shows the 3D views of the 2D slices of the open string processes shown in fig 3.4.

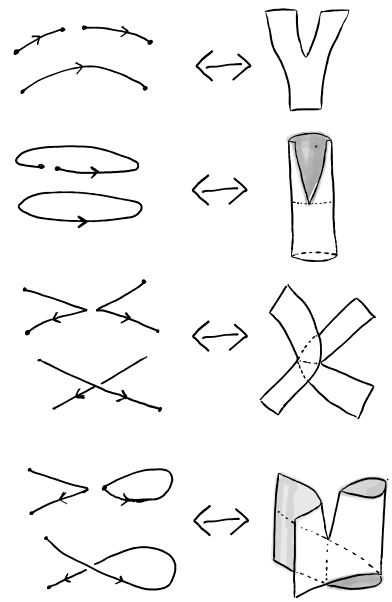


Figure 3.1: 3D view of open string processes

3.2 p 82: Eq (3.2.3b) The Weyl Invariance of the Euler Number

We wish to show that the Euler term

$$\chi = \frac{1}{4\pi}\int_M d^2\sigma\,\sqrt{g}R + \frac{1}{2\pi}\int_{\partial M} ds\,k \tag{3.1}$$

with k the geodesic curvature given by

$$k = \pm t^a n_b \nabla_a t^b \tag{3.2}$$

is invariant under a Weyl rescaling. From (1.2.32) we know that under a Weyl rescaling $g'_{ab} = e^{2\omega(\sigma)} g_{ab}$ the Riemann curvature transforms, in Euclidean spacetime, as

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega) = \sqrt{g}(R - 2\nabla^a\partial_a\omega) \tag{3.3}$$

We have used the fact that ω is a scalar so the $\nabla_a\omega = \partial_a\omega$. From [1.7] we also know how a connection transforms under a Weyl rescaling

$$\begin{aligned}\Gamma'^a_{bc} &= \Gamma^a_{bc} + g^{ad}\left(g_{cd}\partial_b\omega + g_{bd}\partial_c\omega - g_{bc}\partial_d\omega\right)\\ &= \Gamma^a_{bc} + \delta^a_c\partial_b\omega + \delta^a_b\partial_c\omega - \gamma^{ad}\gamma_{bc}\partial_d\omega\end{aligned} \tag{3.4}$$

The tangent and normal vectors t^a and n^a are normalised, $g_{ab}t^a t^b = \mp 1$ (with the minus sign for timelike boundaries and the plus sign for spacelike boundaries). Therefore

$$\mp 1 = g'_{ab}t'^a t'^b = e^{2\omega} t'^a t'^b \tag{3.5}$$

and similarly for n^a. This can be satisfied if

$$t'^a = e^{-\omega} t^a \qquad \text{and} \qquad n'^a = e^{-\omega} n^a \tag{3.6}$$

From this we find that

$$n'_a = g'_{ab} n'^b = e^{2\omega}\gamma_{ab} e^{-\omega} n^b = e^{\omega} n_a \tag{3.7}$$

The term with the geodesic curvature then becomes

$$\begin{aligned} k' &= \pm\, t'^a n'_b \nabla_a t'^b = \pm t'^a n'_b(\partial_a t'^b + \Gamma'^b_{ac} t'^c) \\ &= \pm\, e^{-\omega} t^a e^{\omega} n_b \left[\partial_a(e^{-\omega} t^b) + (\Gamma^b_{ac} + \delta^b_c \partial_a\omega + \delta^b_a\partial_c\omega - \gamma^{bd}\gamma_{ac}\partial_d\omega) e^{-\omega} t^c\right] \\ &= \pm\, t^a n_b e^{-\omega}\left(-\partial_a\omega t^b + \partial_a t^b + \Gamma^b_{ac} t^c + \partial_a\omega t^b + \delta^b_a \partial_c\omega t^c - \gamma^{bd}\gamma_{ac}\partial_d\omega t^c\right) \\ &= e^{-\omega}(k \mp t^a n_b \gamma^{bd}\gamma_{ac}\partial_d\omega t^c) = e^{-\omega}(k \mp t^a t_a n_b \partial^b\omega) \end{aligned} \tag{3.8}$$

where we have used that the tangent and normal vectors are orthonormal $t^a n_a = 0$. Now, if $t^a t_a = -1$ then we are to chose the upper sign, which is minus and so the second term in the brackets gets a plus sign. If $t^a t_a = +1$ we need to take the lower sign and we once again find a plus sign for the second term. Thus

$$k' = e^{-\omega}(k + n^a\partial_a\omega) \tag{3.9}$$

It remains to work out the transformation of ds. We have

$$ds'^2 = g'_{ab} dx'^a dx'^b = e^{2\omega} dx^a dx^b = e^{2\omega} ds^2 \tag{3.10}$$

and thus

$$ds' = e^{\omega} ds \tag{3.11}$$

Bringing everything together we have

$$\begin{aligned} \chi' &= \frac{1}{4\pi}\int_M d^2\sigma\,\sqrt{g'}R' + \frac{1}{2\pi}\int_{\partial M} ds'\, k' \\ &= \frac{1}{4\pi}\int_M d^2\sigma\,\sqrt{g}(R - 2\nabla^2\omega) + \frac{1}{2\pi}\int_{\partial M} e^{\omega} ds e^{-\omega}(k + n^a\partial_a\omega) \\ &= \frac{1}{4\pi}\int_M d^2\sigma\,\sqrt{g}R + \frac{1}{2\pi}\int_{\partial M} ds\, k + \frac{1}{2\pi}\left[-\int_M d^2\sigma\sqrt{g}\nabla^a\partial_a\omega + \int_{\partial M} ds\, n^a\partial_a\omega\right] \\ &= \chi \end{aligned} \tag{3.12}$$

where the term between brackets vanishes due to Stokes' theorem

$$\int_M d^2\sigma\sqrt{g}\nabla^a v_a = \int_{\partial M} ds\, n^a v_a \tag{3.13}$$

3.3 p 83: Eq (3.2.7) String Coupling Constants

The figure below shows the appearance of a closed string handle on an open string and the contributions of the Euler number to the open and closed string interactions.

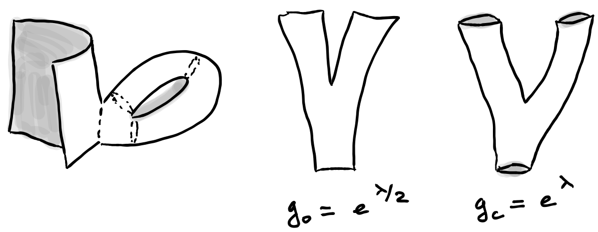


Figure 3.2: String coupling constants and handles

3.4 p 85: Eq (3.3.6) The Relations Between the Ricci Scalar and the Riemann Tensor in 2D

Just as we showed for p16 that $R_{ab} = \frac{1}{2} g_{ab} R$ using Mathematica, it is also convenient to show that

$$R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R \qquad [3.14]$$

using Mathematica. One just needs to be careful that $R_{abcd} = g_{ae} R^{e}_{bcd}$. Here our test functions are $tt[a, b, c, d]$ and we check that the sum of their absolute values is zero, which will of course ensure that they are all zero by themselves.

```
In[151]:= Clear[G, dG, gu, g, dg, ddg, dgu, R, Rd, RR, m, mu, tt];
m = {{g11[x, y], g12[x, y]}, {g12[x, y], g22[x, y]}};
g21[x_, y_] := g12[x, y];
mu = Inverse[m];
g[a_, b_] := m[[a, b]]
gu[a_, b_] := mu[[a, b]]
dg[1, a_, b_] := D[g[a, b], x]
dg[2, a_, b_] := D[g[a, b], y]
dgu[1, a_, b_] := D[gu[a, b], x]
dgu[2, a_, b_] := D[gu[a, b], y]
ddg[1, 1, a_, b_] := D[D[g[a, b], x], x]
ddg[1, 2, a_, b_] := D[D[g[a, b], x], y]
ddg[2, 1, a_, b_] := D[D[g[a, b], y], x]
ddg[2, 2, a_, b_] := D[D[g[a, b], y], y]
G[a_, b_, c_] := (1/2) * Sum[gu[a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c]), {d, 2}]
dG[e_, a_, b_, c_] := Simplify[(1/2) * Sum[dgu[e, a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c])
+ gu[a, d] * (ddg[e, b, c, d] + ddg[e, c, b, d] - ddg[e, d, b, c]), {d, 2}]]
R[a_, b_, c_, d_] := Simplify[dG[c, a, d, b] - dG[d, a, c, b]
+ Sum[G[a, c, e] * G[e, d, b] - G[a, d, e] * G[e, c, b], {e, 2}]]
R[a_, b_] := Simplify[Sum[R[c, a, c, b], {c, 2}]]
RR = Simplify[Sum[ gu[a, b] * R[a, b], {a, 2}, {b, 2}]];
tt[a_, b_, c_, d_] :=
 Sum[g[a, e] * R[e, b, c, d], {e, 2}] - (1/2) * (g[a, c] * g[b, d] - g[a, d] * g[b, c]) * RR

In[171]:= Sum[ Simplify[Abs[tt[a, b, c, d]]], {a, 2}, {b, 2}, {c, 2}, {d, 2}]

Out[171]= 0
```

Figure 3.3: Mathematica code for the relationship between R and R_{abcd} in 2D

3.5 p 85: Eq (3.3.8) The Residual Conformal Symmetry after Gauge Fixing

We first consider a diffeomorphism $z' = f(z)$ with $f(z)$ a holomorphic function. Under this transformation we have

$$ds'^2 = dz'd\bar{z}' = \frac{\partial f(z)}{\partial z}dz\frac{\bar{\partial}\bar{f}(\bar{z})}{\partial \bar{z}}d\bar{z} = |\partial f(z)|^2 dzd\bar{z} \qquad [3.15]$$

Next we also perform a Weyl transformation $g''_{ab} = e^{-2\omega(z,\bar{z})}g'_{ab}$. This gives

$$ds''^2 = e^{-2\omega(z,\bar{z})}ds'^2 = e^{-2\omega(z,\bar{z})}|\partial f(z)|^2 dzd\bar{z} = e^{-2\omega(z,\bar{z})}|\partial f(z)|^2 ds^2 \qquad [3.16]$$

If we now chose $\omega = \ln|\partial f(z)|$ we get

$$ds''^2 = \exp\left[-2\ln|\partial f(z)|\right]|\partial f(z)|^2 ds^2 = ds^2 \qquad [3.17]$$

so that the combinations of a holomorphic diffeomorphism and a Weyl transformation leaves the metric invariant.

3.6 p 87: Footnote 2 The Gauge Invariance of the Delta Function

The delta function $\delta(g-g')$ forces g'_{ab} to be equal to g_{ab} at every point on the worldsheet. But the diffeomorphism (3.3.10) is invertible as we can as well express g in terms of g^{ξ}. So by definition if $g'_{ab}(\sigma') = g_{ab}(\sigma)$ then so is $g'^{\xi}_{ab}(\sigma') = g^{\xi}_{ab}(\sigma)$ and vice versa.

3.7 p 88: Eq (3.3.16) The Infinitesimal Transformation of the Metric

We need to work out

$$\delta g_{ab}(\sigma) = g'_{ab}(\sigma) - g_{ab}(\sigma) \qquad [3.18]$$

for an infinitesimal version of (3.3.10), i.e. with $e^{2\omega} = 1+2\omega$ and $\sigma'_a = \sigma_a + \delta\sigma_a$ First, we see that

$$\frac{\partial\sigma^a}{\partial\sigma'^b} = \delta^a_b - \partial_b\delta\sigma^a \qquad [3.19]$$

Thus, to first order,

$$\begin{aligned}
\delta g_{ab}(\sigma) &= g'_{ab}(\sigma) - g_{ab}(\sigma) = g'_{ab}(\sigma'-\delta\sigma) - g_{ab}(\sigma) \\
&= g'_{ab}(\sigma') - \delta\sigma^c\partial_c g'_{ab}(\sigma') - g_{ab}(\sigma) \\
&= (1+2\omega)(\delta^c_a - \partial_a\delta\sigma^c)(\delta^d_b - \partial_b\delta\sigma^d)g_{cd}(\sigma) - \delta\sigma^c\partial_c g_{ab}(\sigma') - g_{ab}(\sigma) \\
&= \delta^c_a\delta^d_b g_{cd} - \delta^d_b\partial_a\delta\sigma^c g_{cd} - \delta^c_a\partial_b\delta\sigma^d g_{cd} + 2\omega\delta^c_a\delta^d_b g_{cd} - \delta\sigma^c\partial_c g_{ab} - g_{ab} \\
&= 2\omega g_{ab} - \partial_a\delta\sigma^c g_{bc} - \partial_b\delta\sigma^d g_{ad} - \delta\sigma^c\partial_c g_{ab} \qquad [3.20]
\end{aligned}$$

We now use $\delta\sigma^a = g^{ab}\delta\sigma_b$:

$$\begin{aligned}
\delta g_{ab}(\sigma) &= 2\omega g_{ab} - \partial_a(g^{cd}\delta\sigma_d)g_{bc} - \partial_b(g^{cd}\delta\sigma_c)g_{ad} - \delta\sigma^c\partial_c g_{ab} \\
&= 2\omega g_{ab} - \partial_a g^{cd}\delta\sigma_d g_{bc} - \partial_a\delta\sigma_d g^{cd}g_{bc} - \partial_b g^{cd}\delta\sigma_c g_{ad} - g^{cd}\partial_b\delta\sigma_c g_{ad} - \delta\sigma^c\partial_c g_{ab} \\
&= 2\omega g_{ab} + \delta\sigma_d g^{cd}\partial_a g_{bc} - \partial_a\delta\sigma_d\delta^d_b + \delta\sigma_c g^{cd}\partial_b g_{ad} - \partial_b\delta\sigma_c\delta^c_a - \delta\sigma^c\partial_c g_{ab} \\
&= 2\omega g_{ab} - \partial_a\delta\sigma_b - \partial_b\delta\sigma_a + \delta\sigma_d g^{cd}\partial_a g_{bc} + \delta\sigma_c g^{cd}\partial_b g_{ad} - g^{cd}\delta\sigma_d\partial_c g_{ab} \\
&= 2\omega g_{ab} - \partial_a\delta\sigma_b - \partial_b\delta\sigma_a + \delta\sigma_d g^{cd}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \qquad [3.21]
\end{aligned}$$

Let us now work out $\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a$ keeping in mind that because the indices are downstairs, the connection gets a minus sign

$$\begin{aligned}\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a &= \partial_a \delta\sigma_b - \Gamma^c_{ab}\delta\sigma_c + \partial_b \delta\sigma_a - \Gamma^c_{ba}\delta\sigma_c \\ &= \partial_a \delta\sigma_b + \partial_b \delta\sigma_a - (\Gamma^c_{ab} + \Gamma^c_{ba})\delta\sigma_c \\ &= \partial_a \delta\sigma_b + \partial_b \delta\sigma_a - 2\Gamma^d_{ab}\delta\sigma_d \\ &= \partial_a \delta\sigma_b + \partial_b \delta\sigma_a - \delta\sigma_d g^{cd}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \end{aligned} \quad [3.22]$$

and we see that indeed

$$\delta g_{ab}(\sigma) = 2\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a \quad [3.23]$$

If we now just fill in the definition of P_1 in (3.3.16) we get

$$\begin{aligned}\delta g_{ab} &= 2\delta\omega g_{ab} - \nabla_c \delta\sigma^c g_{ab} - (\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab}\nabla_c \delta\sigma^c) \\ &= 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a \end{aligned} \quad [3.24]$$

Which shows that the first and second line of (3.3.16) are equal.

Let us now show that P_1 takes vectors into traceless symmetric tensors. First, it is obvious from the definition (3.3.17) that

$$(P_1\delta\sigma)_{ab} = (P_1\delta\sigma)_{ba} \quad [3.25]$$

Next, also the tracelessness is obvious

$$\begin{aligned} g^{ab}(P_1\delta\sigma)_{ab} &= g^{ab}\frac{1}{2}(\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab}\nabla_c \delta\sigma^c) \\ &= \frac{1}{2}(2\nabla_a \delta\sigma^a - \delta^a_a \nabla_c \delta\sigma^c) = 0 \end{aligned} \quad [3.26]$$

But notice that the tracelessness is only valid in two dimensions.

3.8 p 88: Eq (3.3.18) The Faddeev-Popov Determinant

Even though the derivation is Joe's book is detailed, let's do it again, just for the sake of it. From (3.3.11) we have that

$$\Delta^{-1}_{\rm FP}(g) = \int [d\delta\omega \, d\delta\sigma]\, \delta(g - \hat{g}^\xi) \quad [3.27]$$

We have a functional integration over the diffeomorphism parameters $\delta\sigma_a$ and over the Weyl parameter $\delta\omega$. Recall that $\hat{g}$ is the fiducial metric and $\hat{g}^\xi$ is the fiducial metric after a gauge transformation. We are thus integrating over all possible gauge transformations, fixing the gauge transformed metric to be the fiducial metric.

Now $\hat{g} - \hat{g}^{\xi}$ is simply the change from the fiducial metric to its gauge transformed. For an infinitesimal transformation this is given by (3.3.16):

$$\hat{g} - \hat{g}^{\xi} = -\delta\hat{g} = -(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma \tag{3.28}$$

Notice that everything has to be w.r.t. the fiducial metric, including ∇ as it contains a connection and hence depends on the fiducial metric as well. Notice also that this is to be viewed as a two-component "tensor", i.e.

$$\hat{g}_{ab} - \hat{g}^{\xi}_{ab} = -\delta\hat{g}_{ab} = -(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g}_{ab} + 2(\hat{P}_1\delta\sigma)_{ab} \tag{3.29}$$

That takes us to the first equation in (3.3.18)

$$\Delta^{-1}_{\rm FP}(g) = \int [d\delta\omega\, d\delta\sigma]\, \delta\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big] \tag{3.30}$$

Next, we rewrite the delta function in its integral representation, more exactly the delta functional,

$$\begin{aligned}&\delta\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big]\\ &\quad = \frac{1}{2\pi}\int [dp]\exp\left\{i\int d^2\sigma\sqrt{\hat{g}}\,p.\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big]\right\}\end{aligned} \tag{3.31}$$

Here p is a two-component variable and $p\cdot X = p^{ab}X_{ab}$. We now set $p_{ab} = 2\pi\beta_{ab}$ and this gives

$$\begin{aligned}&\delta\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big]\\ &\quad = \int [d\beta]\exp\left\{2\pi i\int d^2\sigma\,\sqrt{\hat{g}}\,\beta.\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big]\right\}\end{aligned} \tag{3.32}$$

and therefore That takes us to the first equation in (3.3.18)

$$\Delta^{-1}_{\rm FP}(g) = \int [d\delta\omega\, d\delta\sigma d\beta]\,\exp\left\{2\pi i\int d^2\sigma\,\sqrt{\hat{g}}\,\beta.\big[-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g} + 2\hat{P}_1\delta\sigma\big]\right\} \tag{3.33}$$

This is the second equation of (3.3.18). We can now perform the β integration

$$\int [d\delta\omega]\exp\left(-4\pi i\int d^2\sigma\sqrt{\hat{g}}\beta^{ab}\hat{g}_{ab}\delta\omega\right) = \frac{1}{2\pi}\delta\left(4\pi\beta^{ab}\hat{g}_{ab}\right) \tag{3.34}$$

The delta function $\delta\left(4\pi\beta^{ab}\hat{g}_{ab}\right)$ now forces β^{ab} to be traceless and takes away one of the three degrees of freedom. Calling the traceless β now β' and ignoring a normalisation factor we have

$$\Delta^{-1}_{\rm FP}(g) = \int [d\delta\sigma d\beta']\,\exp\left\{2\pi i\int d^2\sigma\,\sqrt{\hat{g}}\,\beta'\cdot\big[(\hat{\nabla}\cdot\delta\sigma)\,\hat{g} + 2\hat{P}_1\delta\sigma\big]\right\} \tag{3.35}$$

but $\beta' \cdot \hat{g} = \beta'^{ab}\hat{g}_{ab} = 0$ as β' is traceless. Thus

$$\Delta_{\mathrm{FP}}^{-1}(g) = \int [d\delta\sigma d\beta'] \exp\left[4\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta'^{ab}(\hat{P}_1\delta\sigma)_{ab}\right] \quad [3.36]$$

which is (3.3.18).

3.9 p 89: Eq (3.3.21) The Faddeev-Popov Action

We use (A.2.28) from the appendix. For x and y c-numbers and ψ and χ Grassmann numbers we have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{2\pi i\lambda xy} = \frac{1}{\lambda} = \left[\int d\psi \int d\chi\, e^{\lambda\psi\chi}\right]^{-1} \quad [3.37]$$

Using this in (3.3.18) gives

$$\Delta_{\mathrm{FP}}^{-1}(g) = \left[\int [db\, dc] \exp\left(2 \int d^2\sigma \sqrt{\hat{g}}\, b_{ab}(\hat{P}_1 c)_{ab}\right)\right]^{-1} \quad [3.38]$$

The sign and normalisation is just a convention, so that we can write

$$\Delta_{\mathrm{FP}}(g) = \int [db\, dc] \exp\left(-\frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab}(\hat{P}_1 c)_{ab}\right) \quad [3.39]$$

Furthermore

$$b_{ab}(\hat{P}_1 c)_{ab} = b_{ab}\frac{1}{2}\left(\hat{\nabla}_a c_b + \hat{\nabla}_b c_b - g_{ab}\hat{\nabla}_c c^c\right) = b_{ab}\hat{\nabla}_a c_b \quad [3.40]$$

where we have used the fact that b_{ab} is traceless, so that the last term vanishes, and symmetric so that the first two terms are identical.

3.10 p 89: Eq (3.3.24) The Faddeev-Popov Action in the Conformal Gauge

From (3.3.21) we have in complex coordinates, where the metric is off-diagonal

$$S_{\mathrm{g}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \left(b_{zz}\hat{\nabla}^z c^z + b_{\bar{z}\bar{z}}\hat{\nabla}^{\bar{z}} c^{\bar{z}}\right) \quad [3.41]$$

Also

$$\hat{\nabla}^z c^z = g^{z\bar{z}}\hat{\nabla}_{\bar{z}} c^z = g^{z\bar{z}}\left(\partial_{\bar{z}} c^z + \hat{\Gamma}^z_{\bar{z}a} c^a\right) = g^{z\bar{z}}\left(\partial_{\bar{z}} c^z + \hat{\Gamma}^z_{\bar{z}z} c^z + \hat{\Gamma}^z_{\bar{z}\bar{z}} c^{\bar{z}}\right) \quad [3.42]$$

But, using the fact that the metric is off-diagonal in complex coordinates,

$$\Gamma^z_{z\bar{z}} = \frac{1}{2}g^{za}(\partial_z g_{\bar{z}a} + \partial_{\bar{z}} g_{za} - \partial_a g_{z\bar{z}}) = \frac{1}{2}g^{z\bar{z}}(\partial_z g_{\bar{z}\bar{z}} + \partial_{\bar{z}} g_{z\bar{z}} - \partial_{\bar{z}} g_{z\bar{z}}) = 0 \qquad [3.43]$$

Similarly

$$\Gamma^z_{\bar{z}\bar{z}} = \frac{1}{2}g^{za}(\partial_{\bar{z}} g_{\bar{z}a} + \partial_{\bar{z}} g_{\bar{z}a} - \partial_a g_{\bar{z}\bar{z}}) = 0 \qquad [3.44]$$

because with $a = z$ we have $g^{zz} = 0$ and with $a = \bar{z}$ we have $g_{\bar{z}\bar{z}} = 0$. Thus

$$\hat{\nabla}^z c^z = g^{z\bar{z}}\hat{\nabla}_{\bar{z}} c^z = g^{z\bar{z}}\hat{\partial}_{\bar{z}} c^z \qquad [3.45]$$

This also proves the claim that in a conformal gauge the covariant $\bar{z}$ derivative of a tensor with only covariant z indices reduces to the ordinary derivative. In the conformal gauge we $\sqrt{\hat{g}} = e^{2\omega}$ and $g^{z\bar{z}} = e^{-\omega}$ so that

$$\begin{aligned} S_{\mathrm{g}} &= \frac{1}{2\pi}\int d^2\sigma\, e^{2\omega}\, \left(b_{zz}e^{-2\omega}\nabla_{\bar{z}} c^z + b_{\bar{z}\bar{z}}e^{-2\omega}\hat{\nabla}_z c^{\bar{z}}\right) \\ &= \frac{1}{2\pi}\int d^2\sigma\, \left(b_{zz}\nabla_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right) \\ &= \frac{1}{2\pi}\int d^2\sigma\, \left(b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right) \end{aligned} \qquad [3.46]$$

which is (3.3.24).

3.11 p 90-91: The Anomaly of a Global Scale Symmetry

This is a rather long detour on the anomaly of a global scale symmetry. It is taken almost verbatim from my QFT notes. Some of the conventions may therefore be different. This is a.o. the case of the signature which is mostly negative, as is pretty standard in QFT texts.

There is an important symmetry of the classical level that can become anomalous at the quantum level. This is the scale invariance of massless field theories with dimensionless couplings. In fact, it is not difficult to understand that these theories can have such an anomaly. Indeed, these theories have no mass scales at all, but when we renormalise the theory we introduce a renormalisation scale and we see that the theory becomes dependent on that mass scale, e.g. via the running of the coupling constants. We can derive this dependence on the renormalisation scale via the Callan-Symanzik equations and the renormalisation equations. Here we will show how this dependence on a mass scale in the quantum theory can be described via the anomaly of a classically conserved current, the energy-momentum tensor.

THE CLASSICAL ENERGY-MOMENTUM TENSOR

Before we discuss the anomalous breaking of scale invariance, we need a better understanding of the energy-momentum tensor. There are actually different ways to derive the energy-momentum tensor.

The traditional way to derive the energy momentum tensor $T^{\mu\nu}$ is from Noether's theorem, where it is the current corresponding translation invariance $x^\mu \to x^\mu + a^\mu$. A Lagrangian invariant under this symmetry transforms as $\mathcal{L} \to \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$ and this leads to the conserved current

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \qquad [3.47]$$

This energy-momentum tensor is called the Canonical Energy-Momentum Tensor. This construction does not guarantee that the energy-momentum tensor is symmetric. There is in fact no reason for it to be symmetric, nor, as it so happens, gauge invariant, but it turns out that we can always make it symmetric and gauge-invariant by adding a total divergence to it. This total divergence does not affect the conserved charge, so the physics remains the same. We define

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial^\sigma \Sigma_{\mu\nu\sigma} \qquad [3.48]$$

for some $\Sigma_{\mu\nu\sigma}$ that is antisymmetric between μ and σ. We the see that $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu \partial^\sigma \Sigma_{\mu\nu\sigma} = \partial_\mu T^{\mu\nu}$ and so $\Theta^{\mu\nu}$ is conserved if $T^{\mu\nu}$ is conserved.

Let us now consider a scale transformation of the space-time coordinates $x \longrightarrow \lambda x$ for some positive λ. We define the scalar field to transform as $\phi(x) \longrightarrow \lambda \phi(\lambda x)$, but we prefer to write this in the more general form

$$\phi(x) \longrightarrow e^{-D\sigma} \phi(e^{-\sigma} x) \qquad [3.49]$$

with $D = 1$ the mass dimension of the field. This definition allows us to generalise the scaling transformation to fermions and to gauge fields. Theories that have no dimensionful couplings will be invariant under such a transformation and we say that the theory has scale invariance. The conserved current corresponding to this invariance is the so-called dilatation current D^μ. The notation should not be confused with the covariant derivative! The dilatation current is related to the symmetric energy-momentum tensor in a simple way

$$D^\mu = \Theta^{\mu\nu} x_\nu \qquad [3.50]$$

Taking the divergence of the dilatation current we find

$$\partial_\mu D^\mu = (\partial_\mu \Theta^{\mu\nu}) x_\nu + \Theta^{\mu\nu} \eta_{\mu\nu} = \Theta^\mu{}_\mu \qquad [3.51]$$

We thus see that scale-invariance of the theory is equivalent to the tracelessness of the energy-momentum tensor. Pay attention to the fact that we have not explicitly derived the form of the dilatation current. It is in fact not straightforward to do so, but we can obtain this result from another method. If we couple the field theory to an gravitational field $g^{\mu\nu}(x)$ we can then define the energy-momentum tensor as the functional derivative of the matter action w.r.t. the gravitational field

$$\Theta^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{\mathrm{M}}[\phi]}{\delta g_{\mu\nu}} \quad [3.52]$$

where $g = \det g_{\mu\nu}$. This is a manifestly symmetric and gauge-invariant tensor. Under a scale transformation $x \longrightarrow e^{\sigma} x$ we can find the transformation of the gravitational field by requiring the invariance of the line element $ds^2 = g'_{\mu\nu}(x')dx'^{\mu}dx'^{\nu} = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}$. We find $g'_{\mu\nu}e^{-2\sigma}dx^{\mu}dx^{\nu} = g_{\mu\nu}dx^{\mu}dx^{\nu}$ or

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = e^{2\sigma} g_{\mu\nu}(x) \quad [3.53]$$

How does the matter action transform under such a scale transformation? We can find this using the chain rule. Consider an infinitesimal rescaling $g_{\mu\nu} \longrightarrow (1+2\sigma)g_{\mu\nu}$ or hence $\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$. We have

$$\delta \mathcal{S}_M = \frac{\delta \mathcal{S}_M}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta \sigma} = \frac{\delta \mathcal{S}_M}{\delta g^{\mu\nu}} 2 g^{\mu\nu} = g^{\mu\nu} \sqrt{g}\, \Theta_{\mu\nu} = \sqrt{g}\, \Theta^{\mu}{}_{\mu} \quad [3.54]$$

and we see indeed that the action is invariant under scaling transformations provided the energy-momentum tensor is traceless.

In QED the symmetric energy-momentum tensor is given by

$$\Theta^{\mu\nu} = -F^{\mu\sigma}F^{\nu}{}_{\sigma} + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + \frac{1}{2}\bar{\psi} i(\gamma^{\mu}D^{\nu} + \gamma^{\nu}D^{\mu})\psi - \eta^{\mu\nu}\bar{\psi}(i\not{D} - m)\psi \quad [3.55]$$

One can check that this is gauge-invariant and that it leads to the classical expression for the total energy

$$H = \int d^3x\, T^{00} = \int d^3x \, \left[\frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \psi^{\dagger}(-i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \gamma^0 m)\psi\right] \quad [3.56]$$

Note that these formulae are valid at the classical level in any dimensions d. Taking the trace of the energy-momentum tensor we find

$$\begin{aligned}
\Theta^{\mu}{}_{\mu} =& - F^{\mu\sigma}F_{\mu\sigma} + \frac{1}{4}\delta^{\mu}_{\mu}F^{\rho\sigma}F_{\rho\sigma} + \frac{1}{2}\bar{\psi} i(2\not{D})\psi - \delta^{\mu}_{\mu}\bar{\psi}(i\not{D} - m)\psi \\
=& \frac{d-4}{4}F^{\mu\nu}F_{\mu\nu} + (1-d)\bar{\psi} i\not{D}\psi + dm\bar{\psi}\psi \\
=& \frac{d-4}{4}F^{\mu\nu}F_{\mu\nu} + (1-d)\bar{\psi} m\psi + dm\bar{\psi}\psi = \frac{d-4}{4}F^{\mu\nu}F_{\mu\nu} + m\bar{\psi}\psi
\end{aligned} \quad [3.57]$$

We have used the equations of motion as the symmetry is only valid for on-shell particles. We see that in four dimensions the energy-momentum tensor is traceless for a massless theory, as we expect for a theory that has no dimensionful coupling.

The Energy-Momentum Tensor and the Scale Anomaly

We know from the previous discussion on renormalisation that the scale invariance is broken at the quantum level and that this shows up a.o. in the running of the coupling constant $\bar{\lambda}(p, \lambda)$. This is given by the renormalisation group equation

$$\frac{d}{d \log p/M} \bar{\lambda}(p, \lambda) = \beta(\bar{\lambda}) \qquad \text{with initial condition} \qquad \bar{\lambda}(M, \lambda) = \lambda \qquad [3.58]$$

Let us translate this to our case at hand. We have $x' = e^{-\sigma} x$, so $p' = e^{+\sigma} p$. A rescaling of the momentum from a renormalisation scale M to a momentum p is thus a multiplication by $e^{+\sigma}$. Thus $\log p/M$ is essentially $\log e^{+\sigma} = \sigma$. Replacing λ by the coupling constant g we find

$$\frac{d\bar{g}}{d\sigma} = \beta \qquad [3.59]$$

To first order we can replace $\beta(\bar{g})$ by $\beta(g)$ and as this is independent of σ we can integrate the differential equation immediately to get $\bar{g} = \sigma\beta + \mathrm{c^{te}}$. The initial conditions are that $\bar{g} = g$ when there is no re-scaling, i.e. when $\sigma = 0$. Therefore $\mathrm{c^{te}} = g$. We thus find that the coupling constant runs as follows

$$g \longrightarrow g' = g + \sigma\beta(g) \qquad [3.60]$$

Under such a rescaling the Lagrangian picks up a change

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g}\delta g = \sigma\beta(g)\frac{\partial\mathcal{L}}{\partial g} \qquad [3.61]$$

All other changes are zero because at fixed coupling constant, i.e. at the classical level, the Lagrangian is assumed to be scale invariant. As the change of the Lagrangian under a scale transformation is, according to [3.51], equal to the total divergence of the dilatation current, or equivalently to the trace of the energy-momentum tensor, we have

$$\partial_\mu D^\mu = \Theta^\mu{}_\mu = \beta(g)\frac{\partial\mathcal{L}}{\partial g} \qquad [3.62]$$

The trace of the energy-momentum tensor is thus proportional to the β function of the theory. This is the general form of the trace anomaly.

Let us work this out for QED. The Lagrangian is

$$\mathcal{L}[\psi, A] = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu A_\mu\psi \qquad [3.63]$$

Let us redefine the gauge field as $eA^\mu \longrightarrow A^\mu$. The Lagrangian then becomes

$$\mathcal{L}[\psi, A] = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\gamma^\mu A_\mu \psi \qquad [3.64]$$

The advantage of this redefinition is that the coupling e now only appears in the kinetic term of the gauge field. We then immediately find

$$\frac{\partial \mathcal{L}}{\partial e} = \frac{\partial}{\partial e}\left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \cdots\right] = \frac{1}{2e^3}F_{\mu\nu}F^{\mu\nu} \qquad [3.65]$$

and thus we find for the trace of the energy momentum tensor

$$\Theta^\mu{}_\mu = \frac{\beta(e)}{2e^3}F_{\mu\nu}F^{\mu\nu} \qquad [3.66]$$

This derivation of the trace anomaly for QED was quite heuristic. We should be able to recover the same result from perturbation theory and so we now evaluate the trace of the energy-momentum tensor explicitly to one-loop order.

We are thus interested in calculating the expectation value $\langle \Theta^\mu{}_\mu \rangle$. Let us first think about how we expect the result to be as it will guide us to the right answer. We will use the background field method. The idea there is to split the gauge field in a fixed background field A^a_μ and a fluctuating field $\mathcal{A}^a_\mu$ and integrate out the fluctuation field in the path integral for the expectation value. From (3.66) we expect the lowest order result to be quadratic in the background field and to be of the form

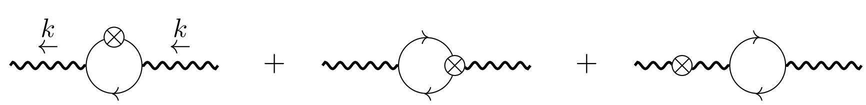
$$\langle \Theta^\mu{}_\mu \rangle = C \int \frac{d^4k}{(2\pi)^4} \mathrm{A}_\mu(-k)(k^2 g^{\mu\nu} - k^\mu k^\nu)\mathrm{A}_\nu(k) \qquad [3.67]$$

with C some constant. In fact, working out the details, one would find $C = \beta(e)/e^3$ to be in full agreement with (3.66).

As we will use dimensional regularisation for the divergent integrals, let us first remind ourselves of the formula for the trace for an arbitrary dimension d, (3.57). For massless fermions this is

$$\Theta^\mu{}_\mu = -\frac{4-d}{4}F^{\mu\nu}F_{\mu\nu} + (1-d)\bar{\psi}iD\!\!\!\!/\,\psi \qquad [3.68]$$

What are the Feynman diagrams contributing to $\langle \Theta^\mu{}_\mu \rangle$ with two external background fields? If we denote the insertion of the trace of the energy-momentum tensor in a Feynman diagram by $\otimes$ and an external background field by 〰 then we can have the insertion at three places (+ symmetric diagrams):



[3.69]

It turns out that the contributions of the first two diagrams cancel. A simple way to understand this is to look at the difference between the two diagrams. The second diagram has the operator $\Theta^{\mu}{}_{\mu}$ inserted in stead of a vertex. Because we need one gauge-field external to that, it takes the gauge field form the covariant derivative, i.e. $(1-d)\times i\times(-ie\bar{\psi}\gamma^{\mu}\psi)A_{\mu}$. The two fermions become propagators and so this operator insertion contributes a factor $(1-d)e\gamma^{\mu}$ to the diagram. Consider now the first diagram. The operator is now replaced by an ordinary vertex $(-ie\gamma^{\mu})$, but we have an additional fermion propagator $i/\not{p}$ and and operator inserted at the end of that vertex. As we have already two gauge fields, the only contribution from that operator now comes from the partial derivative $(1-d)\bar{\psi}i\gamma^{\mu}\partial\psi$. In momentum space the partial derivate brings down a factor ip_{μ} and so we have a factor

$$(-ie\gamma^{\mu})\frac{i}{\not{p}}(1-d)\times i\times i\not{p}=-(1-d)e\gamma^{\mu} \tag{3.70}$$

contributing to the Feynman diagram. All other factors in the matrix elements of the first two diagrams are identical, so that they are indeed equal and opposite and hence sum to zero. The third diagram corresponds to two external fields emanating from the operator so the lowest order contribution comes from the quadratic field strength term. This term is actually zero in four dimensions, but we see that it combines with a fermion loop that is divergent in four dimensions and so the product may be finite! This fermion loop is the photon self-energy and we only need its divergent term by virtue of the fact that the self-energy gets multiplied by the $(4-d)$ in front of the quadratic field strength in the inserted operator. The one-loop photon self-energy is a standard calculation in any QED text book. The details are e.g. available in my QFT Notes. The result is $i(k^2g^{\mu\nu}-k^{\mu}k^{n}u)\Pi_2(k^2)$ with

$$\Pi_2(k^2)\underset{d\to4}{=}\frac{-2\alpha}{\pi}\int_0^1 dx\, x(1-x)(\frac{2}{\varepsilon}-\log\Delta(k^2)+\log4\pi-\gamma+\mathsf{o}(\varepsilon)) \tag{3.71}$$

The divergent part is then simply

$$\Pi_2(k^2)\sim-\frac{2}{\pi}\frac{e^2}{4\pi}\int_0^1 x(1-x)\frac{2}{4-d}=\frac{e^2}{2\pi^2}\left(\frac{1}{2}-\frac{1}{3}\right)\frac{2}{4-d}=-\frac{e^2}{12\pi^2}\frac{2}{4-d} \tag{3.72}$$

We can now write the amplitude for the third diagram. We have two external background fields from the $-(4-d)/4\times F^{\mu\nu}F_{\mu\nu}$. There is a factor of two for symmetry reasons so that gives in momentum space $-2\times(4-d)/4\times A_{\mu}(-k)(k^2g^{\mu\sigma}-k^{\mu}k^{\sigma})A_{\sigma}(k)$. Next we have a gauge field propagator $-i/k^2$ and then we have the self-energy $i(k^2g^{\nu\sigma}-k^{\nu}k^{\sigma})\times(-e^2/12\pi^2\times2/(4-d))$. Bringing it together we find

$$\begin{aligned}\mathcal{M}_3=&\int\frac{d^4k}{(2\pi)^4}\left(-2\frac{4-d}{4}A^{\mu}(-k)(k^2g^{\mu\sigma}-k^{\mu}k^{\sigma})A_{\sigma}(k)\right)\left(\frac{-i}{k^2}\right)\\&\times\left(-i\frac{e^2}{12\pi^2}\frac{2}{4-d}(k^2\delta^{\nu}_{\sigma}-k_{\sigma}k^{\nu})\right)\\=&\frac{e^2}{12\pi^2}\int\frac{d^4k}{(2\pi)^4}A^{\mu}(-k)k^2\left(g^{\mu\sigma}-\frac{k^{\mu}k^{\sigma}}{k^2}\right)\left(\delta^{\nu}_{\sigma}-\frac{k_{\sigma}k^{\nu}}{k^2}\right)A_{\sigma}(k)\end{aligned} \tag{3.73}$$

Now $P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu/k^2$ is a projection operator, i.e.

$$P^{\mu\sigma}P_\sigma{}^\nu = \left(g^{\mu\sigma} - \frac{k^\mu k^\sigma}{k^2}\right)\left(\delta^\nu_\sigma - \frac{k_\sigma k^\nu}{k^2}\right) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} - \frac{k^\mu k^\nu}{k^2} + \frac{k^2 k^\mu k^\nu}{(k^2)^2}$$
$$= g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = P^{\mu\nu} \qquad [3.74]$$

Therefore

$$\mathcal{M}_3 = \frac{e^2}{12\pi^2}\int \frac{d^4k}{(2\pi)^4} A^\mu(-k)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)A^\nu(k) \qquad [3.75]$$

This is indeed of the form (3.76)

$$\langle \Theta^\mu{}_\mu \rangle = C\int \frac{d^4k}{(2\pi)^4} \mathrm{A}_\mu(-k)(k^2 g^{\mu\nu} - k^\mu k^\nu)\mathrm{A}_\nu(k) \qquad [3.76]$$

To find C let us recall that we absorbed the electric charge in the gauge field, so that

$$C = \frac{1}{12\pi^2} \qquad [3.77]$$

and we find that this agrees with (3.66) when we use the QED β function, $\beta = e^3/12\pi^2$, i.e.

$$C = \frac{\beta(e)}{e^3} = \frac{e^3}{12\pi^2}e^3 = \frac{1}{12\pi^2} \qquad [3.78]$$

It is now straightforward to generalise this discussion to QCD. In a non-abelian gauge theory the energy-momentum tensor is the obvious generalisation of (3.55) where we replace the abelian field strength by the non-abelian one. In the massless case it becomes

$$\Theta^{\mu\nu} = -F^{a\,\mu\sigma}F^{a\,\nu}_{\ \sigma} + \frac{1}{4}g^{\mu\nu}F^{a\,\rho\sigma}F^a_{\rho\sigma} + \frac{1}{2}\bar\psi i(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - g^{\mu\nu}\bar\psi i \not{D}\psi \qquad [3.79]$$

The trace is then the generalisation of (3.68) with $m = 0$

$$\Theta^\mu{}_\mu = -\frac{4-d}{4}F^{a\,\mu\nu}F^a_{\mu\nu} + (1-d)\bar\psi i\not{D}\psi \qquad [3.80]$$

where the fermion contributions vanishes because of the equation of motion, but also does not contribute to the one-loop result as in the case of QED. In an expectation value this becomes the generalisation of (3.76)

$$\langle \Theta^\mu{}_\mu \rangle = C\int \frac{d^4k}{(2\pi)^4} \mathrm{A}^a_\mu(-k)(k^2 g^{\mu\nu} - k^\mu k^\nu)\mathrm{A}^a_\nu(k) \qquad [3.81]$$

with $C = \beta(g)/g^3$ and $\beta(g)$ the QCD β function given by $\beta = -b_0 g^3/(4\pi)^2$ and $b_0 = (11 - 2n_F/3)$ for QCD.

Just as it is the case for the axial anomaly, we can find the trace anomaly using different methods. The anomaly will then always occur as a certain "pathology" of the chosen regularisation scheme. For example, if one uses Pauli-Villars regularisation, which introduces heavy fermions, we will find that the trace anomaly comes from the mass term $M\Phi\Phi$ from the heavy fermions. The loop diagram with this term inserted will have a finite result as we take M to infinity. It may seem at first sight that each regularisation scheme brings out the trace anomaly in a different way, just as it happens for the axial anomaly, but at the end the anomaly cannot be gotten rid of. It is an inherent part of those field theories and the classical symmetries cannot all automatically be promoted to quantum symmetries. One would need special cases of the theory, e.g. by combining certain fields, to ensure the symmetry still exists at the quantum level.

3.12 p 92: Eq (3.4.6) Weyl Invariance of an Expectation Value

This is straightforward, but sometimes it is good to emphasise the straightforward. A Weyl transformation has $g'_{ab}(\sigma') = e^{2\omega(\sigma)}g_{ab}(\sigma)$ and $\sigma' = \sigma$. Thus under an infinitesimal Weyl transformation

$$\begin{aligned}\delta g_{ab}(\sigma) &= g'_{ab}(\sigma) - g_{ab}(\sigma) = g'_{ab}(\sigma') - g_{ab}(\sigma) = [1 + 2\omega(\sigma)]\, g_{ab}(\sigma) - g_{ab}(\sigma) \\ &= 2\omega(\sigma) g_{ab}(\sigma)\end{aligned} \qquad [3.82]$$

Therefore

$$\begin{aligned}\delta_{\mathrm{W}} \left\langle \cdots \right\rangle_{\mathrm{g}} &= \left\langle -\frac{1}{4\pi}\int d^2\sigma\, \sqrt{g} 2\delta\omega g_{ab} T^{ab} \cdots \right\rangle \\ &= -\frac{1}{2\pi}\int d^2\sigma\, \sqrt{g}\delta\omega \left\langle g_{ab}T^{ab}\cdots\right\rangle\end{aligned} \qquad [3.83]$$

3.13 p 92: Eq (3.4.8) The General Form of the Weyl Anomaly

Let us count some dimensions. The Polyakov action is

$$S = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \qquad [3.84]$$

We need to make a difference between world-sheet dimensions, which we will denote by [ ], and space-time dimensions which we will denote by $[\![\ ]\!]$. Here α' has unit of space-time length squared, see page 11, i.e. $[\![\alpha']\!]$, but world-sheet dimension zero, $[\alpha'] = 0$. The metric had worldsheet dimension $[g_{ab}] = [g^{ab}] = 0$. This follows e.g. from the line element

$ds^2 = g_{ab}d\sigma^a d\sigma^b$ as $[ds] = [d\sigma^a] = 1$ as they are both a (worldsheet) length. Thus from the Polyakov action we have that $[X^\mu] = 0$. This means that the energy-momentum tensor

$$T^{ab} = -\frac{1}{\alpha'}\left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}g^{ab}\partial^c X^\mu \partial_c X_\mu\right) \qquad [3.85]$$

has $[T] = -2$, as $[\partial^a] = -1$. The Riemann curvature is of the form $\partial\Gamma + \Gamma\Gamma$ with Γ of the form $g^{ad}\partial_b g_{dc}$. Thus R has two derivatives and $[R] = -2$. From $T^a_a = a_1 R$ we thus find that a_1 is dimensionless.

Let us next consider other terms that could be added to T^a_a. A term of the form $a_2 R^2$ would have two more derivatives and so $[a_2] = +2$. If we would put in a high momentum cut-off Λ in the theory then a_2 would scale as Λ^{-2} and so vanish as we take the cut-off to infinity. The same reasoning is valid for higher order terms of R of for terms that include (worldsheet invariant) combinations of ∂^a. They would all lead to coefficients a with a positive dimension and so these coefficients would all vanish in a high momentum cut-off.

3.14 p 92: Eq (3.4.9) The General Form of the Weyl Anomaly in Complex Coordinates, I

In the conformal gauge we have $g_{ab} = e^{2\omega}$ this means that the only non-zero components of the metric tensor in complex coordinates are

$$g_{z\bar{z}} = \frac{1}{2}e^{2\omega} \qquad \text{and} \qquad g^{z\bar{z}} = 2e^{-2\omega} \qquad [3.86]$$

Therefore

$$g^{ab}T_{ab} = 2g^{z\bar{z}}T_{z\bar{z}} = 4e^{-2\omega}T_{z\bar{z}} \qquad [3.87]$$

and thus

$$4e^{-2\omega}T_{z\bar{z}} = a_1 R \quad \Rightarrow \quad T_{z\bar{z}} = \frac{e^{2\omega}}{4}a_1 R = \frac{a_1}{2}g_{z\bar{z}}R \qquad [3.88]$$

3.15 p 92: Eq (3.4.10) The General Form of the Weyl Anomaly in Complex Coordinates, II

$$\nabla^{\bar{z}}T_{z\bar{z}} = \frac{a_a}{2}\nabla^{\bar{z}}(g_{z\bar{z}}R) = \frac{a_a}{2}g_{z\bar{z}}\nabla^{\bar{z}}R = \frac{a_a}{2}g_{z\bar{z}}\partial^{\bar{z}}R = \frac{a_a}{2}\partial_z R \qquad [3.89]$$

We have used that the covariant derivative of the metric is zero and that the curvature is a scalar so that its covariant derivative is equal to its ordinary derivative.

3.16 p 93: Eq (3.4.11) The General Form of the Weyl Anomaly in Complex Coordinates, III

The conservation equation in curved metric is (for $b = z$)

$$0 = \nabla^a T_{ab} = \nabla^z T_{zz} + \nabla^{\bar{z}} T_{\bar{z}z} \qquad [3.90]$$

In case you are wondering, as I did for a brief moment, why the conservation equation is in terms of covariant derivatives and not ordinary derivatives, remember that the conserved charge follows from Stokes' theorem, which requires the covariant derivative.

3.17 p 93: Eq (3.4.12) The Actual Form of the Weyl Anomaly in Complex Coordinates, I

Using (1.2.32) we find that

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega) \qquad [3.91]$$

For an infinitesimal Weyl transformation $g'_{ab}(\sigma') = e^{2\delta\omega} g_{ab}(\sigma) = (1 + 2\delta\omega) g_{ab}(\sigma)$ we have for the RHS of (3.4.11)

$$\delta_{\mathrm{W}} RHS = -\frac{a_1}{2}\partial_z \delta_{\mathrm{W}} R \qquad [3.92]$$

now

$$\begin{aligned}\delta_{\mathrm{W}} R =& R'(\sigma) - R(\sigma) = \sqrt{\frac{g}{g'}}(R - 2\nabla^2\omega) - R \\ =& e^{-2\omega}(R - 2\nabla^2\omega) - R = (1 - 2\delta\omega)(R - 2\nabla^2\delta\omega) - R \\ =& -2\delta\omega R - 2\nabla^2\delta\omega \end{aligned} \qquad [3.93]$$

Therefore

$$\delta_{\mathrm{W}} RHS = -\frac{a_1}{2}\partial_z(-2\delta\omega R - 2\nabla^2\delta\omega) \qquad [3.94]$$

We now expand this near a flat worldsheet, where we have $R = 0$ and $\nabla^2 = 2g^{z\bar{z}}\nabla_z\nabla_{\bar{z}} = 4\partial_z\partial_{\bar{z}}$. This gives

$$\delta_{\mathrm{W}} RHS = -\frac{a_1}{2}\partial_z(-2 \times 4\partial_z\partial_{\bar{z}}\delta\omega) = 4a_1\partial_z^2\partial_{\bar{z}}\delta\omega \qquad [3.95]$$

3.18 p 93: Eq (3.4.15) The Actual Form of the Weyl Anomaly in Complex Coordinates, II

Combining (3.4.12) and (3.4.14) we find

$$4a_1\partial_z^2\partial_{\bar{z}}\delta\omega = \nabla^z\left(-\frac{c}{6}\partial_z^2\delta\omega\right) = -\frac{c}{6}g^{z\bar{z}}\partial_{\bar{z}}\partial_z^2\delta = -\frac{c}{3}\partial_{\bar{z}}\partial_z^2\delta\omega \qquad [3.96]$$

and so

$$a_1 = -\frac{c}{12} \qquad [3.97]$$

3.19 p 93: Eq (3.4.16a) The Ricci Scalar in the Conformal Gauge

The conformal gauge is

$$g_{ab} = e^{2\omega(\sigma)}\delta_{ab} \qquad [3.98]$$

Direct calculation gives the following values for the connections

$$\begin{aligned}\Gamma^1_{11} &= -\,\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \partial_1\omega\\ \Gamma^1_{12} &= \Gamma^1_{21} = -\Gamma^2_{12} = \Gamma^2_{22} = \partial_2\omega\end{aligned} \qquad [3.99]$$

The non-zero components of the Riemann curvature tensor are

$$R^1_{221} = -R^1_{212} = R^2_{112} = R^2_{121} = (\partial_1^2 + \partial_2^2)\omega = \partial_a\partial_a\omega \qquad [3.100]$$

and of the Ricci tensor

$$R_{11} = R_{22} = -\partial_a\partial_a\omega \qquad [3.101]$$

This then leads to the Ricci scalar

$$R = -2e^{-2\omega}\partial_a\partial_a\omega \qquad [3.102]$$

3.20 p 93: Eq (3.4.16b) The Laplacian in the Conformal Gauge

We know that in complex coordinates, see e.g. [3.96],

$$\nabla^2 = g^{ab}\nabla_a\nabla_b = 2g^{z\bar{z}}\nabla_z\nabla_{\bar{z}} = 2g^{z\bar{z}}\partial_z\partial_{\bar{z}} \qquad [3.103]$$

Thus

$$\nabla^2 = g^{ab}\partial_a\partial_b = e^{-2\omega}\partial_a\partial_a \qquad [3.104]$$

3.21 p 93: Eq (3.4.17) The Weyl Variation of $Z[g]$

From (3.4.6) we have, where $\hat{g}$ denotes the conformal gauge, $\hat{g}_{ab} = e^{2\omega}\delta_{ab}$,

$$\begin{aligned}
\delta_{\mathrm{W}} Z[\hat{g}] &= -\frac{1}{2\pi}\left\langle \int d^2\sigma \sqrt{\hat{g}}\,\delta\omega\, a_1 R\cdots \right\rangle \\
&= \frac{a_1}{\pi}\left\langle \int d^2\sigma e^{-2\omega}\delta\omega\, e^{-2\omega}\partial_a\partial_a\omega\cdots \right\rangle \\
&= \frac{a_1}{\pi}\int d^2\sigma\, \delta\omega\,\partial_a\partial_a\omega\, Z[\hat{g}] \qquad [3.105]
\end{aligned}$$

3.22 p 93: Eq (3.4.18) $Z[g]$ in the Conformal Gauge

It is easiest to check that (3.4.18) is a solution of (3.4.17):

$$\begin{aligned}
\delta_{\mathrm{W}} Z[e^{2\omega}\delta_{..}] &= \delta_{\mathrm{W}} Z[\delta_{..}] e^{-\frac{a_1}{2\pi}\int d^2\sigma\,\partial_a\omega\partial_a\omega} \\
&= Z[\delta_{..}] e^{-\frac{a_1}{2\pi}\int d^2\sigma\,\partial_a\omega\partial_a\omega}\delta_{\mathrm{W}}\left(-\frac{a_1}{2\pi}\int d^2\sigma\,\partial_a\omega\partial_a\omega\right) \\
&= Z[e^{2\omega}\delta_{..}]\left(-\frac{a_1}{2\pi}\int d^2\sigma\, 2\partial_a\delta_\omega\partial_a\omega\right) \\
&= \frac{a_1}{\pi} Z[e^{2\omega}\delta_{..}]\int d^2\sigma\,\delta_\omega\partial_a\partial_a\omega \qquad [3.106]
\end{aligned}$$

We have used partial integration in the last line.

3.23 p 94: Eq (3.4.19) $Z[g]$ for an Arbitrary Metric

Let us check that (3.4.19) reduces to (3.4.18) in the conformal gauge

$$\begin{aligned}
Z[g]\Big|_{g=e^{2\omega}\delta_{..}} &= Z[\delta_{..}]\,\exp\left\{\frac{a_1}{8\pi}\int d^2\sigma\int d^2\sigma'\, e^{2\omega(\sigma)}\left[-2e^{-2\omega(\sigma)}\partial_a\partial_a\omega(\sigma)\right]\right. \\
&\qquad\left.\times\; G(\sigma,\sigma')e^{2\omega(\sigma')}\left[-2e^{-2\omega(\sigma')}\partial'_a\partial'_a\omega(\sigma')\right]\right\} \\
&= Z[\delta_{..}]\,\exp\left[\frac{a_1}{2\pi}\int d^2\sigma\int d^2\sigma'\,\partial_a\partial_a\omega(\sigma)G(\sigma,\sigma')\partial'_a\partial'_a\omega(\sigma')\right] \\
&= Z[\delta_{..}]\,\exp\left[\frac{a_1}{2\pi}\int d^2\sigma\int d^2\sigma'\,\omega(\sigma)\partial_a\partial_a G(\sigma,\sigma')\partial'_a\partial'_a\omega(\sigma')\right] \qquad [3.107]
\end{aligned}$$

where we have used partial integration in the last line. We now use (3.4.16b) and (3.4.20) to rewrite

$$\begin{aligned}\partial_a\partial_a G(\sigma,\sigma') &= e^{2\omega}\nabla^2 G(\sigma,\sigma') = e^{2\omega} g^{-1/2}\delta^2(\sigma-\sigma') \\ &= e^{2\omega}e^{-2\omega}\delta^2(\sigma-\sigma') = \delta^2(\sigma-\sigma')\end{aligned} \qquad [3.108]$$

Thus

$$\begin{aligned}Z[g]\Big|_{g=e^{2\omega}\delta_{..}} &= Z[\delta_{..}]\,\exp\left[\frac{a_1}{2\pi}\int d^2\sigma\int d^2\sigma'\,\omega(\sigma)\delta^2(\sigma-\sigma')\partial'_a\partial'_a\omega(\sigma')\right] \\ &= Z[\delta_{..}]\,\exp\left[\frac{a_1}{2\pi}\int d^2\sigma\,\omega(\sigma)\partial_a\partial_a\omega(\sigma)\right] \\ &= Z[\delta_{..}]\,\exp\left[-\frac{a_1}{2\pi}\int d^2\sigma\,\partial_a\omega(\sigma)\partial_a\omega(\sigma)\right]\end{aligned} \qquad [3.109]$$

where we have, once more, used partial integration in the last line. Note also that (3.4.19) is manifestly diffeomorphism invariant. Indeed R and G are scalar functions and the measure in the integrals is the diffeomorphism invariant measure $d^2\sigma\sqrt{g}$.

3.24 p 94: Eq (3.4.21) The Second Way to Calculate the Variation of $Z[g]$, I

We first compute the Ricci scalar in the linear limit. If $g_{ab} = \delta_{ab} + h_{ab}$ then in complex coordinates we need a linear deformation from $g_{zz} = g_{\bar z\bar z} = 0$ and $g_{z\bar z} = 1/2$. Thus

$$g_{..} = \begin{pmatrix} h_{zz} & \frac{1}{2}+h_{z\bar z} \\ \frac{1}{2}+h_{z\bar z} & h_{\bar z\bar z}\end{pmatrix} \qquad [3.110]$$

The inverse metric is

$$g^{..} = \begin{pmatrix} -4h_{\bar z\bar z} & 2(1-2h_{z\bar z}) \\ 2(1-2h_{z\bar z}) & -4h_{zz}\end{pmatrix} \qquad [3.111]$$

This is easily checked by multiplying the two matrices with one another and showing that they are equal to the identity matrix plus terms of second order in h. To calculate the determinant $\sqrt{g}$ we first revert to the ordinary worldsheet coordinates. We have

$$\begin{aligned}ds^2 &= g_{zz}dz\,dz + g_{\bar z\bar z}d\bar z\,d\bar z + 2g_{z\bar z}dz\,d\bar z \\ &= h_{zz}(d\sigma^1+id\sigma^2)^2 + h_{\bar z\bar z}(d\sigma^1-id\sigma^2)^2 + (1+2h_{z\bar z}(d\sigma^1+id\sigma^2)(d\sigma^1-id\sigma^2) \\ &= (1+h_{zz}+h_{\bar z\bar z}+2h_{z\bar z})d\sigma^1 d\sigma^1 + (1-h_{zz}-h_{\bar z\bar z}+2h_{z\bar z})d\sigma^2 d\sigma^2 \\ &\quad + 2i(h_{zz}-h_{\bar z\bar z})d\sigma^1 d\sigma^2\end{aligned} \qquad [3.112]$$

and therefore

$$
\begin{aligned}
g_{11} &= 1 + h_{zz} + h_{\bar z\bar z} + 2h_{z\bar z} \\
g_{22} &= 1 - h_{zz} - h_{\bar z\bar z} + 2h_{z\bar z} \\
g_{12} &= i(h_{zz} - h_{\bar z\bar z})
\end{aligned} \qquad [3.113]
$$

The determinant is therefore

$$
\begin{aligned}
g &= (1 + h_{zz} + h_{\bar z\bar z} + 2h_{z\bar z})(1 - h_{zz} - h_{\bar z\bar z} + 2h_{z\bar z}) + (h_{zz} - h_{\bar z\bar z})^2 \\
&= 1 + 4h_{z\bar z} + \mathsf{o}(h^2)
\end{aligned} \qquad [3.114]
$$

and

$$
\sqrt{g} = 1 + 2h_{z\bar z} + \mathsf{o}(h^2) \qquad [3.115]
$$

Messy calculations are best left to Mathematica. The calculation of R is rather messy; the code is shown in fig. 3.24. The result of this Mathematica calculation is given in fig. 3.4. Just focussing on the linear terms, the result is quite simple

$$
R = 4\partial_{\bar z}^2 h_{zz} + 4\partial_z^2 h_{\bar z\bar z} - 8\partial_z\partial_{\bar z} h_{z\bar z} + \mathsf{o}(h^2) \qquad [3.116]
$$

As per Joe's book we now focus on the terms with $h_{\bar z\bar z}$ only. This means that to first order we can take $\sqrt{g} = 1$ and $R = 4\partial_z^2 h_{\bar z\bar z}$. Moreover the solution of (3.4.20) is something we already know; it is given by (2.1.24), i.e. $\partial\bar\partial \ln|z|^2 = 2\pi\delta^2(z,\bar z)$. Using $\nabla^2 = 2\partial\bar\partial + \mathsf{o}(h)$ we get

$$
G(\sigma,\sigma') = \frac{1}{4\pi}\ln|z - z'|^2 \qquad [3.117]
$$

We can now use all this in (3.4.19):

$$
\begin{aligned}
Z[g] &= Z[\delta]\exp\frac{a_1}{8\pi}\int \frac{1}{2}d^2z \int \frac{1}{2}d^2z' \times 1 \times 4\partial_z^2 h_{\bar z\bar z}(z,\bar z) \times \frac{1}{4\pi}\ln|z - z'|^2 \times 1 \times 4\partial_{z'}^2 h_{\bar z\bar z}(z',\bar z') \\
&= Z[\delta]\exp\frac{a_1}{8\pi^2}\int d^2z \int d^2z'\, \partial_z^2 h_{\bar z\bar z}(z,\bar z)\ln|z - z'|^2 \partial_{z'}^2 h_{\bar z\bar z}(z',\bar z')
\end{aligned} \qquad [3.118]
$$

Using partial integration and $g = \delta + h$ this gives

$$
\ln\frac{Z[\delta + h]}{Z[\delta]} = \frac{a_1}{8\pi^2}\int d^2z \int d^2\,z' h_{\bar z\bar z}(z,\bar z)\partial_{z'}^2\partial_z^2(\ln|z - z'|^2)h_{\bar z\bar z}(z',\bar z') \qquad [3.119]
$$

Now $\partial_{z'}^2\partial_z^2\ln|z - z'|^2 = -6/(z - z')^4$ so that we find (3.4.21).

$$
\ln\frac{Z[\delta + h]}{Z[\delta]} = -\frac{3a_1}{4\pi^2}\int d^2z \int d^2\,z' \frac{h_{\bar z\bar z}(z,\bar z)h_{\bar z\bar z}(z',\bar z')}{(z - z')^4} \qquad [3.120]
$$

```
In[201]:= (* 2D gravity: R linearised *)

Clear[h, G, dG, gu, g11, g22, g12, g21, g, dg, ddg, dgu, R, RR, m, mu];
m = {{g11[x, y], g12[x, y]}, {g12[x, y], g22[x, y]}};
g11[x_, y_] := hz[x, y]
g22[x_, y_] := hb[x, y]
g12[x_, y_] := 1/2 + h[x, y]
g21[x_, y_] := g12[x, y];
mu = {{-4 hb[x, y], 2 - 4 h[x, y]}, {2 - 4 h[x, y], -4 hz[x, y]}};
g[a_, b_] := m[[a, b]]
gu[a_, b_] := mu[[a, b]]
dg[1, a_, b_] := D[g[a, b], x]
dg[2, a_, b_] := D[g[a, b], y]
dgu[1, a_, b_] := D[gu[a, b], x]
dgu[2, a_, b_] := D[gu[a, b], y]
ddg[1, 1, a_, b_] := D[D[g[a, b], x], x]
ddg[1, 2, a_, b_] := D[D[g[a, b], x], y]
ddg[2, 1, a_, b_] := D[D[g[a, b], y], x]
ddg[2, 2, a_, b_] := D[D[g[a, b], y], y]
G[a_, b_, c_] := (1/2) * Sum[gu[a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c]), {d, 2}]
dG[e_, a_, b_, c_] := Simplify[(1/2) * Sum[dgu[e, a, d] * (dg[b, c, d] + dg[c, b, d] - dg[d, b, c])
+ gu[a, d] * (ddg[e, b, c, d] + ddg[e, c, b, d] - ddg[e, d, b, c]), {d, 2}]]
R[a_, b_, c_, d_] := Simplify[dG[c, a, d, b] - dG[d, a, c, b]
+ Sum[G[a, c, e] * G[e, d, b] - G[a, d, e] * G[e, c, b], {e, 2}]]
R[a_, b_] := Simplify[Sum[R[c, a, c, b], {c, 2}]]
RR = Expand[Sum[ gu[a, b] * R[a, b], {a, 2}, {b, 2}]];

In[225]:= RR
```

Out[225]=

$$-8\, h^{(0,1)}[x,y]\, hz^{(0,1)}[x,y] + 16\, h[x,y]\, h^{(0,1)}[x,y]\, hz^{(0,1)}[x,y] - 8\, hz[x,y]\, hb^{(0,1)}[x,y]\, hz^{(0,1)}[x,y] -$$
$$8\, hb[x,y]\, hz^{(0,1)}[x,y]^2 + 4\, hz^{(0,2)}[x,y] - 16\, h[x,y]\, hz^{(0,2)}[x,y] + 16\, h[x,y]^2\, hz^{(0,2)}[x,y] -$$
$$16\, hb[x,y]\, hz[x,y]\, hz^{(0,2)}[x,y] + 16\, h^{(0,1)}[x,y]\, h^{(1,0)}[x,y] - 32\, h[x,y]\, h^{(0,1)}[x,y]\, h^{(1,0)}[x,y] +$$
$$16\, hz[x,y]\, hb^{(0,1)}[x,y]\, h^{(1,0)}[x,y] - 32\, h[x,y]\, hz[x,y]\, hb^{(0,1)}[x,y]\, h^{(1,0)}[x,y] +$$
$$32\, h[x,y]^2\, hz[x,y]\, hb^{(0,1)}[x,y]\, h^{(1,0)}[x,y] - 32\, hb[x,y]\, hz[x,y]^2\, hb^{(0,1)}[x,y]\, h^{(1,0)}[x,y] +$$
$$32\, h[x,y]\, hb[x,y]\, hz^{(0,1)}[x,y]\, h^{(1,0)}[x,y] - 32\, h[x,y]^2\, hb[x,y]\, hz^{(0,1)}[x,y]\, h^{(1,0)}[x,y] +$$
$$32\, hb[x,y]^2\, hz[x,y]\, hz^{(0,1)}[x,y]\, h^{(1,0)}[x,y] + 32\, h[x,y]\, hz[x,y]\, h^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] -$$
$$32\, h[x,y]^2\, hz[x,y]\, h^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] + 32\, hb[x,y]\, hz[x,y]^2\, h^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] -$$
$$4\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] - 8\, h[x,y]\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] + 48\, h[x,y]^2\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] -$$
$$32\, h[x,y]^3\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] - 16\, hb[x,y]\, hz[x,y]\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] +$$
$$32\, h[x,y]\, hb[x,y]\, hz[x,y]\, hz^{(0,1)}[x,y]\, hb^{(1,0)}[x,y] - 8\, h^{(1,0)}[x,y]\, hb^{(1,0)}[x,y] +$$
$$16\, h[x,y]\, h^{(1,0)}[x,y]\, hb^{(1,0)}[x,y] - 8\, hz[x,y]\, hb^{(1,0)}[x,y]^2 + 16\, hb[x,y]\, h^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] -$$
$$32\, h[x,y]\, hb[x,y]\, h^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] + 32\, h[x,y]^2\, hb[x,y]\, h^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] -$$
$$32\, hb[x,y]^2\, hz[x,y]\, h^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] + 4\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] +$$
$$8\, h[x,y]\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] - 48\, h[x,y]^2\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] +$$
$$32\, h[x,y]^3\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] + 16\, hb[x,y]\, hz[x,y]\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] -$$
$$32\, h[x,y]\, hb[x,y]\, hz[x,y]\, hb^{(0,1)}[x,y]\, hz^{(1,0)}[x,y] - 8\, hb[x,y]\, hb^{(1,0)}[x,y]\, hz^{(1,0)}[x,y] -$$
$$8\, h^{(1,1)}[x,y] + 32\, h[x,y]\, h^{(1,1)}[x,y] - 32\, h[x,y]^2\, h^{(1,1)}[x,y] + 32\, hb[x,y]\, hz[x,y]\, h^{(1,1)}[x,y] +$$
$$4\, hb^{(2,0)}[x,y] - 16\, h[x,y]\, hb^{(2,0)}[x,y] + 16\, h[x,y]^2\, hb^{(2,0)}[x,y] - 16\, hb[x,y]\, hz[x,y]\, hb^{(2,0)}[x,y]$$

Figure 3.4: Mathematica code and result for R with a linearised metric

3.25 p 94: Eq (3.4.22) The Second Way to Calculate the Variation of $Z[g]$, II

Let us do this "symbolically" as it will be a mess to track all indices. We can extract the exact contribution, i.e. the contribution quadratic $h_{\bar{z}\bar{z}}$ in order to compare with (3.4.21) at the appropriate times. From (3.3.22) we have

$$Z[\delta + h] = \int [dX\, db\, dc\,] e^{-S[X,b,c,\delta+h]} \tag{3.121}$$

Here $S[X, b, c, \delta + h]$ is the sum of the matter and ghost actions. Now

$$S[X, b, c, \delta + h] = S[X, b, c, \delta] + h \frac{\delta S[X, b, c, g]}{\delta g}\bigg|_{g=\delta} + \mathsf{o}(h^2) \tag{3.122}$$

But from (3.4.5) we know that $\delta S/\delta g = (\sqrt{g}/4\pi)T$ by definition of the energy-momentum. From [3.115] we know that $\sqrt{g} = 1 + 2h_{z\bar{z}} + \mathsf{o}(h^2)$. As we are only interested in contributions that are second order in $h_{\bar{z}\bar{z}}$, we can simply take $\sqrt{g} = 1$ for our purposes.

Thus

$$S[X, b, c, \delta + h] = S[X, b, c, \delta] + \frac{h}{4\pi} T + \mathsf{o}(h^2) \tag{3.123}$$

and so

$$Z[\delta + h] = \int [dX\, db\, dc\,] \exp - \left(S[X, b, c, \delta] + \frac{h}{4\pi} T + \mathsf{o}(h^2) \right) \tag{3.124}$$

From this we have

$$\begin{aligned} \ln \frac{Z[\delta + h]}{Z[\delta]} &= \ln \frac{\int [dX\, db\, dc\,] e^{-\left(S[X,b,c,\delta] + \frac{h}{4\pi} T + \mathsf{o}(h^2)\right)}}{Z[\delta]} \\ &= \ln \frac{\int [dX\, db\, dc\,] e^{-S[X,b,c,\delta]} \left[1 - (1/2)(hT/4\pi)^2 + \cdots\right]}{Z[\delta]} \\ &\sim \ln \left[1 - \frac{1}{2(4\pi)^2 Z[\delta]} \int [dX\, db\, dc\,] e^{-S[X,b,c,\delta]} h^2 T^2 \right] \\ &= \frac{1}{2(4\pi)^2 Z[\delta]} \int [dX\, db\, dc\,] e^{-S[X,b,c,\delta]} h^2 T^2 \\ &= \frac{1}{2(4\pi)^2 Z[\delta]} \left\langle h^2 T^2 \right\rangle_\delta \end{aligned} \tag{3.125}$$

Here $\langle\ \rangle_\delta$ means that the expectation value is taken with a Euclidean metric, $g = \delta$. In the second line, we have only written down explicitly the contribution quadratic in h; all the other contributions are in the dots. In the third line we have ignored all these other contributions as they play no rôle in what we wish to show. Let us now plug in the indices. We wish to compare with (3.4.21) so need the contribution $h_{\bar{z}\bar{z}}$. Now $hT = h^{ab}T_{ab}$ and as $h^{zz} \sim h_{\bar{z}\bar{z}}$ we have

$$h^{zz}T_{zz} = g^{z\bar{z}}g^{z\bar{z}}h_{\bar{z}\bar{z}}T_{zz} = 4h_{\bar{z}\bar{z}}T_{zz} + \mathsf{o}(h^2) \qquad [3.126]$$

Thus the contribution we are looking for is

$$\begin{aligned}\ln\frac{Z[\delta+h]}{Z[\delta]} &= \frac{1}{2(4\pi)^2 Z[\delta]}\left\langle 4\int\frac{1}{2}d^2z h_{\bar{z}\bar{z}}(z,\bar{z})T_{zz}(z)4\int\frac{1}{2}d^2z' h_{\bar{z}\bar{z}}(z',\bar{z}')T_{zz}(z')\right\rangle_\delta \\ &= \frac{1}{8\pi^2 Z[\delta]}\int d^2z\int d^2z'\, h_{\bar{z}\bar{z}}(z,\bar{z})h_{\bar{z}\bar{z}}(z',\bar{z}')\left\langle T_{zz}(z)T_{zz}(z')\right\rangle_\delta \qquad [3.127]\end{aligned}$$

where we have used $2d^2z = d^2\sigma$.

But assuming this is correct, we can now use the $T(z)T(z')$ OPE

$$T(z)T(z') \sim \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial T(z')}{z-z'} \qquad [3.128]$$

The last two terms don't contribute because $\langle T(z')\rangle = \langle \partial T(z')\rangle = 0$ as in terms of creation and annihilation operators either $|0\rangle$ or $\langle 0|$ will be annihilated by one of the Virasoro generators. We thus conclude that

$$\begin{aligned}\ln\frac{Z[\delta+h]}{Z[\delta]} &= \frac{1}{8\pi^2 Z[\delta]}\int d^2z\int d^2z'\,\frac{h_{\bar{z}\bar{z}}(z,\bar{z})h_{\bar{z}\bar{z}}(z',\bar{z}')}{(z-z')^4}\langle c/2\rangle \\ &= \frac{c}{16\pi^2}\int d^2z\int d^2z'\,\frac{h_{\bar{z}\bar{z}}(z,\bar{z})h_{\bar{z}\bar{z}}(z',\bar{z}')}{(z-z')^4} \qquad [3.129]\end{aligned}$$

where we have used the fact that

$$\langle c/2\rangle = \frac{c}{2}\int[dX\,db\,dc\,]e^{-S[X,b,c,\delta]} = \frac{c}{2}Z[\delta] \qquad [3.130]$$

and the $Z[\delta]$ cancels the $Z[\delta]$ in the denominator of the pre-factor.

Comparing this with (3.4.21) gives

$$-\frac{3a_1}{4\pi^2} = \frac{c}{16\pi^2} \qquad \Rightarrow \qquad c = -12a_1 \qquad [3.131]$$

which is again (3.4.15).

All of this derivation seems ok, but there is one concern. The expansion of the action [3.123] has a second order term in h, viz. $(1/2)h^2\delta^2 S/\delta g^2|_{g=\delta}$. That term should have a contribution in [3.124] and in the subsequent formulae. It isn't clear to me why we can ignore it.

3.26 p 95: Theories with a Quantum Anomaly

One may well wonder why there is so much emphasis that string theory needs to be anomaly free, and have a zero β function, whilst that is not the case in QED and QCD. Indeed in (massless) QED and QCD there is no dimensional scale, but the renormalisation group introduces one and this cause a scale anomaly. Why is it no problem there?

The difference between string theory and QED/QCD is that in the former case the scale invariance is local; it is a gauge redundancy and if there is an anomaly the result is that the gauge symmetry is broken, with implications for e.g. unitarity. In QED/QCD the scale invariance is global. If it is broken, it has no impact on unitarity. It only indicates that, even if in the classical Lagrangian, there is no length scale, the quantum theory has an effective length scale. This then is an indication that these theories are probably just effective field theories of a more fundamental theory. String theory?

3.27 p 95: Eq (3.4.26) The Energy-Momentum Tensor of the Cosmological Term

The cosmological constant adds an additional term to the energy momentum tensor:

$$\begin{aligned}
T^{ab}_{\text{ct}}(\sigma) &= \frac{4\pi}{\sqrt{g}}\frac{\delta S_{\text{ct}}}{\delta g_{ab}(\sigma)} = \frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g_{ab}(\sigma)} b \int d^2\sigma' \sqrt{g} \\
&= \frac{4\pi}{\sqrt{g}} b \int d^2\sigma' \, \frac{1}{2} g^{-1/2}(\sigma') \frac{\delta g(\sigma')}{\delta g_{ab}(\sigma)} \\
&= \frac{2\pi b}{\sqrt{g}} \int d^2\sigma' \, g^{-1/2}(\sigma') g(\sigma') g^{ab}(\sigma') \delta^2(\sigma - \sigma') \\
&= \frac{2\pi b}{\sqrt{g}} \sqrt{g} g^{ab}(\sigma) = 2\pi b g^{ab}
\end{aligned} \qquad [3.132]$$

The extra factor in the trace becomes

$$g_{ab} T^{ab}_{\text{ct}}(\sigma) = 2\pi b \delta^a_b = 4\pi b \qquad [3.133]$$

3.28 p 96: Eq (3.4.27) The Most General Form $\delta_{\rm W} \ln Z[g]$ with Boundary Terms

Let us identify all the contributions in (3.4.27). The term with a_1 is just the contribution (3.4.8) that is linear in the Ricci scalar. The term with a_2 is the contribution from a possible cosmological constant. The terms with a_3, a_4 and a_5 come from the boundary term in Euler number contribution to the action, i.e. the second term in (3.2.3b)

$$\xi = \frac{1}{4\pi}\int_M d^2\sigma\,\sqrt{g}R + \frac{1}{2\pi}\int_{\partial M} kds \qquad [3.134]$$

To see this note that [3.9] and [3.11] imply that under a Weyl transformation kds transforms as

$$kds \to e^{-\omega}(k + n^a\partial_a\omega)e^{\omega}ds = (k + n^a\partial_a\omega)ds \qquad [3.135]$$

which explains the possible appearance of the a_4 and the a_5 term. The a_3 term is just a constant contribution, similar to a_2.

3.29 p 96: Eq (3.4.29) The Weyl Transformation of the Counterterms

Let us look at the transformation of the first term to start with:

$$\begin{aligned}\delta_{\rm W}\left(\int_M d^2\sigma\, b_1\sqrt{g}\right) &= \int_M d^2\sigma\, b_1\frac{1}{2}g^{-1/2}\delta_{\rm W}g = \int_M d^2\sigma\, b_1\frac{1}{2}g^{-1/2}gg^{ab}\delta_{\rm W}g_{ab}\\ &= \int_M d^2\sigma\, b_1\frac{1}{2}\sqrt{g}g^{ab}2\delta\omega g_{ab} = 2\int_M d^2\sigma\, b_1\sqrt{g}\delta\omega \end{aligned}\qquad [3.136]$$

The first term in the boundary part of the counterterms transforms as

$$\int_{\partial M} b_2 ds \to \int_{\partial M} b_2(1+\delta\omega)ds \quad \Rightarrow \quad \delta_{\rm W}\left(\int_{\partial M} b_2 ds\right) = \int_{\partial M} b_2\delta\omega ds \qquad [3.137]$$

where we have again used [3.11], i.e. $ds \to e^{\omega}ds$. The second term in the boundary part of the counterterms transforms, using [3.135], as

$$\int_{\partial M} b_3 kds \to \int_{\partial M} b_3(k + n^a\partial_a\delta\omega)ds \qquad [3.138]$$

Hence

$$\delta_{\rm W}\left(\int_{\partial M} b_3 kds\right) = \int_{\partial M} b_3 n^a\partial_a\delta\omega ds \qquad [3.139]$$

Bringing the three transformations together we find indeed (3.4.29).

3.30 p 96: Eq (3.4.30) The Wess-Zumino Consistency Condition

From (3.4.27) with only the terms with a_1 and a_4 we have, using the known transformation of $\sqrt{g}$, R and kds,

$$
\begin{aligned}
\delta_{\omega_1}\delta_{\omega_2} Z[G] = \delta_{\omega_1}&\left[-\frac{1}{2\pi}\int_M d^2\sigma\,\sqrt{g}\,a_1 R\delta\omega_2 - \frac{1}{2\pi}\int_{\partial M} ds\,a_4 k\delta\omega_2\right] \\
= &-\frac{1}{2\pi}\int_M d^2\sigma\,\frac{1}{2}g^{-1/2}gg^{ab}2\delta\omega_1 g_{ab}\,a_1 R\delta\omega_2 - \frac{1}{2\pi}\int_M d^2\sigma\,\sqrt{g}\,a_1(-2\nabla^2\delta\omega_1)\delta\omega_2 \\
&-\frac{1}{2\pi}\int_{\partial M} ds\,a_4 n^a(\partial_a\delta\omega_1)\delta\omega_2 \\
= &-\frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,R\delta\omega_1\delta\omega_2 + \frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,a_1\delta\omega_2\nabla^2\delta\omega_1 \\
&-\frac{a_4}{2\pi}\int_{\partial M} ds\,n^a\delta\omega_2\partial_a\delta\omega_1\delta\omega_2 \\
= &-\frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,R\delta\omega_1\delta\omega_2 - \frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,a_1(\partial^a\delta\omega_2)(\partial_a\delta\omega_1) \\
&+\frac{a_1}{\pi}\int_M d^2\sigma\,\nabla^a\big[\sqrt{g}\,a_1\delta\omega_2(\partial_a\delta\omega_1)\big] - \frac{a_4}{2\pi}\int_{\partial M} ds\,n^a\delta\omega_2\partial_a\delta\omega_1
\end{aligned}
\tag{3.140}
$$

In the last line we have used partial integration in the second term and the used the fact that the covariant derivative of the metric is zero so that also $\nabla_a\sqrt{g} = 0$ and also the fact that $\delta\omega_i$ are scalars so that $\nabla_a\delta\omega_i = \partial_a\delta\omega_i$. We can now use Stokes theorem on the third term and find

$$
\begin{aligned}
\delta_{\omega_1}\delta_{\omega_2} Z[G] = &-\frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,R\delta\omega_1\delta\omega_2 - \frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\,a_1(\partial^a\delta\omega_2)(\partial_a\delta\omega_1) \\
&+\frac{2a_1 - a_4}{\pi}\int_{\partial M} ds\,n^a\delta\omega_2\partial_a\delta\omega_1
\end{aligned}
\tag{3.141}
$$

The first two terms are symmetric in $\delta\omega_1$ and $\delta\omega_2$, the latter is not and requires $2a_1 = a_4$.

3.31 p 97: Eq (3.4.31) The Central Charge is Constant

This follows immediately from the last line of [3.140] where we have used partial integration. If we replace a_1 by $-\mathcal{C}(\sigma)/12$ one would get an extra term

$$
-\frac{a_1}{\pi}\int_M d^2\sigma\,\sqrt{g}\big(-\frac{\partial^a\mathcal{C}(\sigma)}{12}\big)\delta\omega_2(\partial_a\delta\omega_1)
\tag{3.142}
$$

where we have used the fact that $\mathcal{C}(\sigma)$ is a scalar so that $\nabla^a\mathcal{C}(\sigma) = \partial^a\mathcal{C}(\sigma)$. This additional term is not symmetric in $\delta\omega_1$ and $\delta\omega_2$ and so needs to vanish by the Wess-Zumino consistency condition. This implies that $\partial^a\mathcal{C}(\sigma) = 0$.

3.32 p 98: Fig 3.8 Scattering of Closed Strings

I had initially struggled with the representation of the representation of a scattering of strings as a sphere with small holes, but it is actually trivial. In the hope of sparing the same anguish for other people, let me explain why very slowly.

We consider a closed string as a semi-infinite cylinder. The cylinder worldsheet coordinates are $w = \sigma^1 + i\sigma^2$. The periodic boundary conditions for the closed string are $\sigma^1 \equiv \sigma^1 + 2\pi$. We rename for convenience $\sigma^2 = -2\pi t$. The asymptotic in-state is just obtained by taking $t \to \infty$.

We now consider the conformal transformation

$$z = e^{-iw} = e^{-i(\sigma^1 + i\sigma^2)} = e^{-i\sigma^1 - 2\pi t} \qquad [3.143]$$

We see that equal t curves corresponds to circles of radius $e^{-2\pi t}$ in the complex plane. The asymptotic state $t \to \infty$ corresponds to $z = 0$. At $t = 0$ the circle is the unit circle, $|z| = 1$. A closed string evolving from its asymptotic in-state to $t = 0$ thus corresponds to point – i.e. a circle of infinitesimal radius – evolving to a unit circle. This is the same discussion we had around (2.8.1), see also fig. 2.4 of these notes. Pictorially we have

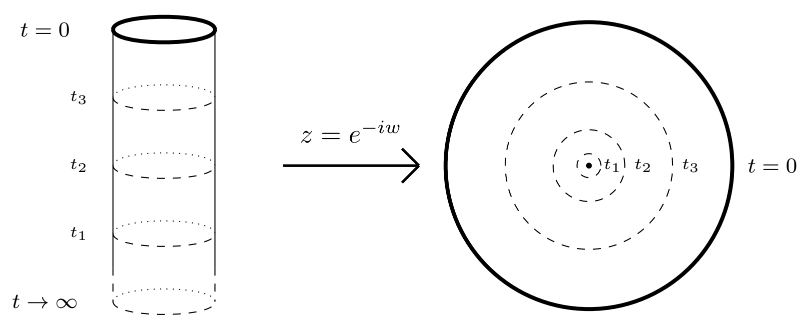


Figure 3.5: Mapping the semi-infinite cylinder to the unit disk. Evolving from $t = \infty$ to $t = 0$ via t_1, t_2 and t_3 corresponds to circles of increasing radius $e^{-2\ pit_i}$ in he complex plane.

Now there is no conformal transformation between fig 3.8a and fig 3.8b and that is what caused my initial confusion. Topologically, both pictures are equivalent. The picture of four cylinders interacting is nothing but a very stretched out sphere. The states at any time correspond to circles on the surface of a sphere. At $t \to \infty$ these are just infinitesimal holes. As t evolves these holes becomes circles of growing radius.

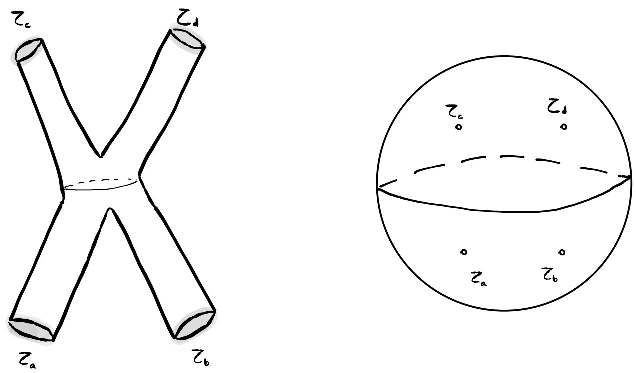


Figure 3.6: Closed string scattering amplitude. The asymptotic states correspond to holes on a sphere.

The process of four strings starting from asymptotic in-states and then interacting, thus corresponds to a worldsheet of a sphere but with four holes corresponding to the insertion of the four asymptotic in-states.

For open strings, almost the same applies word for word. The only difference is that the conformal transformation from w to z has an extra minus sign, i.e.

$$z = -e^{-iw} = -e^{-i\sigma^1+\sigma^2} \tag{3.144}$$

As the open string has $0 \leq \sigma^1 \leq \pi$ this then means that for a given $\sigma^2 = -2\pi t$ this corresponds to a semi-circle in the upper half plane. As per fig 3.9. a scattering of four open strings can then be represented as a desk with small dents at the boundary, corresponding to the asymptotic in-states.

3.33 p 100: Compact Connected Topologies

Let us, for convenience summarise the different compact connected 2D topologies, oriented or unoriented. This is done in table 3.3. In that table "o" stands for an oriented surface and "u" for an unoriented surface. The first line, a surface without boundaries, holes or cross-caps is just a sphere with g handles. The other surfaces are a sphere with extra holes and cross-caps.

boundaries	orientation	handles	holes	cross-caps
–	o	g	0	0
✓	o	g	h	0
–	u	g	0	c
✓	u	g	h	c

Table 3.1: 2D compact connected surfaces with $g =$ # handles, $h =$ # holes and $c =$ # cross-caps

Let us illustrate the effect of boundaries this with some examples when $g = 0$ in fig. 3.33. We will ignore cross-caps for now. When $g = 0$ then we simply have the sphere. With one boundary, the surface is just the disk. Where is the hole, you may ask? Just perform the conformal transformation $z \to 1/z$. Thus $z = e^{-i\sigma^1+\sigma^2}$ which for σ^2 between $-\infty$ and 0 describes the unit disk becomes $1/z = e^{i\sigma^1-\sigma^2}$ which becomes the full complex plane, outside of the unit disk. In that representation, the unit disk is hole. Similarly when there are two boundaries, then the surface is the annulus. With three boundaries, the surface looks like a pair of pants.

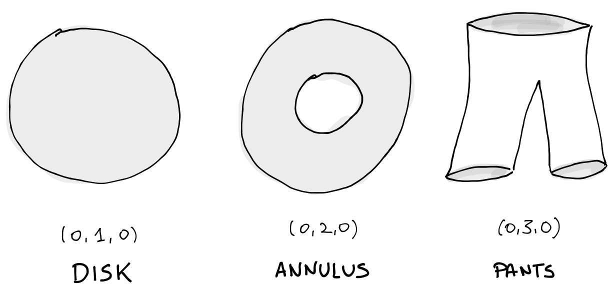


Figure 3.7: 2D compact connected surfaces with (g, h, c) and $g =$ # handles, $h =$ # holes and $c =$ # cross-caps

Let us get back to the cross-cap. Around that point, which for convenience we take to be at $z = 0$ we identify z with $-1/\bar{z}$. In this case a point on a sphere with radius r is identified with an antipodal point on a sphere with radius $1/r$. This is explained in more detail later in these notes, see fig. 6.5.

Let us illustrate this with two more examples. The first example has $(g, b, c) = (0, 1, 1)$. It is an unoriented surface with no handles and one boundary. It is simply the Möbius strip.

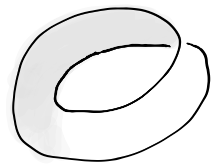


Figure 3.8: Möbius strip: unoriented surface with no handles and one boundary

The second example has $(g, b, c) = (0, 0, 2)$. It is an unoriented surface with two cross-caps, no handles and no boundaries. It is the infamous Klein bottle.

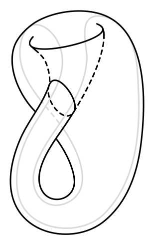


Figure 3.9: Klein bottle: unoriented surface with two cross-caps, no handles and no boundaries

3.34 p 102: Eq (3.6.3) The Normalisation of the First Excited States

The coefficient g_c is just the same string coupling constant as for the tachyon vertex operator. The coefficient $2/\alpha'$ comes from the state-operator mapping (2.8.7). Each holomorphic and anti-holomorphic modes contributes a factor $\sqrt{2/\alpha'}$.

3.35 p 102: Eq (3.6.4) The On-Shell Condition for the First Excited States

The first excited stated are tensors of weight $h = \tilde{h} = 1 + \alpha' k^2/4$. By conformal invariance, these weights have to be equa to one, and so this implies that $m^2 = -k^2 = 0$ and we recover the massless states.

3.36 p 103: Eq (3.6.7) The Weyl Transformation of a Renormalised Operator

This is just Leibniz:

$$\begin{aligned}
\delta_{\mathrm{W}}[\mathcal{F}]_{\mathrm{r}} &= \delta_{\mathrm{W}} e^{\frac{1}{2}\int d^2\sigma\, d^2\sigma'\, \Delta(\sigma,\sigma') \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')}} \mathcal{F} \\
&= \delta_{\mathrm{W}} \left[\frac{1}{2} \int d^2\sigma\, d^2\sigma'\, \Delta(\sigma,\sigma') \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')} \right] e^{\frac{1}{2}\int d^2\sigma\, d^2\sigma'\, \Delta(\sigma,\sigma') \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')}} \mathcal{F} \\
&\quad + e^{\frac{1}{2}\int d^2\sigma\, d^2\sigma'\, \Delta(\sigma,\sigma') \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')}} \delta_{\mathrm{W}} \mathcal{F} \\
&= \frac{1}{2} \int d^2\sigma\, d^2\sigma' \left(\delta_{\mathrm{W}} \Delta(\sigma,\sigma') \right) \frac{\delta}{\delta X^\mu(\sigma)} \frac{\delta}{\delta X_\mu(\sigma')} [\mathcal{F}]_{\mathrm{r}} + [\delta_{\mathrm{W}} \mathcal{F}]_{\mathrm{r}}
\end{aligned} \qquad [3.145]$$

3.37 p 103: Eq (3.6.8) The Weyl Transformation for the Tachyon Vertex for the Polyakov String

with $\delta_{\mathrm{W}} g_{ab} = 2\delta\omega g_{ab}$ we find, using (3.6.7)

$$\begin{aligned}
\delta_{\mathrm{W}} V_0 &= 2g_c \int d^2\sigma \delta_{\mathrm{W}} \sqrt{g} \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\
&= 2g_c \int d^2\sigma \Bigg\{ \frac{1}{2} g^{-1/2} g g^{ab} 2\delta\omega g_{ab} \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} + \sqrt{g} \left[\delta_{\mathrm{W}} e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\
&\quad + \frac{1}{2}\sqrt{g} \int d^2\sigma' d^2\sigma''\, \delta_{\mathrm{W}} \Delta(\sigma',\sigma'') \frac{\delta}{\delta X^\mu(\sigma')} \frac{\delta}{\delta X_\mu(\sigma'')} \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \Bigg\} \\
&= 2g_c \int d^2\sigma \Bigg\{ \sqrt{g} 2\delta\omega \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\
&\quad + \frac{1}{2}\sqrt{g} \int d^2\sigma' d^2\sigma''\, \delta_{\mathrm{W}} \Delta(\sigma',\sigma")(-k^2)\delta^2(\sigma-\sigma')\delta^2(\sigma-\sigma'') \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \Bigg\} \\
&= 2g_c \int d^2\sigma \left\{ \sqrt{g} 2\delta\omega \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} - \frac{k^2}{2} \sqrt{g} \delta_{\mathrm{W}} \Delta(\sigma,\sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \right\} \\
&= 2g_c \int d^2\sigma\, \sqrt{g} \left(2\delta\omega - \frac{k^2}{2} \delta_{\mathrm{W}} \Delta(\sigma,\sigma) \right) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}}
\end{aligned} \qquad [3.146]$$

We have also used the fact that $\delta_{\mathrm{W}} e^{ik\cdot X} = 0$ as there is no metric component in that expression.

3.38 p 103: Eq (3.6.11) The Weyl Transformation of the Geodesic Distance, I

This formula is a simplified case of (3.6.16). We derive the latter formula in detail and (3.6.11) will be derived along the way. We thus refer to that note for details. See in particular [3.157].

3.39 p 105: Eq (3.6.15) The Weyl Transformation of the Geodesic Distance, II

This explanation is from the Physics StackExchange[1] and is from "Trimok". Credit to him for working this out.

As is explained in the text, the diffeomorphism symmetry is not anomalous as we can find a regularisation scheme that preserves that symmetry. We are thus only conserved with the Weyl symmetry and can chose a metric in the conformal gauge $g_{ab} = e^{2\omega}\delta_{ab}$ and investigate if under a transformation $\omega \to \omega + \delta\omega$ the theory is anomalous.

Note that in the conformal gauge the connection becomes

$$\begin{aligned}\Gamma^c_{ab} &= \frac{1}{2} g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \\ &= \frac{1}{2} e^{-2\omega}\delta^{cd} 2e^{2\omega}(\delta_{bd}\partial_a\omega + \delta_{ad}\partial_b\omega - \delta_{ab}\partial_d\omega) \\ &= \delta^c_b\partial_a\omega + \delta^c_a\partial_b\omega - \delta_{ab}\partial^c\omega \end{aligned} \qquad [3.147]$$

and so

$$\nabla_a\partial_b\omega = \partial_a\partial_b\omega - \Gamma^c_{ab}\partial_c\omega = \partial_a\partial_b\omega + \mathsf{o}\big((\partial\omega)^2\big) \qquad [3.148]$$

We will later argue that at very short separation we can ignore terms of second and higher order in $\partial\omega$ for the Weyl transformations of the objects we are interested in, so that we can freely interchange the covariant and the partial derivative when acting on $\partial_a\omega$.

At very short separation, the distance between two points on the world-sheet is given in the conformal gauge by (3.6.9)

$$d^2(\sigma, \sigma') \approx (\sigma - \sigma')^2 e^{2\omega(\sigma)} \qquad [3.149]$$

We can make this more precise as

$$d(\sigma, \sigma') = \int_\sigma^{\sigma'} dz\, e^{\omega(z)} \qquad [3.150]$$

[1] https://physics.stackexchange.com/questions/73393/a-question-about-the-higher-order-weyl-variation-for-the-geodesic-distance

where z is some parameter describing the geodesic between the two points on the world-sheet. First we expand $\omega(z)$ around σ:

$$\begin{aligned} d(\sigma,\sigma') &= e^{\omega(\sigma)} \int_\sigma^{\sigma'} dz \, \exp\left[(z-\sigma)^a\partial_a\omega + \frac{1}{2}(z-\sigma)^a(z-\sigma^b)\partial_a\partial_b\omega + \mathsf{o}\big((z-\sigma)^3\big)\right] \\ &= e^{\omega(\sigma)} \int_\sigma^{\sigma'} dz \, \left[1 + (z-\sigma)^a\partial_a\omega + \frac{1}{2}(z-\sigma)^a(z-\sigma^b)\partial_a\partial_b\omega + \mathsf{q}\big((\partial\omega)^2\big)\right] \end{aligned} \quad [3.151]$$

Here $\mathsf{q}\big((\partial\omega)^2\big)$ denotes contributions that are quadratic in $\partial\omega$ or higher order. As already mentioned, we will argue later that they don't contribute that the Weyl transformation of the objects we are investigating. We can easily integrate this. Use

$$\begin{aligned} &\int_\sigma^{\sigma'} dz \; = \sigma' - \sigma = |\sigma' - \sigma| \\ &\int_\sigma^{\sigma'} dz\,(z-\sigma)^a\partial_a\omega = \int_0^{\sigma'-\sigma} dy\, y^a\partial_a\omega = \frac{1}{2}|\sigma'-\sigma| \times (\sigma-\sigma')^a\partial_a\omega \\ &\int_\sigma^{\sigma'} dz\,(z-\sigma)^a\omega(z-\sigma)^b\partial_a\partial_b\omega = \int_0^{\sigma'-\sigma} dy\, y^a y^b\partial_a\partial_b\omega \\ &\qquad\qquad = \frac{1}{3}|\sigma'-\sigma| \times (\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega \end{aligned} \quad [3.152]$$

to find

$$\begin{aligned} d(\sigma,\sigma') &= e^{\omega(\sigma)}|\sigma'-\sigma| \times \left[1 + \frac{1}{2}(\sigma-\sigma')^a\partial_a\omega + \frac{1}{6}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega + \mathsf{o}\big((\partial\omega)^2\big)\right] \\ &= e^{\omega(\sigma)}|\sigma'-\sigma|\, e^{\frac{1}{2}(\sigma-\sigma')^a\partial_a\omega + \frac{1}{6}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega + \mathsf{q}\left((\partial\omega)^2\right)} \end{aligned} \quad [3.153]$$

In the last line, we have raised it again to an exponential. From this we find that

$$\begin{aligned} \Delta(\sigma,\sigma') &= \frac{\alpha'}{2}\ln d^2(\sigma,\sigma') \\ &= \frac{\alpha'}{2}\ln\left[e^{2\omega(\sigma)}(\sigma'-\sigma)^2\, e^{(\sigma-\sigma')^a\partial_a\omega + \frac{1}{3}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega + \mathsf{q}\left((\partial\omega)^2\right)}\right] \\ &= \alpha'\left[\omega(\sigma) + \ln\big(\sigma-\sigma'\big) + \frac{1}{2}(\sigma-\sigma')^a\partial_a\omega + \frac{1}{6}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega + \mathsf{q}\big((\partial\omega)^2\big)\right] \\ &= \alpha'\left[\omega(\sigma) + \frac{1}{2}(\sigma-\sigma')^a\partial_a\omega + \frac{1}{6}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\partial_b\omega + \tilde{\mathsf{q}}\big((\partial\omega)^2\big)\right] \end{aligned} \quad [3.154]$$

Here $\tilde{\mathsf{q}}\big((\partial\omega)^2\big)$ also includes any possible contributions that don't depend on ω, viz $\ln(\sigma-\sigma')$. We now have a concern as $\Delta(\sigma,\sigma')$ should be symmetric for the interchange of σ with σ', but the above expression is manifestly not. This is easily remedied by making it symmetric, giving

$$\Delta(\sigma,\sigma') = \alpha'\left[\frac{1}{2}\left(\omega(\sigma)+\omega(\sigma')\right) + \frac{1}{4}(\sigma'-\sigma)^a\left(\partial_a\omega(\sigma) - \partial'_a\omega(\sigma')\right)\right.$$
$$\left. + \frac{1}{12}(\sigma'-\sigma)^a(\sigma'-\sigma)^b\left(\partial_a\partial_b\omega(\sigma) + \partial'_a\partial'_b\omega(\sigma')\right) + \tilde{\mathsf{q}}\big((\partial\omega)^2\big)\right] \qquad [3.155]$$

We can now this equation to calculate the Weyl transformation of $\Delta(\sigma,\sigma')$ and its derivatives. First we have

$$\begin{aligned}\delta_{\rm W}\Delta(\sigma,\sigma') &= \alpha'\left[\frac{1}{2}\left(\delta\omega(\sigma)+\delta\omega(\sigma')\right) + \mathsf{o}(\sigma-\sigma')\right]\\ &= \alpha'\left[\delta\omega(\sigma) + \mathsf{o}(\sigma-\sigma')\right]\end{aligned} \qquad [3.156]$$

Note that the ω dependent terms in $\tilde{\mathsf{q}}\big((\partial\omega)^2\big)$ are of the order $\mathsf{o}\big((\sigma'-\sigma)^2\big)$. Thus in the limit $\sigma'\to\sigma$ we have

$$\delta_{\rm W}\Delta(\sigma,\sigma')\Big|_{\sigma'\to\sigma} = \alpha'\delta\omega(\sigma) \qquad [3.157]$$

which is (3.6.11). Next we have from [3.155], expanding σ' around σ

$$\begin{aligned}\partial_a\delta_{\rm W}\Delta(\sigma,\sigma') &= \alpha'\left[\frac{1}{2}\partial_a\delta\omega(\sigma) + \frac{1}{4}\big(\partial_a\omega(\sigma) - \partial'_a\omega(\sigma')\big) + \mathsf{o}(\sigma'-\sigma)\right]\\ &= \frac{1}{2}\alpha'\partial_a\delta\omega(\sigma) + \mathsf{o}(\sigma'-\sigma)\end{aligned} \qquad [3.158]$$

which is when $\sigma'\to\sigma$

$$\partial_a\delta_{\rm W}\Delta(\sigma,\sigma')\Big|_{\sigma'\to\sigma} = \frac{1}{2}\alpha'\partial_a\delta\omega(\sigma) \qquad [3.159]$$

i.e. (3.6.15a).

Let us now go back to [3.155] and calculate

$$\partial'_b\delta_{\rm W}\ \Delta(\sigma,\sigma') = \alpha'\left[\frac{1}{2}\partial'_b\delta\omega(\sigma') + \frac{1}{4}\left(\partial_b\delta\omega(\sigma) - \partial'_b\delta\omega(\sigma')\right) - \frac{1}{4}(\sigma'-\sigma)^a\partial'_a\partial'_b\delta\omega(\sigma')\right.$$
$$\left. + \frac{1}{6}(\sigma'-\sigma)^a\left(\partial_a\partial_b\delta\omega(\sigma) + \partial'_a\partial'_b\delta\omega(\sigma')\right) + \mathsf{o}\big((\sigma'-\sigma)^2\big)\right] \qquad [3.160]$$

From this we have

$$\begin{aligned}\partial_a\partial'_b\delta_{\rm W}\ \Delta(\sigma,\sigma') &= \alpha'\left[\frac{1}{4}\partial_a\partial_b\delta\omega(\sigma) + \frac{1}{4}\partial'_a\partial'_b\delta\omega(\sigma') - \frac{1}{6}\big(\partial_a\partial_b\delta\omega(\sigma) + \partial'_a\partial'_b\delta\omega(\sigma')\big) + \mathsf{o}(\sigma'-\sigma)\right]\\ &= \alpha'\left[\frac{1}{6}\partial_a\partial_b\delta\omega(\sigma) + \mathsf{o}(\sigma'-\sigma)\right]\end{aligned} \qquad [3.161]$$

and in the limit $\sigma' \to \sigma$ this is

$$\partial_a \partial'_b \delta_{\mathrm{W}}\ \Delta(\sigma,\sigma')\Big|_{\sigma'\to\sigma} = \frac{1}{6}\alpha' \partial_a \partial_b \delta\omega(\sigma) \qquad [3.162]$$

Replacing the partial derivative by a covariant derivative, see [3.148] we get

$$\partial_a \partial'_b \delta_{\mathrm{W}}\ \Delta(\sigma,\sigma')\Big|_{\sigma'\to\sigma} = \frac{1}{6}\alpha' \nabla_a \partial_b \delta\omega(\sigma) \qquad [3.163]$$

which is (3.6.15b) with $\gamma = -3/2$.

Let us go back to [3.155] and calculate

$$\begin{aligned}
\partial_b \partial_a \delta_{\mathrm{W}}\ \Delta(\sigma,\sigma') &= \alpha' \partial_b \Bigg[\frac{1}{2} \partial_a \delta\omega(\sigma) - \frac{1}{4}\big(\partial_a \delta\omega(\sigma) - \partial'_a \delta\omega(\sigma')\big) + \frac{1}{4}(\sigma'-\sigma)^c \partial_a \partial_c \delta\omega(\sigma) \\
&\qquad - \frac{1}{6}(\sigma'-\sigma)^c \big(\partial_a \partial_c \delta\omega(\sigma) + \partial'_a \partial'_c \delta\omega(\sigma')\big) + \mathrm{o}\big((\sigma'-\sigma)^2\big) \Bigg] \\
&= \alpha' \Bigg[\frac{1}{2} \partial_a \partial_b \delta\omega(\sigma) - \frac{1}{4} \partial_a \partial_b \delta\omega(\sigma) - \frac{1}{4} \partial_a \partial_b \delta\omega(\sigma) \\
&\qquad + \frac{1}{6}\big(\partial_a \partial_b \delta\omega(\sigma) + \partial'_a \partial'_b \delta\omega(\sigma')\big) + \mathrm{o}(\sigma'-\sigma) \Bigg] \\
&= \frac{1}{3}\alpha' \partial_a \partial_b \delta\omega(\sigma) + \mathrm{o}(\sigma'-\sigma)
\end{aligned} \qquad [3.164]$$

Taking $\sigma' \to \sigma$ and using the fact that we can freely replace the covariant derivative with the partial derivative in this case again, we find

$$\partial_a \partial_b \delta_{\mathrm{W}}\ \Delta(\sigma,\sigma')\Big|_{\sigma'\to\sigma} = \frac{1}{3}\alpha' \nabla_a \partial_b \delta\omega(\sigma) \qquad [3.165]$$

which is (3.6.15c) with $\gamma = -2/3$.

We have claimed that we can ignore terms quadratic (and higher) in $\partial_a\omega$ and in terms that don't depend on ω. The latter we can trivially ignore as we are concerned with the Weyl variation of Δ and so terms that don't depend on ω don't change under a Weyl transformation. To argue that we can ignore the terms quadratic in $\partial_a\omega$ requires a bit more thinking. Every such term necessarily contains something like $(\sigma'-\sigma)^a(\sigma'-\sigma)^b\partial_a\omega\partial_b\omega$. Now if we take one derivative, then we are left with a $\sigma'-\sigma$ factor and that vanishes when $\sigma' \to \sigma$. If we take two derivatives, the only term that does not automatically vanish when $\sigma' \to \sigma$ is when the two derivatives act on $(\sigma'-\sigma)^a(\sigma'-\sigma)^b$. We are then left with something of the form $\partial_a\omega\partial_b\omega$. But we can always select an inertial frame, i.e. one with $\omega = c^{\mathrm{te}}$ so that $\partial_a\omega = 0$ and this term does not contribute as well. As our results have all been expressed in a covariant form (we wrote them in terms of $\nabla_a\partial_b\omega$ and not in terms of $\partial_a\partial_b\omega$), the result obtained in an inertial frame, is valid in any other frame. This shows that we were indeed justified in ignoring terms in quadratic and higher order of $\partial_a\omega$.

3.40 p 105: Eq (3.6.16) The Weyl Transformation for the Massless Vertex Operator for the Polyakov String

This is a subtle and long calculation, so fasten you seatbelts. There were some points that were obscure to me and I posted them on the Physics Stack Exchange. This time my gratitude goes to "Wakabaloola" for helping clarifying these issues. At the end of the day it is always gratifying to see how all the details of a calculation conspire to give the final result. This was, if I am permitted a personal note, particularly the case for this calculation.

Some preliminaries first. Under an infinitesimal Weyl transformation

$$\delta_{\mathrm{W}} g_{ab} = 2\delta\omega g_{ab} \tag{3.166}$$

we also have

$$\delta_{\mathrm{W}} g^{ab} = -2\delta\omega g^{ab} \tag{3.167}$$

and

$$\delta_{\mathrm{W}}\sqrt{g} = \frac{1}{2}g^{-1/2}\delta_{\mathrm{W}} g = \frac{1}{2}g^{-1/2} g g^{ab}\delta_{\mathrm{W}} g_{ab} = \frac{1}{2}g^{-1/2} g g^{ab} 2\delta\omega g_{ab} = \sqrt{g}\,2\delta\omega \tag{3.168}$$

This implies that

$$\delta_{\mathrm{W}}(\sqrt{g}g^{ab}) = \delta_{\mathrm{W}}(\sqrt{g})g^{ab} + \sqrt{g}\delta_{\mathrm{W}} g^{ab} = \sqrt{g}2\delta\omega g^{ab} - 2\sqrt{g}\delta\omega g^{ab} = 0 \tag{3.169}$$

The antisymmetric tensor ϵ^{ab} also transforms non-trivially under a Weyl transformation. Indeed, from $\sqrt{g}\,\epsilon^{12} = 1$ we have

$$(\delta_{\mathrm{W}}\sqrt{g})\epsilon^{12} + \sqrt{g}(\delta_{\mathrm{W}}\epsilon^{12}) = 0 \qquad \Rightarrow \delta_{\mathrm{W}}\epsilon^{12} = -2\delta\omega\epsilon^{12} \tag{3.170}$$

and thus more generally

$$\delta_{\mathrm{W}}\epsilon^{ab} = -2\delta\omega\epsilon^{ab} \tag{3.171}$$

But we will only need the relation

$$\delta_{\mathrm{W}}(\sqrt{g}\epsilon^{ab}) = 0 \tag{3.172}$$

From (1.2.32) we also know that $\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\delta\omega)$ so that

$$\delta_{\mathrm{W}}(\sqrt{g}R) = -2\sqrt{g}\nabla^2\delta\omega \tag{3.173}$$

After all these preliminaries, let us now work out Weyl transformation of (3.6.14), using the above and also (3.6.7)

$$\begin{aligned}\delta_{\rm W}V_1 = \frac{g_c}{\alpha'}\int d^2\sigma \Bigg\{ &\Big(\delta_{\rm W}(\sqrt{g}g^{ab})s_{\mu\nu} + i\delta_{\rm W}(\sqrt{g}\epsilon^{ab})a_{\mu\nu}\Big)\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ &+ \alpha'\phi\delta_{\rm W}\big(\sqrt{g}R\big)\left[e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ + \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})\frac{1}{2}&\int d^2\sigma' d^2\sigma''\delta_{\rm W}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ + \sqrt{g}\alpha'\phi R\frac{1}{2}&\int d^2\sigma' d^2\sigma''\delta_{\rm W}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\Bigg\}\end{aligned} \quad [3.174]$$

Recall that $\delta_{\rm W} e^{ik\cdot X(\sigma)} = \delta_{\rm W}\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)} = 0$, as there is no metric dependence in these operators. We have just seen that the first line vanishes and so can write

$$\begin{aligned}\delta_{\rm W}V_1 = \frac{g_c}{\alpha'}\int d^2\sigma \Bigg\{ &-2\alpha'\phi\sqrt{g}(\nabla^2\delta\omega)\left[e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ + \frac{1}{2}\sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})&\int d^2\sigma' d^2\sigma''\delta_{\rm W}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ + \frac{1}{2}\sqrt{g}\alpha'\phi R&\int d^2\sigma' d^2\sigma''\delta_{\rm W}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\Bigg\}\end{aligned} \quad [3.175]$$

Let us for convenience work out the different lines separately.

We start with the first line

$$\begin{aligned}\mathcal{J}_1 =& -2g_c\phi\int d^2\sigma\sqrt{g}(\nabla^2\delta\omega)\left[e^{ik\cdot X(\sigma)}\right]_{\rm r} = -2g_c\phi\int d^2\sigma\sqrt{g}(\nabla_a\partial^a\delta\omega)\left[e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ =& -2g_c\phi\int d^2\sigma\sqrt{g}\left\{\nabla_a\left(\partial^a\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\right) - \partial^a\delta\omega\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\right\} \\ =& -2g_c\phi\int d^2\sigma\partial_a\left(\sqrt{g}\partial^a\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\right) + 2g_c\phi\int d^2\sigma\sqrt{g}\partial^a\delta\omega\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_{\rm r} \\ =& +2g_c\phi\int d^2\sigma\sqrt{g}\partial^a\delta\omega\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\end{aligned} \quad [3.176]$$

where we have used $\partial_a(\sqrt{g}v^a) = \sqrt{g}\,\nabla_a v^a$ and $\nabla_a\sqrt{g} = 0$. We can now replace the covariant derivative by a partial derivative as it acts on a scalar and partial integrate the other derivative

$$\begin{aligned}\mathcal{J}_1 =& -2g_c\phi\int d^2\sigma\delta\omega\partial^a\left(\sqrt{g}\partial_a\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\right) \\ =& -2g_c\phi\int d^2\sigma\delta\omega\sqrt{g}\,\nabla^a\partial_a\left[e^{ik\cdot X(\sigma)}\right]_{\rm r}\end{aligned} \quad [3.177]$$

We now have

$$\begin{aligned}\nabla^a\partial_a e^{ik\cdot X} &= \nabla^a(ik^\mu\partial_a X_\mu)e^{ik\cdot X}\\ &= ik^\mu(\nabla^a\partial_a X_\mu)e^{ik\cdot X} + (ik^\mu\partial_a X_\mu)(ik^\nu\nabla^a X_\nu)e^{ik\cdot X}\\ &= ik^\mu\nabla^2 X_\mu e^{ik\cdot X} - k^\mu k^\nu\partial_a X_\mu\partial^a X_\nu e^{ik\cdot X}\end{aligned} \quad [3.178]$$

We have freely replaced ∂_a by ∇_a and vice-versa when they are acting on worldsheet scalars. Therefore, using (3.6.18),

$$\begin{aligned}\left[\nabla^a\partial_a e^{ik\cdot X}\right]_{\mathrm{r}} &= ik^\mu\left[\nabla^2 X_\mu e^{ik\cdot X}\right]_{\mathrm{r}} - k^\mu k^\nu\left[\partial_a X_\mu\partial^a X_\nu e^{ik\cdot X}\right]_{\mathrm{r}}\\ &= ik^\mu\times\left(-i\frac{\alpha'}{6}k_\mu R\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right) - k^\mu k^\nu\left[\partial_a X_\mu\partial^a X_\nu e^{ik\cdot X}\right]_{\mathrm{r}}\\ &= \frac{\alpha'}{6}k^2 R\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} - k^\mu k^\nu g^{ab}\left[\partial_a X_\mu\partial_b X_\nu e^{ik\cdot X}\right]_{\mathrm{r}}\end{aligned} \quad [3.179]$$

and thus

$$\begin{aligned}\mathcal{J}_1 &= -\frac{g_c\alpha'}{3}k^2\phi\int d^2\sigma\,\sqrt{g}\,\delta\omega\,R\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + 2g_c k^\mu k^\nu\phi\int d^2\sigma\,\sqrt{g}\,\delta\omega\,g^{ab}\left[\partial_a X_\mu\partial_b X_\nu e^{ik\cdot X}\right]_{\mathrm{r}}\\ &= \frac{g_c}{2}\int d^2\sigma\,\sqrt{g}\,\delta\omega\left\{-\frac{2\alpha'}{3}k^2\phi R\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + g^{ab}(4k^\mu k^\nu\phi)\left[\partial_a X_\mu\partial_b X_\nu e^{ik\cdot X}\right]_{\mathrm{r}}\right\}\end{aligned} \quad [3.180]$$

Before we tackle the second, more difficult, line of [3.175], let us do the easier third line

$$\begin{aligned}\mathcal{J}_3 &= \frac{g_c}{2}\phi\int d^2\sigma d^2\sigma' d^2\sigma''\sqrt{g}R\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &= \frac{g_c}{2}\phi\int d^2\sigma d^2\sigma' d^2\sigma''\sqrt{g}R\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')(-k^2)\delta^2(\sigma-\sigma')\delta^2(\sigma-\sigma'')\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &= -\frac{g_c}{2}\phi k^2\int d^2\sigma\sqrt{g}R\delta_{\mathrm{W}}\Delta(\sigma,\sigma)\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} = -\frac{g_c}{2}\phi k^2\int d^2\sigma\sqrt{g}R\alpha'\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &= \frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega(-\alpha' k^2\phi R)\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\end{aligned} \quad [3.181]$$

Finally, the second line of [3.175], is

$$\mathcal{J}_2 = \frac{g_c}{2\alpha'}\int d^2\sigma\sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})\tilde{\mathcal{J}}_2 \quad [3.182]$$

with

$$\tilde{\mathcal{J}}_2 = \int d^2\sigma' d^2\sigma''\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \quad [3.183]$$

Let us first consider the contribution where the functional derivatives both act on the exponential This gives

$$\begin{aligned}\tilde{\mathcal{J}}_{2a} =& \int d^2\sigma' d^2\sigma'' \delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\big(ik^\lambda\delta^2(\sigma'-\sigma)\big)\big(ik_\lambda\delta^2(\sigma''-\sigma)\big)\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ =& -k^2\delta_{\mathrm{W}}\Delta(\sigma,\sigma)\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ =& -\alpha' k^2\delta\omega(\sigma)\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\end{aligned} \tag{3.184}$$

where we have used (3.6.11). Thus we have a contribution

$$\begin{aligned}\mathcal{J}_{2a} =& -\frac{g_c}{2}k^2\int d^2\sigma\sqrt{g}\delta\omega(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ =&\frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega\big(g^{ab}(-k^2 s_{\mu\nu})+i\epsilon^{ab}(-k^2 a_{\mu\nu})\big)\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\end{aligned} \tag{3.185}$$

Next, take the case where only one of the functional derivatives acts on the exponential. There are four possible combinations

$$\begin{aligned}\tilde{\mathcal{J}}_{2b} = \int d^2\sigma' d^2\sigma'' \delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\Big\{ & \delta^\mu_\lambda \partial_a\delta^2(\sigma'-\sigma) ik^\lambda\delta^2(\sigma''-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &+\delta^\nu_\lambda \partial_b\delta^2(\sigma'-\sigma) ik^\lambda\delta^2(\sigma''-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &+\eta^{\lambda\mu}\partial_a\delta^2(\sigma''-\sigma) ik_\lambda\delta^2(\sigma'-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &+\eta^{\lambda\nu}\partial_b\delta^2(\sigma''-\sigma) ik_\lambda\delta^2(\sigma'-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\}\end{aligned} \tag{3.186}$$

We can already perform one of the integrations:

$$\begin{aligned}\tilde{\mathcal{J}}_{2b} =& \, i\int d^2\sigma' \delta_{\mathrm{W}}\Delta(\sigma',\sigma)\Big\{ k^\mu \partial_a\delta^2(\sigma'-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &\qquad + k^\nu \partial_b\delta^2(\sigma'-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\}\\ &+i\int d^2\sigma'' \delta_{\mathrm{W}}\Delta(\sigma,\sigma'')\Big\{ k^\mu \partial_a\delta^2(\sigma''-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\ &\qquad + k^\nu \partial_b\delta^2(\sigma''-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\}\end{aligned} \tag{3.187}$$

Change the integration variable from σ'' to σ' and use the symmetry $\Delta(\sigma,\sigma') = \Delta(\sigma',\sigma)$

$$\begin{aligned}
\mathcal{J}_{2b} &= \frac{g_c}{2\alpha'}\int d^2\sigma d^2\sigma' \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \times 2i\delta_{\mathrm{W}}\Delta(\sigma',\sigma) \\
&\quad \times \left[k^\mu \partial_a \delta^2(\sigma'-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + k^\nu \partial_b \delta^2(\sigma'-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right] \\
&= \frac{ig_c}{\alpha'}\int d^2\sigma d^2\sigma' \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})\delta_{\mathrm{W}}\Delta(\sigma',\sigma) \\
&\quad \times \left[k^\mu \partial_a \delta^2(\sigma'-\sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + k^\nu \partial_b \delta^2(\sigma'-\sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right] \qquad [3.188]
\end{aligned}$$

We can now perform another partial integration to free up the last delta function. However, it is convenient to first use the chain rule for derivatives of delta functions[2] in order to change the ∂_a into a ∂'_a:

$$\partial_a \delta^2(\sigma'-\sigma) = -\partial'_a \delta^2(\sigma'-\sigma) \qquad [3.189]$$

We perform the chain rule on the derivative of the delta function and partially integrate.

[2]To see that this is correct, evaluate $I = \int dx\, dy\, f(x)g(y)\partial_x\delta(x-y)$ in two ways. First

$$I_1 = -\int dx\, dy\, f'(x)g(x)\delta(x-y) = -\int dx\, f'(x)g(x)$$

Alternatively

$$I_2 = -\int dx\, dy\, f(x)g(y)\partial_y\delta(x-y) = \int dx\, dy\, f(x)g'(y)\delta(x-y) = \int dx\, f(x)g'(x)$$

Now

$$I_2 - I_1 = \int dx\left[f(x)g'(x) + f'(x)g(x)\right] = \int dx\,\frac{d}{dx}\left[f(x)g(x)\right] = 0$$

which shows that $\partial_x\delta(x-y) = -\partial_y\delta(x-y)$

The $\partial' a$ and $\partial' b$ now only act on $\delta_{\mathrm{W}}\Delta(\sigma', \sigma)$. Two minus signs give a plus and so

$$\begin{aligned}\mathcal{J}_{2b} &= \frac{ig_c}{\alpha'}\int d^2\sigma d^2\sigma' \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \\ &\qquad \times \left\{ \partial'_a \delta_{\mathrm{W}}\Delta(\sigma', \sigma) k^\mu \delta^2(\sigma' - \sigma)\left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right. \\ &\qquad \left. + \partial'_b \delta_{\mathrm{W}}\Delta(\sigma', \sigma) k^\nu \delta^2(\sigma' - \sigma)\left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right\} \\ &= \frac{ig_c}{\alpha'}\int d^2\sigma \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \\ &\qquad \times \left\{ \partial'_a \delta_{\mathrm{W}}\Delta(\sigma', \sigma)\Big|_{\sigma'=\sigma} k^\mu \left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right. \\ &\qquad \left. + \partial'_b \delta_{\mathrm{W}}\Delta(\sigma', \sigma)\Big|_{\sigma'=\sigma} k^\nu \left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right\}\end{aligned} \qquad [3.190]$$

Use (3.6.15a), which is, by symmetry, equally valid if we replace ∂_a by ∂'_a

$$\begin{aligned}\mathcal{J}_{2b} &= \frac{ig_c}{\alpha'}\int d^2\sigma \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \\ &\qquad \times \left\{ \frac{1}{2}\alpha'\partial_a\delta\omega k^\mu \left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + \frac{1}{2}\alpha'\partial_a\delta\omega k^\nu \left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right\} \\ &= \frac{ig_c}{2}\int d^2\sigma \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \\ &\qquad \times \left\{ \partial_a\delta\omega k^\mu \left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + \partial_a\delta\omega k^\nu \left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \right\}\end{aligned} \qquad [3.191]$$

Yet another partial integration gives

$$\begin{aligned}\mathcal{J}_{2b} = &-\frac{ig_c}{2}\int d^2\sigma\delta\omega \left\{ \partial_a\left[\sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})k^\mu \left[\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right] \right. \\ &\left. + \partial_b\left[\sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})k^\nu \left[\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right] \right\}\end{aligned} \qquad [3.192]$$

We use $\partial_a(\sqrt{g}\, v^a) = \sqrt{g}\, \nabla_a v^a$ and the fact that the covariant derivative of g^{ab}, $\sqrt{g}$ and ϵ^{ab}

is zero:

$$\begin{aligned}\mathcal{J}_{2b} = &-\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}\Bigg\{\nabla_a\left[(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\mu}\left[\partial_b X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right] \\ &+\nabla_b\left[(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\nu}\left[\partial_a X^{\mu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right]\Bigg\} \\ = &-\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}\Bigg\{(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\mu}\nabla_a\left[\partial_b X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \\ &+(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\nu}\nabla_b\left[\partial_a X^{\mu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Bigg\}\end{aligned} \quad [3.193]$$

Let us now focus on

$$\begin{aligned}\mathcal{I}_a &= (g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\mu}\nabla_a\left[\partial_b X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \\ &= (g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\mu}\left\{\left[\nabla_a\partial_b X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+\left[\partial_b X^{\nu}ik_{\lambda}\nabla_a X^{\lambda}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right\}\end{aligned} \quad [3.194]$$

We replace ∂_b by ∇_b in the first term and ∇_a by ∂_a in the second term. This is allowed as they both act on worldsheet scalars.

$$\mathcal{I}_a = (g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^{\mu}\left\{\left[\nabla_a\nabla_b X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+\left[\partial_b X^{\nu}ik_{\lambda}\partial_a X^{\lambda}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right\} \quad [3.195]$$

Because $\nabla_a\nabla_b = \nabla_b\nabla_a$ when acting on scalars[3] the ϵ^{ab} part vanishes with the first term. Thus, also changing dummy variables for the second term,

$$\begin{aligned}\mathcal{I}_a &= s_{\mu\nu}k^{\mu}\left[\nabla^2 X^{\nu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+(g^{ab}s_{\mu\lambda}+i\epsilon^{ab}a_{\mu\lambda})ik^{\mu}k^{\lambda}\left[\partial_b X^{\nu}\partial_a X^{\mu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \\ &= -\frac{i\alpha'}{6}Rs_{\mu\nu}k^{\mu}k^{\nu}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+(g^{ab}s_{\mu\lambda}+i\epsilon^{ab}a_{\mu\lambda})ik_{\nu}k^{\lambda}\left[\partial_b X^{\nu}\partial_a X^{\mu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\end{aligned} \quad [3.196]$$

Similarly

$$\mathcal{I}_a = -\frac{i\alpha'}{6}Rs_{\mu\nu}k^{\mu}k^{\nu}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+(g^{ab}s_{\lambda\nu}+i\epsilon^{ab}a_{\lambda\nu})ik_{\mu}k^{\lambda}\left[\partial_b X^{\nu}\partial_a X^{\mu}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} \quad [3.197]$$

3

$$\nabla_a\nabla_b S-\nabla_b\nabla_a S = \partial_a\partial_b S-\Gamma^c_{ab}\partial_c S-\partial_b\partial_a S+\Gamma^c_{ba}\partial_c S = 0$$

as $\Gamma^c_{ab} = \Gamma^c_{ba}$.

and therefore

$$\mathcal{J}_{2b} = \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \Bigg\{ \alpha' R \left(-\frac{1}{3} s_{\mu\nu} k^\mu k^\nu \right) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} + (g^{ab} + i\epsilon^{ab}) \left(s_{\mu\lambda} k_\nu k^\lambda + s_{\lambda\nu} k_\mu k^\lambda \right) \left[\partial_b X^\nu \partial_a X^\mu e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \Bigg\} \quad [3.198]$$

Let us now consider the case where both functional derivatives act on the $\partial_a X^\mu \partial_b X^\mu$. This gives

$$\mathcal{J}_{2c} = \frac{g_c}{2\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \delta_{\mathrm{W}} \Delta(\sigma', \sigma'') \times \left[\delta^\mu_\lambda \partial_a \delta^2(\sigma' - \sigma) \eta^{\lambda\nu} \partial_b \delta^2(\sigma'' - \sigma) + \delta^\nu_\lambda \partial_b \delta^2(\sigma' - \sigma) \eta^{\lambda\mu} \partial_a \delta^2(\sigma'' - \sigma) \right] \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \quad [3.199]$$

We interchange integration variables σ' and σ'' in the second term between brackets and use $\Delta(\sigma', \sigma'') = \Delta(\sigma'', \sigma')$

$$\mathcal{J}_{2c} = \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \delta_{\mathrm{W}} \Delta(\sigma', \sigma'') \times \eta^{\mu\nu} \partial_a \delta^2(\sigma' - \sigma) \partial_b \delta^2(\sigma'' - \sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \quad [3.200]$$

We use $a_{\mu\nu}\eta^{\mu\nu} = 0$ and the chain rule for derivatives of the delta function, then perform our first partial integration

$$\begin{aligned} \mathcal{J}_{2c} &= \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} g^{ab} s^\mu_\mu \delta_{\mathrm{W}} \Delta(\sigma', \sigma'') \partial'_a \delta^2(\sigma' - \sigma) \partial''_b \delta^2(\sigma'' - \sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\ &= -\frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} g^{ab} s^\mu_\mu \partial''_b \delta_{\mathrm{W}} \Delta(\sigma', \sigma'') \partial'_a \delta^2(\sigma' - \sigma) \delta^2(\sigma'' - \sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\ &= -\frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} g^{ab} s^\mu_\mu \partial_b \delta_{\mathrm{W}} \Delta(\sigma', \sigma) \partial'_a \delta^2(\sigma' - \sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \end{aligned} \quad [3.201]$$

We follow with the second partial integration

$$\begin{aligned} \mathcal{J}_{2c} &= \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} g^{ab} s^\mu_\mu \partial'_a \partial_b \delta_{\mathrm{W}} \Delta(\sigma', \sigma) \delta^2(\sigma' - \sigma) \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \\ &= \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} g^{ab} s^\mu_\mu \partial'_a \partial_b \delta_{\mathrm{W}} \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} \left[e^{ik\cdot X(\sigma)} \right]_{\mathrm{r}} \end{aligned} \quad [3.202]$$

We now use (3.6.15b) and then partially integrate two more times

$$\begin{aligned}\mathcal{J}_{2c} &= \frac{g_c}{\alpha'}\int d^2\sigma\sqrt{g}g^{ab}s^\mu_\mu\frac{1}{6}\alpha'\nabla_a\partial_b\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r} = -\frac{g_c}{6}\int d^2\sigma\partial_b\delta\omega\nabla_a\Big\{\sqrt{g}g^{ab}s^\mu_\mu\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r}\Big\}\\ &= -\frac{g_c}{6}\int d^2\sigma\partial_b\delta\omega\sqrt{g}g^{ab}s^\mu_\mu\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r} = +\frac{g_c}{6}\int d^2\sigma\delta\omega\partial_b\big\{\sqrt{g}g^{ab}s^\mu_\mu\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r}\big\}\\ &= +\frac{g_c}{6}\int d^2\sigma\delta\omega\sqrt{g}\nabla_b\big\{g^{ab}s^\mu_\mu\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r}\big\}\\ &= +\frac{g_c}{6}\int d^2\sigma\delta\omega\sqrt{g}g^{ab}s^\mu_\mu\nabla_b\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r}\end{aligned} \qquad [3.203]$$

use [3.179] and the fact that $g^{ab}\nabla_b\nabla_a\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r} = \nabla^a\partial_a\left[e^{ik\cdot X}\right]_\mathrm{r}$, remember scalars!

$$\begin{aligned}\mathcal{J}_{2c} &= \frac{g_c}{6}\int d^2\sigma\delta\omega\sqrt{g}s^\lambda_\lambda\Big\{\frac{\alpha'}{6}k^2R\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r} - k_\mu k_\nu g^{ab}\left[\partial_aX^\mu\partial_bX^\nu e^{ik\cdot X}\right]_\mathrm{r}\Big\}\\ &= \frac{g_c}{2}\int d^2\sigma\delta\omega\sqrt{g}\Big\{\alpha'R\left(\frac{1}{18}k^2s^\lambda_\lambda\right)\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r} + g^{ab}\left(-\frac{1}{3}k_\mu k_\nu s^\lambda_\lambda\right)\left[\partial_aX^\mu\partial_bX^\nu e^{ik\cdot X}\right]_\mathrm{r}\Big\}\end{aligned} \qquad [3.204]$$

We can now bring all the contributions together. They are of the form

$$\delta_\mathrm{W}V_1 = \frac{g_c}{2}\int d^2\sigma\,\sqrt{g}\,\delta\omega\Big\{(g^{ab}\mathbb{s}_{\mu\mu} + i\epsilon^{ab}\mathbb{a}_{\mu\nu})\left[\partial_aX^\mu\partial_aX^\mu e^{ik\cdot X(\sigma)}\right]_\mathrm{r} + \alpha'R\,\mathbb{f}\left[e^{ik\cdot X(\sigma)}\right]_\mathrm{r}\Big\} \qquad [3.205]$$

The contributions form the different parts to $\mathbb{s}$, $\mathbb{a}$ and $\mathbb{f}$ are summarised in the table below.

	$\mathbb{s}_{\mu\nu}$	$\mathbb{a}_{\mu\nu}$	$\mathbb{f}$
$\mathcal{J}_1$	$4k_\mu k_\nu\phi$	–	$-\frac{2}{3}k^2\phi$
$\mathcal{J}_{2a}$	$-k^2s_{\mu\nu}$	$-k^2a_{\mu\nu}$	–
$\mathcal{J}_{2b}$	$k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\nu\lambda}$	$k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda}$	$-\frac{1}{3}k^\mu k^\nu s_{\mu\nu}$
$\mathcal{J}_{2c}$	$-\frac{1}{3}s^\lambda_\lambda k_\mu k_\nu$	–	$\frac{1}{18}k^2s^\lambda_\lambda$
$\mathcal{J}_3$	–	–	$-k^2\phi$

Table 3.2: Weyl transformation of the massless vertex operator; contributions to [3.205]

Thus

$$\begin{aligned}\mathbb{s}_{\mu\nu} &= -k^2s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} - \frac{1}{3}s^\lambda_\lambda k_\mu k_\nu + 4k_\mu k_\nu\phi\\ \mathbb{a}_{\mu\nu} &= -k^2a_{\mu\nu} + k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda}\\ &= -k^2a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda}\\ \mathbb{f} &= -\frac{5}{3}k^2\phi - \frac{1}{3}k^\mu k^\nu s_{\mu\nu} + \frac{1}{18}k^2s^\lambda_\lambda\end{aligned} \qquad [3.206]$$

We need to compare this with Polchinski's (3.6.17), with $\gamma = -2/3$, i.e.

$$\begin{aligned} S_{\mu\nu} &= -k^2 s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} - \frac{1}{3} s^\lambda_\lambda k_\mu k_\nu + 4 k_\mu k_\nu \phi \\ A_{\mu\nu} &= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda} \\ F &= -\frac{5}{3} k^2 \phi - \frac{1}{3} k^\mu k^\nu s_{\mu\nu} + \frac{1}{18} k^2 s^\lambda_\lambda \end{aligned} \qquad [3.207]$$

Lo and behold! These are identical!

3.41 p 105: Eq (3.6.18) Linking $\nabla^2 X^\mu$ with $k^\mu R$

This is actually a highly non-trivial equation. I haven't found a simple derivation of it, but I will show in a very roundabout way that it is correct. For this I use a derivation provided by "Wakabaloola" on the Physics Stack Exchange, for which I am extremely grateful. As a return gesture, the least I can do is to refer to one of his papers

D. Luest and D. Skliros, Handle Operators in String Theory, arXiv:1912.0155 [hep-th]

which we will denote by **[LS]** here. The building blocks of most of what follows comes from their section 2. Any errors in my explanation are, of course, solely due to me. It is also worth mentioning that the basis of this work is Polchinski's earlier work published in NBP307 (1988) 61-92.

Because of the length of the calculation, it is useful to summarise what we are going to do. Our first goal is to find transition functions of conformal and Weyl transformations. We will first introduce the concept of holomorphic normal coordinates. These are coordinates in a patch around a point σ_1 of the manifold that are chosen to be "as flat as possible", implying that the connections at that point vanish. This suggests that we introduce the concept of Weyl normal ordering, which is similar to Polchinki's conformal normal ordering but where both points z_1 and z_2 are taken at the same base point. We will then derive the transition function for a change of base points from σ_1 to σ_1'. This will be achieved in [3.236]. We will then proceed to use this result to work out how a change of coordinates, this time keeping the base point fixed, impacts a local, Weyl normal ordered, operator. I.e we will derive and expression for the derivative of such an operator. This result is achieved in[3.247] and shows that we cannot just bring in the derivative into the normal ordering as there is an additional contribution that depends on the Ricci scalar and its covariant derivatives. As an illustration of this we will show that the formula [3.260] is valid

$$\nabla_a \circledcirc \nabla^a X^\mu e^{ik\cdot X}(\sigma_1) \circledcirc_z = \circledcirc \nabla_a X^\mu \nabla^a e^{ik\cdot X}(\sigma_1) \circledcirc_z - \frac{i\alpha' k^\mu}{4} R(\sigma_1) \circledcirc e^{ik\cdot X}(\sigma_1) \circledcirc_z \qquad [3.208]$$

where the $\circledcirc \cdots \circledcirc_z$ denote Weyl normal ordering. This shows how the Ricci scalar can appear in expressions with normal ordering and derivatives.

Next we turn to the case of interest, which is the case of Weyl transformations and derive the transition function for such a transformation. This is given in [3.271]. It is then an easy step to determine how a local operator transforms under a Weyl transformation. This is given in [3.274]. We then derive expression for the propagator and its derivatives in this new scheme. This allows us to focus on requiring Weyl invariance of the vertex operators, giving us a set of equations that lead to the same mass-shell conditions (3.6.22). This then establishes the validity for (3.6.18).

The appendix to this chapter contains a brief introduction to complex structures where we will also derive some results needed in this section. There is a lot of work to be done, so let's get cracking!

Holomorphic Normal Coordinates and Weyl Normal Ordering

We have the usual complex coordinates z and $\bar{z}$ on the Riemann surface M under consideration. But let us be very precise. We have a chart on our manifold in which we define holomorphic and anti-holomorphic coordinates, z and $\bar{z}$. We define the complex coordinates in such a way that at a given point σ_1 these complex coordinates vanish. Restricting ourselves to the holomorphic side, as the anti-holomorphic is similar, we denote this set of coordinates by $z_{\sigma_1}(\sigma)$, meaning the holomorphic coordinates based on a a chart around the point σ_1 where these coordinates vanish, i.e. where

$$z_{\sigma_1}(\sigma_1) = 0 \qquad [3.209]$$

We now define holomorphic normal coordinates as follows. In that local patch around σ_1 go to the conformal gauge $g_{ab} = e^{2\omega}\delta_{ab} = \rho(\sigma)\delta_{ab}$.[4] We know that in the conformal gauge the metric and the Ricci scalar are given by

$$\begin{aligned} ds^2 &= \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \\ R &= -4\rho(z_{\sigma_1}, \bar{z}_{\sigma_1})^{-1} \partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \end{aligned} \qquad [3.210]$$

Here we have used the fact that $e^{2\omega} = \rho$ and that $\partial_a \partial_a = 4\partial_{z\sigma_1}\partial_{\bar{z}_{\sigma_1}}$. Note that we are explicitly mentioning that these are holomorphic coordinates around the point σ_1, because soon we will turn our attention to the question what happens in a point nearby σ_1. We now chose the $z_{\sigma_1}(\sigma)$ in such a way that at $\sigma = \sigma_1$, i.e. where the patch is based and only at that point, the metric is as flat as possible. i.e.

$$\begin{aligned} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1})\Big|_{\sigma=\sigma_1} &= 1 \\ \partial^n_{z_{\sigma_1}} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1})\Big|_{\sigma=\sigma_1} &= 0 \qquad \text{for } n \geq 1 \end{aligned} \qquad [3.211]$$

[4] That it is always possible to chose such a conformal gauge is explained in the appendix to this chapter, see in particular, the discussion around [3.444].

so that all holomorphic derivatives of ρ vanish at σ_1. Mixed derivatives (holomorphic plus anti-holomorphic) need nor vanish as we cannot generally make the Ricci scalar zero by a choice of coordinates. Holomorphic coordinates satisfying these conditions are called holomorphic normal coordinates,[5] a term that has been coined by the authors of **[LS]**, which I find very appropriate and hope will stick.

We now define Weyl Normal Ordering by (2.2.7), using $d^2z = 2d^2\sigma$

$$\circledcirc\mathcal{F}\circledcirc = \exp\left(\frac{1}{8}\int d^2z_1\, d^2z_2\, \Delta(z_1,z_2)\frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\mathcal{F}\right)\Bigg|_{\substack{z_1=z_{\sigma_1}(\sigma_1)\\ z_2=z_{\sigma_1}(\sigma_2)}} \qquad [3.212]$$

I.e. for an operator based at σ_1 we take the subtractions at the points $z_{\sigma_1}(\sigma_1)$ and $z_{\sigma_1}(\sigma_2)$.

Note that the whole toolkit of CFT can be applied, e.g. the mode expansion of the field X^μ is given by

$$i\partial_{z_{\sigma_1}}X(\sigma) = \sum_{n=-\infty}^{\infty}\frac{\alpha_n^{(z_{\sigma_1})}(\sigma_1)}{z_{\sigma_1}(\sigma)^{n+1}} \qquad [3.213]$$

This is heavy on notation, but it emphasises that this is taken in the patch around σ_1 and that the modes depend on that patch and the base point. The same, of course applies for the expansion of the energy-momentum tensor in the Virasoro generators.

The Transition Function of a Holomorphic Coordinate

Let us now consider a point nearby σ_1, say $\sigma_1' = \sigma_1 + \delta\sigma_1$ and construct a set of holomorphic coordinates $z_{\sigma_1'}$ around that point. Recall again that by construction $z_{\sigma_1'}(\sigma_1') = 0$. If σ_1 and σ_1' are close enough that the two holomorphic charts based on them overlap, then the transition function between the two is, by definition of a complex manifold, holomorphic:

$$\begin{aligned} z_{\sigma_1'}(\sigma) &= f_{\sigma_1'\sigma_1}\big(z_{\sigma_1}(\sigma)\big) \\ &= z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma) \end{aligned} \qquad [3.214]$$

This defines $\delta z_{\sigma_1}(\sigma)$, which is a holomorphic function as well, since $f_{\sigma_1'\sigma_1}$ is holomorphic. Our first task will be to determine an equation for this transition function. This is not a straightforward thing to do and we will only achieve this in [3.236].

[5] The fact that we can chose such a set if coordinates that are as flat as possible is not trivial. For a discussion on how to go from a general coordinate system to a holomorphic normal coordinates see the appendix to this chapter and in particular [3.468] which gives an explicit formula for the required transition function from a set of coordinates in the conformal gauge to a set of holomorphic normal coordinates.

The above holomorphic transformation induces a change in metric at a generic point σ that lives in the overlap of the two patches of the form

$$\delta \ln \rho(\sigma) = \nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \qquad [3.215]$$

This transformation is derived in the appendix, see [3.452]. Let us now take the $(n-1)^{\text{th}}$ derivative of this, with $n \geq 2$ and evaluate this at the point $\sigma = \sigma_1$. We use the fact that in the conformal gauge the only non-vanishing connections are

$$\begin{aligned} \Gamma^z_{zz}(\sigma) &= \frac{\partial_{z_{\sigma_1}} \rho(\sigma)}{\rho(\sigma)} = \partial_{z_{\sigma_1}} \ln \rho(\sigma) \\ \Gamma^{\bar{z}}_{\bar{z}\bar{z}}(\sigma) &= \frac{\partial_{\bar{z}_{\sigma_1}} \rho(\sigma)}{\rho(\sigma)} = \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \end{aligned} \qquad [3.216]$$

We find

$$\begin{aligned} \partial^{n-1}_{z_{\sigma_1}} \delta \ln \rho(\sigma)\Big|_{\sigma=\sigma_1} &= \partial^{n-1}_{z_{\sigma_1}} \left(\nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \right) \Big|_{\sigma=\sigma_1} \\ &= \partial^{n-1}_{z_{\sigma_1}} \Big[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}} \ln \rho(\sigma) \delta z_{\sigma_1}(\sigma) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \\ &\qquad + \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \delta \bar{z}_{\sigma_1}(\sigma) \Big] \Big|_{\sigma=\sigma_1} \\ &= \Big[\partial^n_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \partial^n_{z_{\sigma_1}} \ln \rho(\sigma) \delta z_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}} \ln \rho(\sigma) \partial^{n-1}_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) \\ + \partial^{n-1}_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) &+ \partial^{n-1}_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \delta \bar{z}_{\sigma_1}(\sigma) + \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \partial^{n-1}_{z_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \Big] \Big|_{\sigma=\sigma_1} \\ &= \Big[\partial^n_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \left(\partial^{n-1}_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \right) \delta \bar{z}_{\sigma_1}(\sigma) \Big] \Big|_{\sigma=\sigma_1} \end{aligned} \qquad [3.217]$$

In the last line we have used the fact that we are working in holomorphic normal coordinates such that $\partial^n_{z_{\sigma_1}} \ln \rho(z_{\sigma_1}, \bar{z}_{\sigma_1})\Big|_{\sigma=\sigma_1} = 0$ for $n \geq 1$ and that $\delta \bar{z}_{\sigma_1}$ is holomorphic so that $\partial^n_{z_{\sigma_1}} \delta \bar{z}_{\sigma_1} = 0$ as well. Note that, as we have mentioned earlier, the mixed derivative term $\partial^{n-1}_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma)$ is not necessarily zero, on the contrary [3.210] links it to the Ricci scalar R. Indeed, note that

$$\partial^{n-2}_{z_{\sigma_1}} \left[\rho^{-1} \partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma_1) \right] \Big|_{\sigma=\sigma_1} = \partial^{n-1}_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \Big|_{\sigma=\sigma_1} \qquad [3.218]$$

This is easily seen that noting that the derivatives $\partial^{n-2}_{z_{\sigma_1}}$ on the ρ^{-1} are zero when taken in σ_1 by the normal holomorphicity condition. We are then just left with a ρ^{-1} which at the point σ_1 is equal to one by the same condition. We can thus write

$$\begin{aligned}\partial^{n-1}_{z_{\sigma_1}}\delta\ln\rho(\sigma)\Big|_{\sigma=\sigma_1} &= \partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1) - \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\partial^{n-2}_{z_{\sigma_1}}\big[-4\rho^{-1}\partial_{z_{\sigma_1}}\partial_{\bar{z}_{\sigma_1}}\ln\rho(\sigma_1)\big]\\ &= \partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1) - \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\partial^{n-2}_{z_{\sigma_1}}R(\sigma_1)\\ &= \partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1) - \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\nabla^{n-2}_{z_{\sigma_1}}R(\sigma_1)\end{aligned} \qquad [3.219]$$

We have here, immediately written the fact that all these quantities are taken at the point σ_1 and in the last line we can replace the ordinary derivatives by the covariant derivatives because as the only non-zero connection $\Gamma^z_{zz}(\sigma) = \partial_{z_{\sigma_1}}\ln\rho(\sigma)$ vanishes at σ_1.

Requiring that both at σ_1 and at σ_1' we have holomorphic normal coordinates implies that $\partial^{n-1}_{z_{\sigma_1}}\ln\rho(\sigma_1)$ vanishes and so [3.219] implies that

$$\partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1) = \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\nabla^{n-2}_{z_{\sigma_1}}R(\sigma_1) \qquad [3.220]$$

Multiply both sides by $\big(z_{\sigma_1}(\sigma)\big)^n/n!$ and sum from $n=2$ to infinity

$$\sum_{n=2}^{\infty}\frac{1}{n!}\big(z_{\sigma_1}(\sigma)\big)^n\partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1) = \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\sum_{n=2}^{\infty}\frac{1}{n!}\big(z_{\sigma_1}(\sigma)\big)^n\nabla^{n-2}_{z_{\sigma_1}}R(\sigma_1) \qquad [3.221]$$

We can rewrite the LHS as

$$LHS = \sum_{n=0}^{\infty}\frac{1}{n!}\big(\partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1)\big)\big(z_{\sigma_1}(\sigma)\big)^n - \delta z_{\sigma_1}(\sigma_1) - \big(\partial_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1)\big)z_{\sigma_1}(\sigma) \qquad [3.222]$$

The sum is just a Taylor expansion. To recognise it, recall that

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\partial^n f(a)(x-a)^n \qquad [3.223]$$

and Taylor expand $\delta z_{\sigma_1}(\sigma)$ around $z_{\sigma_1}(\sigma_1)$. We thus have $x = z_{\sigma_1}(\sigma)$ and $a = z_{\sigma_1}(\sigma_1)$, with $f \equiv \delta$. It follows that $x - a = z_{\sigma_1}(\sigma) - z_{\sigma_1}(\sigma_1)$. Now recall from [3.209] that the base point is chosen so that $z_{\sigma_1}(\sigma_1) = 0$, albeit it that $\delta z_{\sigma_1}(\sigma_1) = 0$ and its derivatives need not be zero. We thus have $x - a = x = z_{\sigma_1}(\sigma)$ and so

$$\delta z_{\sigma_1}(\sigma) = \sum_{n=0}^{\infty}\frac{1}{n!}\big(\partial^{n}_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1)\big)\big(z_{\sigma_1}(\sigma)\big)^n \qquad [3.224]$$

We therefore have

$$\delta z_{\sigma_1}(\sigma) - \delta z_{\sigma_1}(\sigma_1) - \big(\partial_{z_{\sigma_1}}\delta z_{\sigma_1}(\sigma_1)\big)z_{\sigma_1}(\sigma) = \frac{1}{4}\delta\bar{z}_{\sigma_1}(\sigma_1)\sum_{n=2}^{\infty}\frac{1}{n!}\big(z_{\sigma_1}(\sigma)\big)^n\nabla^{n-2}_{z_{\sigma_1}}R(\sigma_1) \qquad [3.225]$$

or, equivalently

$$\delta z_{\sigma_1}(\sigma) = \delta z_{\sigma_1}(\sigma_1) + \big(\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\big) z_{\sigma_1}(\sigma) + \frac{1}{4}\delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla^{n-1}_{z_{\sigma_1}} R(\sigma_1)}{(n+1)!} \big(z_{\sigma_1}(\sigma)\big)^{n+1} \tag{3.226}$$

where we have changed the summation index $n \to n+1$. Let us see what we have achieved so far. We have found an expression for $\delta z_{\sigma_1}(\sigma)$ as a power series in $z_{\sigma_1}(\sigma)$ with coefficients that depend only on the base point σ_1. In detail

$$\delta z_{\sigma_1}(\sigma) = \sum_{n=0}^{\infty} a_n(\sigma_1)\big(z_{\sigma_1}(\sigma)\big)^n \tag{3.227}$$

with

$$\begin{aligned} a_0 &= \delta z_{\sigma_1}(\sigma_1) \\ a_1 &= \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\ a_n &= \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) \nabla^{n-2}_{z_{\sigma_1}} R(\sigma_1)}{4(n+1)!} \qquad \text{for } n \geq 2 \end{aligned} \tag{3.228}$$

We know add $z_{\sigma_1}(\sigma)$ to both sides. On the left hand side we get $z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma)$, which by [3.214] is exactly $z_{\sigma_1'}(\sigma)$. On the RHS this means that the coefficient a_1 changes to $1 + \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)$. Now, to first order in the change δ we can rewrite this as $e^{partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)}$. We can also multiply a_0 and a_n for $n \geq 2$ with this exponential as these coefficients are already of the order $\delta z_{\sigma_1}(\sigma_1)$ and so the difference would just be a second order correction. The upshot of this is that we can write

$$z_{\sigma_1'}(\sigma) = e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \sum_{n=0}^{\infty} \tilde{a}_n(\sigma_1)\big(z_{\sigma_1}(\sigma)\big)^n \tag{3.229}$$

with

$$\begin{aligned} \tilde{a}_0 &= \delta z_{\sigma_1}(\sigma_1) \\ \tilde{a}_1 &= 1 \\ \tilde{a}_n &= \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) \nabla^{n-2}_{z_{\sigma_1}} R(\sigma_1)}{4(n+1)!} \qquad \text{for } n \geq 2 \end{aligned} \tag{3.230}$$

or explicitly

$$z_{\sigma_1'}(\sigma) = e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \left[\delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4}\delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla^{n-1}_{z_{\sigma_1}} R(\sigma_1)}{(n+1)!} \big(z_{\sigma_1}(\sigma)\big)^{n+1}\right] \tag{3.231}$$

Let us now go back to [3.264] and evaluate it at σ_1. Since $\rho(\sigma_1) = 1$ by the definition of holomorphic normal coordinates we have $\ln \rho(\sigma_1) = 0$. We furthermore have that all connections vanish at σ_1, see [3.216]. Therefore

$$\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma_1) = 0 \qquad [3.232]$$

For a general complex number $w = x + iy$ we have $w + \bar{w} = 2x = 2\mathrm{Re}\, w$, thus the above equation says that

$$\mathrm{Re}\, \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\right] = 0 \qquad [3.233]$$

This means that we can write

$$\begin{aligned} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) &= \mathrm{Re}\, \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\right] + i\, \mathrm{Im}\, \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\right] \\ &= i\, \mathrm{Im}\, \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\right] \end{aligned} \qquad [3.234]$$

and [3.235] becomes

$$z_{\sigma_1'}(\sigma) = e^{i\, \mathrm{Im}\left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)\right]} \left[\delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} \big(z_{\sigma_1}(\sigma)\big)^{n+1}\right] \qquad [3.235]$$

The exponential pre-factor is now just a phase factor. We will shove this phase factor under the rug, together with the one that determines the transition functions from coordinates in the conformal gauge to holomorphic normal coordinates, see [3.467] in the appendix to this chapter or for more details **[LS]**.

Our final result is then

$$z_{\sigma_1'}(\sigma) = \delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} \big(z_{\sigma_1}(\sigma)\big)^{n+1} \qquad [3.236]$$

This is the holomorphic transition function between two charts with nearby base points σ_1 and $\sigma_1{}'$. It is holomorphic in σ, any any point in the overlap of the two charts, but due to the appearance of $\delta \bar{z}_{\sigma_1}(\sigma_1)$ it is not holomorphic in the base points σ_1 and σ_1'.

The Derivative of a Normal Ordered Operator

Using the result [3.236] and the chain rule we can now work out the derivative w.r.t. the base point σ_1 for a given point σ

$$\frac{\partial}{\partial z_{\sigma_1}(\sigma_1)}\Big|_{\sigma} = \frac{\partial z_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)}\Big|_{\sigma} \frac{\partial}{\partial z_{\sigma_1}(\sigma)} + \frac{\partial \bar{z}_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)}\Big|_{\sigma} \frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \qquad [3.237]$$

What does this mean? $\partial z_{\sigma_1}(\sigma)/\partial z_{\sigma_1}(\sigma_1)$ is just the how $z_{\sigma_1}(\sigma)$ varies when we vary the the base point. We can read this off from [3.236]. It is just

$$\begin{aligned}\frac{\partial z_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} &= \lim_{\delta z_{\sigma_1}(\sigma_1)\to 0} \frac{z_{\sigma_1'}(\sigma) - z_{\sigma_1}(\sigma)}{\delta z_{\sigma_1}(\sigma_1)} \\ &= \lim_{\delta z_{\sigma_1}(\sigma_1)\to 0} \frac{\delta z_{\sigma_1}(\sigma_1) + \frac{1}{4}\delta \bar{z}_{\sigma_1}(\sigma_1)\sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\big(z_{\sigma_1}(\sigma)\big)^{n+1}}{\delta z_{\sigma_1}(\sigma_1)} \\ &= 1\end{aligned} \tag{3.238}$$

But we have also a non-vanishing term for the complex conjugate of [3.236]

$$\begin{aligned}\frac{\partial \bar{z}_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} &= \lim_{\delta z_{\sigma_1}(\sigma_1)\to 0} \frac{\bar{z}_{\sigma_1'}(\sigma) - \bar{z}_{\sigma_1}(\sigma)}{\delta z_{\sigma_1}(\sigma_1)} \\ &= \lim_{\delta z_{\sigma_1}(\sigma_1)\to 0} \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) + \frac{1}{4}\delta z_{\sigma_1}(\sigma_1)\sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\big(\bar{z}_{\sigma_1}(\sigma)\big)^{n+1}}{\delta z_{\sigma_1}(\sigma_1)} \\ &= +\frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\big(\bar{z}_{\sigma_1}(\sigma)\big)^{n+1}\end{aligned} \tag{3.239}$$

and thus

$$\frac{\partial}{\partial z_{\sigma_1}(\sigma_1)}\Big|_{\sigma} = \frac{\partial}{\partial z_{\sigma_1}(\sigma)} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\big(\bar{z}_{\sigma_1}(\sigma)\big)^{n+1}\frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \tag{3.240}$$

In this expression we recognise the Virasoro generatos

$$L_n^{(z_{\sigma_1})} = -\big(z_{\sigma_1}(\sigma)\big)^{n+1}\frac{\partial}{\partial z_{\sigma_1}(\sigma)} \qquad \text{and} \qquad \tilde{L}_n^{(z_{\sigma_1})} = -\big(\bar{z}_{\sigma_1}(\sigma)\big)^{n+1}\frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \tag{3.241}$$

We have explicitly added the superscript $^{(z_{\sigma_1})}$ to remind ourselves that these are the Virasoro generators for the patch around the base point σ_1. We can thus write

$$\frac{\partial}{\partial z_{\sigma_1}(\sigma_1)}\Big|_{\sigma} = -L_{-1}^{(z_{\sigma_1})} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n^{(\bar{z}_{\sigma_1})} \tag{3.242}$$

This is an expression to the derivative w.r.t. to the variation base point keeping the point σ fixed, i.e. w.r.t. $\delta z_{\sigma_1}(\sigma_1) = \big(z_{\sigma_1'}(\sigma) - z_{\sigma_1}(\sigma)\big)\big|_{\sigma=\sigma_1}$. This is a so-called passive variation where we move the frame of reference σ_1 to σ_1' and keep the coordinate σ fixed. We can obtain the active variation of changing the coordinate but keeping the base point fixed simply by introducing a minus sign. We thus have that the derivative w.r.t. coordinate for a fixed base point is

$$\partial_{z_{\sigma_1}} = L_{-1}^{(z_{\sigma_1})} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n^{(\bar{z}_{\sigma_1})} \qquad [3.243]$$

From here on we are working with a fixed base-point σ_1 and so it is superfluous to keep on reminding ourselves that the coordinates z are taking with base point σ_1. The above equation thus becomes

$$\partial_z = L_{-1} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n \qquad [3.244]$$

Let us now see what the corresponding derivative ∂_z does when acting on a local operator that is Weyl normal ordered and is inserted at the base point σ_1 with holomorphic normal coordinates z, viz an operator ${}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z}$. We have

$$\partial_z {}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} = \left[L_{-1} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n\right]{}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} \qquad [3.245]$$

We can evaluate $L_{-1}{}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z}$ via the OPE of the energy-momentum tensor with the operator ${}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z}$

$$L_{-1}{}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} = \oint \frac{dz}{2\pi}T(z)\mathcal{A}(w) = {}^{\odot}_{\odot}\partial_z\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} \qquad [3.246]$$

and so we just pick up the residue of the simple pole of the simple pole of the OPE which is, as we have seen before by translation invariance ${}^{\odot}_{\odot}\partial_z\mathcal{A}(\sigma_1){}^{\odot}_{\odot z}$.

We have now achieved our end result for this part. We know how to take the derivative of a normal ordered local operator:

$$\partial_z {}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} = {}^{\odot}_{\odot}\partial_z\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n{}^{\odot}_{\odot}\mathcal{A}(\sigma_1){}^{\odot}_{\odot z} \qquad [3.247]$$

The key point here, and it is worth emphasising it, is that one cannot just take a derivative in a normal ordered product. There is an additional contribution that depends on the anti-holomorphic part of the theory. Note also that these results are very general. They are valid for an arbitrary background and CFT with total central charge zero. These results are also valid off-shell.

An Explicit Example

After all this hard work, let us consider a specific example. We wish to compute

$$\mathfrak{T}^\mu(\sigma_1) = \nabla_a {}^{\circ}_{\circ} \nabla^a X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ}{}_z \tag{3.248}$$

As at the end of the last section, we have not written all the necessary indices, but we are working with holomorphic normal coordinates at base point σ_1. This means that covariant derivatives $\nabla_z = \nabla_{z_{\sigma_1}}$ are equal to ordinary derivatives and and $\rho(\sigma_1) = 1$ with all its derivatives at σ_1 being zero. Replacing the covariant derivative by ordinary derivatives, going to complex coordinates, and using our formula [3.247] for the derivative of a normal ordered operator, we find

$$\begin{aligned}\mathfrak{T}^\mu(\sigma_1) =& 4\partial_z {}^{\circ}_{\circ} \partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ}{}_z \\ =& 4 {}^{\circ}_{\circ} \partial_z \left(\partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1)\right) {}^{\circ}_{\circ}{}_z + \frac{1}{4}\sum_{n=1}^{\infty} \frac{\nabla_{\bar z}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n 4 {}^{\circ}_{\circ} \partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ}{}_z\end{aligned} \tag{3.249}$$

We now use the OPE to compute

$$\tilde{L}_n {}^{\circ}_{\circ} \partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ}{}_z = \oint_{\mathcal{C}_z} \frac{dw}{2\pi i} (w-z)^{n+1} \tilde{T}(\bar w) {}^{\circ}_{\circ} \partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ}{}_z \tag{3.250}$$

Let us consider first what we need before we start blindly calculating. Because $n \geq 1$ we need a pole in the OPE $\tilde{T}(\bar w)\partial_{\bar z} X^\mu e^{ik\cdot X}(z)$ of order three or higher to have a non-zero contribution. Here we have not written the normal ordering symbols as these are understood in OPEs. We will also focus further on the holomorphic twin of this expressions, and put conjugates at the end. We thus need the third and higher order poles of

$$T(w)\partial X^\mu e^{ik\cdot X}(z) = -\frac{1}{\alpha'}\partial X^\nu \partial X_\nu(w) \partial X^\mu e^{ik\cdot X}(z) \tag{3.251}$$

The highest order pole we get is when we contract one of the $\partial X^\nu(w)$ with the $\partial X^\mu(z)$ and the other $\partial X^\nu(w)$ with an $X^\sigma(z)$ from the exponential. Using

$$\begin{aligned} X^\mu(w) X^\nu(z) =& -\frac{1}{2}\alpha' \ln|w-z|^2 \\ \partial X^\mu(w) X^\nu(z) =& -\frac{1}{2}\alpha' \frac{1}{w-z} \\ \partial X^\mu(w) \partial X^\nu(z) =& -\frac{1}{2}\alpha' \frac{1}{(w-z)^2} \end{aligned} \tag{3.252}$$

we see that the highest order pole is a third order pole and is given by

$$\begin{aligned} T(w)\partial X^\mu e^{ik\cdot X}(z) =& -\frac{1}{\alpha'}\Big[2\left(-\frac{1}{2}\alpha' \frac{\eta^{\mu\nu}}{(w-z)^2}\right)\left(-\frac{1}{2}\alpha' \frac{ik^\sigma \eta_{\nu\sigma}}{w-z}\right) e^{ik\cdot X}(z)\Big] + \cdots \\ =& -\frac{ik^\mu \alpha'}{2}\frac{1}{(w-z)^3} e^{ik\cdot X}(z) + \cdots \end{aligned} \tag{3.253}$$

Thus

$$
\begin{aligned}
L_n {}^{\circ}_{\circ} \partial_z X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} &= \oint_{\mathcal{C}_z} \frac{dw}{2\pi i}(w-z)^{n+1}\left[-\frac{ik^\mu \alpha'}{2}\frac{1}{(w-z)^3}e^{ik\cdot X}(z)+\cdots\right] \\
&= -\frac{i\alpha' k^\mu}{2}\delta_{n,1}\oint_{\mathcal{C}_z}\frac{dw}{2\pi i(w-z)}e^{ik\cdot X}(z) \\
&= -\frac{i\alpha' k^\mu}{2}\delta_{n,1}e^{ik\cdot X}(z) \qquad [3.254]
\end{aligned}
$$

We see that the result is the same for the conjugate twin and therefore

$$
\begin{aligned}
\mathfrak{T}^\mu(\sigma_1) =& 4 {}^{\circ}_{\circ} \partial_z \left(\partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1)\right) {}^{\circ}_{\circ z} + \sum_{n=1}^{\infty}\frac{\nabla_{\bar z}^{n-1}R(\sigma_1)}{(n+1)!}\left(-\frac{i\alpha' k^\mu}{2}\delta_{n,1}e^{ik\cdot X}(z)\right) \\
=& 4 {}^{\circ}_{\circ} \partial_z \left(\partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1)\right) {}^{\circ}_{\circ z} - \frac{i\alpha' k^\mu}{4}R(\sigma_1) {}^{\circ}_{\circ} e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} \qquad [3.255]
\end{aligned}
$$

In the last line we have once more added the bells and whistles of the normal ordering. Consider now the first term. From [3.246] we have

$$
4 {}^{\circ}_{\circ} \partial_z \left(\partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1)\right) {}^{\circ}_{\circ z} = 4L_{-1} {}^{\circ}_{\circ} \partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} \qquad [3.256]
$$

We now need

$$
L_{-1}\partial_{\bar z} X^\mu e^{ik\cdot X}(z) = \oint_{\mathcal{C}_z}\frac{dw}{2\pi i}(w-z)^{-1+1}T(w)\partial_{\bar z}X^\mu e^{ik\cdot X}(z) \qquad [3.257]
$$

The only non-vanishing contribution comes from the single pole in the OPE $T(w)\partial_{\bar z}X^\mu e^{ik\cdot X}(z)$, which is easily computed as we know that $e^{ik\cdot X}$ is a primary field, and hence so is $\partial_{\bar z}X^\mu e^{ik\cdot X}(z)$ as $T(w)\partial_{\bar z}X^\mu(\bar z)$ has no singular parts in the contour. We can thus write down immediately

$$
L_{-1}\partial_{\bar z} X^\mu e^{ik\cdot X}(z) = \oint_{\mathcal{C}_z}\frac{\partial_{\bar z}X^\mu \partial_z e^{ik\cdot X}(w)}{w-z} = {}^{\circ}_{\circ} \partial_{\bar z} X^\mu \partial_z e^{ik\cdot X}(z) {}^{\circ}_{\circ z} \qquad [3.258]
$$

where we have added the bells and whistles of normal ordering again at the end. This is actually nothing but a complicated and rigorous way to explain that bring the ∂_z through the $\partial_{\bar z}X^\mu$ in a normal ordering, i.e. that $\partial_{\bar z}X^\mu$ is anti-holomorphic.

Recall that we are using holomorphic normal coordinates so that we can replace the ordinary derivatives by covariant ones. Thus

$$
4 {}^{\circ}_{\circ} \partial_z \left(\partial_{\bar z} X^\mu e^{ik\cdot X}(\sigma_1)\right) {}^{\circ}_{\circ z} = 4 {}^{\circ}_{\circ} \partial_{\bar z} X^\mu \partial_z e^{ik\cdot X}(z) {}^{\circ}_{\circ z} = {}^{\circ}_{\circ} \nabla_a X^\mu \nabla^a e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} \qquad [3.259]
$$

We thus find our final result for $\mathfrak{T}^\mu(\sigma_1)$:

$$
\nabla_a {}^{\circ}_{\circ} \nabla^a X^\mu e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} = {}^{\circ}_{\circ} \nabla_a X^\mu \nabla^a e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} + \frac{i\alpha'\gamma k^\mu}{4}R(\sigma_1) {}^{\circ}_{\circ} e^{ik\cdot X}(\sigma_1) {}^{\circ}_{\circ z} \qquad [3.260]
$$

Where we have introduce a parameter γ that is equal to -1 in our case.

WEYL TRANSFORMATIONS IN WEYL NORMAL ORDERING

Let us consider a patch with base point σ_1 holomorphic normal coordinates $z\sigma_1(\sigma)$. We now wish to perform an infinitesimal but general holomorphic change of coordinates

$$z_{\sigma_1}(\sigma) \to w_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \sum_{n=0}^{\infty} \varepsilon_n \big(z_{\sigma_1}(\sigma)\big)^n \qquad [3.261]$$

Note that $\delta z_{\sigma_1}(\sigma)$ has a different meaning than in [3.214]. Here it is a change of coordinates with a fixed base point σ_1, whereas previously it denoted the impact on the coordinate of a change of base point σ_1 to σ_1'.

As we are working in holomorphic normal coordinates, the following relations are valid, see [3.210] and [3.211]

$$ds^2 = \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \qquad \text{with} \qquad \partial^n_{z_{\sigma_1}} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1})\Big|_{\sigma=\sigma_1} = \delta_{n,0} \qquad [3.262]$$

Our aim is to find out how vertex operators transform under Weyl transformations – remember that we have a regularisation scheme, either Pauli-Villars or dimensional regularisation, that is manifestly diffeomorphism invariant and so the survival of the Weyl invariance in the quantum theory is what should concern us – so let us find the coefficient ε_n for a Weyl transformation.

Consider, therefore, the Weyl transformation

$$ds^2 \to d\hat{s}^2 = e^{\delta\phi(\sigma_1)} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \qquad [3.263]$$

In order to satisfy the nitpickers amongst us, let us point out that by $\delta\phi(\sigma_1)$ we actually mean $\delta\phi\big(z_{\sigma_1}(\sigma), \bar{z}_{\sigma_1}(\sigma)\big)$. From [3.264] we have that, using $\ln\rho = \delta\phi$

$$\delta\phi(\sigma) = \nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta\bar{z}_{\sigma_1}(\sigma) \qquad [3.264]$$

Because both z_{σ_1} and w_{σ_1} are chosen to be holomorphic normal coordinates we have $z_{\sigma_1}(\sigma_1) = w_{\sigma_1}(\sigma_1) = 0$ and hence also $\delta z_{\sigma_1}(\sigma_1) = 0$. But recall that the derivatives of $\delta z_{\sigma_1}(\sigma)$ taken at sic are not necessarily zero.

In order to compute $\delta z_{\sigma_1}(\sigma)$ we perform the same trick as before: we take the $(n-1)$th derivative of $\delta z_{\sigma_1}(\sigma)$ evaluated at $\sigma\sigma_1$, multiply it by $\big(z_{\sigma_1}(\sigma)\big)^n/n!$ and sum from $n = 2$ to infinity

$$\sum_{n=2}^{\infty} \frac{1}{n!} \partial^{n-1}_{z_{\sigma_1}} \delta\phi(\sigma)\Bigg|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma)\big)^n = \sum_{n=2}^{\infty} \frac{1}{n!} \partial^{n-1}_{z_{\sigma_1}} \Big(\nabla_{z_{\sigma_1}} \delta z_{\sigma_1} + \nabla_{\bar{z}_{\sigma_1}} \delta\bar{z}_{\sigma_1}\Big)(\sigma)\Bigg|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma)\big)^n \qquad [3.265]$$

We now use [3.216] for the non-vanishing connections and the RHS becomes

$$\begin{aligned}
RHS = &\sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \Big(\partial_{z_{\sigma_1}} \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}} \ln \rho) \delta z_{\sigma_1} + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1} + (\partial_{\bar{z}_{\sigma_1}} \ln \rho) \delta \bar{z}_{\sigma_1} \Big)(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n \\
= &\sum_{n=2}^{\infty} \frac{1}{n!} \Big(\partial_{z_{\sigma_1}}^{n} \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}}^{n} \ln \rho) \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}} \ln \rho) \partial_{z_{\sigma_1}}^{n-1} \delta z_{\sigma_1} \\
&\qquad + (\partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho) \delta \bar{z}_{\sigma_1} + (\partial_{\bar{z}_{\sigma_1}} \ln \rho) \partial_{z_{\sigma_1}}^{n-1} \delta \bar{z}_{\sigma_1} \Big)(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n \\
= &\sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n} \delta z_{\sigma_1}(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n
\end{aligned} \qquad [3.266]$$

Indeed, all but one term survives due to the choice of holomorphic normal coordinates and the fact that $\delta \bar{z}_{\sigma_1}$ is anti-holomorphic. We complete the sum and extract the Taylor expansion to find

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n} \delta z_{\sigma_1}(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n - \delta z_{\sigma_1}(\sigma_1) - \big(\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \big) z_{\sigma_1}(\sigma) \\
&= \delta z_{\sigma_1}(\sigma) - \big(\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \big) z_{\sigma_1}(\sigma)
\end{aligned} \qquad [3.267]$$

where we have also used $\delta z_{\sigma_1}(\sigma_1) = 0$. We thus have

$$\delta z_{\sigma_1}(\sigma) = \big(\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \big) z_{\sigma_1}(\sigma) + \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^n \qquad [3.268]$$

We add $z_{\sigma_1}(\sigma)$ to both sides and use the fact that $\delta z_{\sigma_1}(\sigma) + z_{\sigma_1}(\sigma) = w_{\sigma_1}(\sigma)$ to find

$$\begin{aligned}
w_{\sigma_1}(\sigma) &= \big[1 + \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \big] \, z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^{n} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^{n+1} \\
&= e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \left[z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^{n} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma) \big)^{n+1} \right]
\end{aligned} \qquad [3.269]$$

The last equation is correct to first order in $\delta\phi$ and thus also δz. The argument of the exponential can be written as

$$\begin{aligned}
\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) &= \mathrm{Re}\, \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + i \, \mathrm{Im}\, \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\
&= \frac{1}{2} \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma_1) \right] + i \, \mathrm{Im}\, \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\
&= \frac{1}{2} \delta \phi(\sigma_1) + i \, \mathrm{Im}\, \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)
\end{aligned} \qquad [3.270]$$

In the last line we have used [3.264] and the fact that at $\sigma = \sigma_1$ the connections vanish. The imaginary part just gives a phase, which we can, just as before, ignore. We thus find our final result, the transition function for a holomorphic change of coordinates $z_{\sigma_1} \to w_{\sigma_1}(z_{\sigma_1})$, with fixed base point, corresponding to a Weyl transformation

$$w_{\sigma_1}(\sigma) = e^{\frac{1}{2}\delta\phi(\sigma_1)} \left[z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial^n_{z_{\sigma_1}} \delta\phi(\sigma) \bigg|_{\sigma=\sigma_1} \big(z_{\sigma_1}(\sigma)\big)^{n+1} \right] \qquad [3.271]$$

We can thus write for a Weyl transformation

$$z_{\sigma_1}(\sigma) \to w_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \sum_{n=0}^{\infty} \varepsilon_n \big(z_{\sigma_1}(\sigma)\big)^n \qquad [3.272]$$

with

$$\begin{aligned} \varepsilon_0 &= 0 \\ \varepsilon_1 &= \frac{1}{2}\delta\phi \\ \varepsilon_n &= e^{\frac{1}{2}\delta\phi(\sigma_1)} \frac{1}{(n)!} \partial^{n-1}_{z_{\sigma_1}} \delta\phi(\sigma) \bigg|_{\sigma=\sigma_1} \qquad \text{for } n \geq 2 \end{aligned} \qquad [3.273]$$

From standard CFT we know that local operator transforms under a general conformal transformation as

$${}^{\odot}_{\odot}\mathcal{A}^{(w_{\sigma_1})}(\sigma_1)^{\odot}_{\odot}{}_{w_{\sigma_1}} = {}^{\odot}_{\odot}\mathcal{A}^{(z_{\sigma_1})}(\sigma_1)^{\odot}_{\odot}{}_{z_{\sigma_1}} - \sum_{n=0}^{\infty} \left(\varepsilon_n L_n^{(z_{\sigma_1})} + \tilde{\varepsilon}_n \tilde{L}_n^{(z_{\sigma_1})} \right) {}^{\odot}_{\odot}\mathcal{A}^{(z_{\sigma_1})}(\sigma_1)^{\odot}_{\odot}{}_{z_{\sigma_1}} \qquad [3.274]$$

As we have the ε_n from [3.273] and we can evaluate $L_n^{(z_{\sigma_1})} {}^{\odot}_{\odot}\mathcal{A}^{(z_{\sigma_1})}(\sigma_1)^{\odot}_{\odot}{}_{z_{\sigma_1}}$ by contour operation of the OPE, we thus have an explicit expression of how a local operator changes under a Weyl transformation.

Weyl Transformations of the Propagator

We are now in a position to really start working on showing that Weyl invariance of the operator V_1 requires the conditions (3.6.16). First we need to work out the Weyl transformations of the propagator $\Delta(\sigma', \sigma)$ and derivatives thereof in our scheme. We have

$$\Delta(\sigma', \sigma) = \frac{\alpha'}{2} \ln \left| z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma) \right|^2 \qquad [3.275]$$

As per our scheme, the geodesic distance is taken with fixed base point. We wish to find the change of this under a Weyl transformation, i.e.

$$\begin{aligned}\delta_{\mathrm{W}}\Delta(\sigma',\sigma) &= \frac{\alpha'}{2}\ln\left|w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)\right|^2 - \frac{\alpha'}{2}\ln\left|z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)\right|^2 \\ &= \frac{\alpha'}{2}\ln\left|\frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)}\right|^2\end{aligned} \quad [3.276]$$

We have worked out an expression for w_{σ_1} under a Weyl transformation in [3.271] which we write to first order as

$$w_{\sigma_1}(\sigma) = \left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right) z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\partial^n_{z_{\sigma_1}}\delta\phi(\sigma_1)\big(z_{\sigma_1}(\sigma)\big)^{n+1} \quad [3.277]$$

We have written $\delta\phi(\sigma_1)$ but it is, of course, understood that we first take the appropriate number of derivatives of $\delta\phi(\sigma)$ and only afterwards take $\sigma = \sigma_1$. Therefore

$$\begin{aligned}\frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)} &= \Big[z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)\Big]^{-1}\Bigg\{\left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right)\Big[z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)\Big] \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\partial^n_{z_{\sigma_1}}\delta\phi(\sigma_1)\Big[\big(z_{\sigma_1}(\sigma')\big)^{n+1} - \big(z_{\sigma_1}(\sigma)\big)^{n+1}\Big]\Bigg\} \\ &= 1 + \frac{1}{2}\delta\phi(\sigma_1) + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\partial^n_{z_{\sigma_1}}\delta\phi(\sigma_1)\frac{\big(z_{\sigma_1}(\sigma')\big)^{n+1} - \big(z_{\sigma_1}(\sigma)\big)^{n+1}}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)}\end{aligned} \quad [3.278]$$

Let us now look at the fraction in the sum. In the end we will be interested in the limit $\sigma' = \sigma = \sigma_1$. First we have

$$\lim_{x\to y}\frac{x^{n+1} - y^{n+1}}{x - y} = \lim_{x-\to y}\sum_{k=0}^{2} x^k y^{n-k} = (n+1)y^n \quad [3.279]$$

We will thus end up with $n\big(z_{\sigma_1}(\sigma)\big)^n$, which we will take at $\sigma = \sigma_1$ and hence this vanishes. Therefore at $\sigma' = \sigma = \sigma_1$

$$\ln\frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)} = \ln\left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right) = \frac{1}{2}\delta\phi(\sigma_1) \quad [3.280]$$

We have a similar contribution of the conjugate part and thus

$$\delta_{\mathrm{W}}\Delta(\sigma',\sigma)\Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2}\times 2\times\frac{1}{2}\delta\phi(\sigma_1) \quad [3.281]$$

This deserves its standalone formula

$$\delta_{\mathrm{W}}\Delta(\sigma,\sigma')\Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2}\delta\phi(\sigma_1) \qquad [3.282]$$

Recalling that our $\delta\phi$ is equal to the $2\delta\omega$ of Joe, we recover (3.6.11).

Let us now work out $\partial_{z'}\delta_{\mathrm{W}}\Delta(\sigma',\sigma)$ in our scheme. As we will see, this has become pretty easy in this scheme. Using [3.276] and [3.278] we have

$$\begin{aligned}\delta_{\mathrm{W}}\Delta(\sigma',\sigma) &= \frac{\alpha'}{2}\ln\left|1+\frac{1}{2}\delta\phi(\sigma_1)+\sum_{n=1}^{\infty}\frac{\partial^n_{z_{\sigma_1}}\delta\phi(\sigma_1)}{(n+1)!}\sum_{k=0}^{n}\big(z_{\sigma_1}(\sigma')\big)^k\big(z_{\sigma_1}(\sigma)\big)^{n-k}\right|^2\\ &= \frac{\alpha'}{2}\left[\frac{1}{2}\delta\phi(\sigma_1)+\sum_{n=1}^{\infty}\frac{\partial^n_{z_{\sigma_1}}\delta\phi(\sigma_1)}{(n+1)!}\sum_{k=0}^{n}\big(z_{\sigma_1}(\sigma')\big)^k\big(z_{\sigma_1}(\sigma)\big)^{n-k}+\mathrm{c.c.}\right] \qquad [3.283]\end{aligned}$$

where $\mathrm{c.c.}$ stands for complex conjugate. We now need to take the $z'=z_{\sigma_1}(\sigma')$ derivative of this and then put $\sigma'=\sigma=\sigma_1$. There is only one term in the expression that then remains, it is the term with $k=1$ and $n=1$ and this gives

$$\partial_{z'}\delta_{\mathrm{W}}\Delta(\sigma',\sigma))\Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2}\left[\frac{\partial_{z_{\sigma_1}}\delta\phi(\sigma_1)}{2}\right] \qquad [3.284]$$

Again, this deserves a standalone formula

$$\partial_{z'}\delta_{\mathrm{W}}\Delta(\sigma',\sigma))\Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{4}\partial_z\delta\phi(\sigma_1) \qquad [3.285]$$

This is (3.6.15a) with again $\delta\phi=2\delta\omega$

Let us do one more derivative: $\partial_{z'}\partial_{\bar z}\delta_{\mathrm{W}}\Delta(\sigma',\sigma)$. This is pretty simple from [3.283]. There are no mixed z' and $\bar z$ terms so that expression vanishes

$$\partial_{z'}\partial_{\bar z}\delta_{\mathrm{W}}\Delta(\sigma',\sigma))\Big|_{\sigma'=\sigma=\sigma_1} = 0 \qquad [3.286]$$

which is (3.6.15b) for $\gamma=-1$.

WEYL TRANSFORMATIONS OF THE VERTEX OPERATOR V_1

We can now calculate the Weyl Variation of the operator V_1 in the new renormalisation scheme. Recall that we already did this for using the renormalisation in Joe's book. But in order to do that we had to assume (3.6.18) was valid. In our new renormalisation scheme we can calculate this Weyl variation, without having to assume (3.6.18). As we have already done most of the calculation, we will not repeat everything. The reader is referred to the derivation of (3.6.16) on p105 for all details.

We can immediately go to [3.175]

$$\delta_{\mathrm{W}} V_1 = \frac{g_c}{\alpha'} \int d^2\sigma \Bigg\{ -2\alpha'\tilde{\phi}\sqrt{g}(\nabla^2\delta\omega)\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}}$$
$$+\frac{1}{2}\sqrt{g}(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})\int d^2\sigma' d^2\sigma''\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}}$$
$$+\frac{1}{2}\sqrt{g}\alpha'\tilde{\phi}R\int d^2\sigma' d^2\sigma''\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\lambda(\sigma')}\frac{\delta}{\delta X_\lambda(\sigma'')}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}}\Bigg\} \quad [3.287]$$

We are denoting the renormalisation based on Weyl normal ordering by $[\cdots]_{\mathrm{w}}$. We are taking into account that due to the fact that we are using a different renormalisation scheme, we have assumed a different $\tilde{\phi}$. Once more we take the three lines separately. The first line gives [3.177]

$$\mathcal{J}_1 = -2g_c\tilde{\phi}\int d^2\sigma\delta\omega\sqrt{g}\,\nabla^a\partial_a\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} \quad [3.288]$$

We know that we need to be very careful now with derivatives and Weyl normal ordering. Let us first check $\partial_z\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}}$. Eq [3.247] tells us how to bring a derivative in a Weyl normal ordered product

$$\partial_z\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} = \left[\partial_z e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\nabla_{\bar{z}}^{n-1}R(\sigma_1)}{(n+1)!}\tilde{L}_n\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} \quad [3.289]$$

Let us first check the second term. We have

$$\begin{aligned}\tilde{L}_n\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} &= \oint_{\mathcal{C}_z}\frac{dw}{2\pi i}(w-z)^{n+1}T(w)e^{ik\cdot X(z)}\\ &= \oint_{\mathcal{C}_z}\frac{dw}{2\pi i}(w-z)^{n+1}\left[\frac{(\alpha' k^2/4)e^{ik\cdot X(z)}}{(z-w)^2}+\frac{\partial_z e^{ik\cdot X(z)}}{z-w}\right]\end{aligned} \quad [3.290]$$

The first term of the OPE gives a $(z-w)^{n-1}$ and the second a $(z-w)^n$. For $n \geq 1$ neither of these contributions give a pole and so

$$\tilde{L}_n\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} = 0 \qquad \text{for } n \geq 1 \quad [3.291]$$

and therefore

$$\partial_z\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} = \left[\partial_z e^{ik\cdot X(\sigma)}\right]_{\mathrm{w}} \quad [3.292]$$

The same holds, of course, for its complex conjugate and hence

$$\nabla^a\partial_a\left[e^{ik\cdot X(\sigma)}\right]_{\text{w}}=\nabla^a\left[\partial_a e^{ik\cdot X(\sigma)}\right]_{\text{w}}=\nabla^a\left[ik_\mu\partial_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\text{w}}$$
$$=ik_\mu\nabla^a\left[\nabla_a X^\mu e^{ik\cdot X(\sigma)}\right]_{\text{w}} \qquad [3.293]$$

But now comes the fruit of all our hard work, because we already calculated this in [3.260]

$$\nabla^a\left[\nabla_a e^{ik\cdot X(\sigma)}\right]_{\text{w}}=ik_\nu\left[\partial_a X^\mu\partial^a X^\nu e^{ik\cdot X(\sigma)}\right]_{\text{w}}+\frac{i\alpha'\gamma k^\mu}{4}R(\sigma_1)\left[e^{ik\cdot X}\right]_{\text{w}} \qquad [3.294]$$

We thus find

$$\mathcal{J}_1=-2g_c\tilde{\phi}\int d^2\sigma\delta\omega\sqrt{g}\,ik_\mu\left[ik_\nu\left[\partial_a X^\mu\partial^a X^\nu e^{ik\cdot X(\sigma)}\right]_{\text{w}}+\frac{i\alpha'\gamma k^\mu}{4}R(\sigma_1)\left[e^{ik\cdot X}\right]_{\text{w}}\right]$$
$$=\frac{g_c}{2}\int d^2\sigma\,\sqrt{g}\,\delta\omega\left\{\left(\gamma\alpha' k^2\tilde{\phi}R\right)\left[e^{ik\cdot X}\right]_{\text{w}}\right.$$
$$\left.+\left(4k_\mu k_\nu\tilde{\phi}\right)\left[\partial_a X^\mu\partial^a X^\nu e^{ik\cdot X)}\right]_{\text{w}}\right\} \qquad [3.295]$$

Note that this is exactly the same result as in our previous calculation, [3.180] if we set $\gamma=-2/3$ and $\tilde{\phi}=\phi$ as per Joe's book.

The third line is identical to the previous result, as there are no derivatives to mess it up:

$$\mathcal{J}_3=\frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega\big(-\alpha' k^2\tilde{\phi}R\big)\left[e^{ik\cdot X}\right]_{\text{w}} \qquad [3.296]$$

Let us finally focus our attention to the second line. If both functional derivatives act on the exponential we have no derivatives to worry about and we can, once more, just take over the result from our previous calculation, [3.185]

$$\mathcal{J}_{2a}=\frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega\big(g^{ab}(-k^2 s_{\mu\nu})+i\epsilon^{ab}(-k^2 a_{\mu\nu})\big)\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X}\right]_{\text{w}} \qquad [3.297]$$

If one functional derivative acts on the exponential and the other on ∂X, then the previous calculation is valid up to the point where we will apply derivatives on the Weyl normal ordered local operators. This means that we can just take over up to [3.193]

$$\mathcal{J}_{2b}=-\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}\left\{(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^\mu\nabla_a\left[\partial_b X^\nu e^{ik\cdot X}\right]_{\text{w}}\right.$$
$$\left.+(g^{ab}s_{\mu\nu}+i\epsilon^{ab}a_{\mu\nu})k^\nu\nabla_b\left[\partial_a X^\mu e^{ik\cdot X}\right]_{\text{w}}\right\} \qquad [3.298]$$

We will consider the symmetric and antisymmetric part separately

$$\mathcal{J}_{2b}^{s} = -\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}\left\{s_{\mu\nu}k^\mu\nabla^a\left[\nabla_a X^\nu e^{ik\cdot X}\right]_{\mathrm{w}} + s_{\mu\nu}k^\nu\nabla^a\left[\nabla_a X^\mu e^{ik\cdot X}\right]_{\mathrm{w}}\right\} \qquad [3.299]$$

We can again use [3.260] and the symmetry of $s_{\mu\nu}$

$$\begin{aligned}\mathcal{J}_{2b}^{s} &= -ig_c\int d^2\sigma\delta\omega\sqrt{g}s_{\mu\nu}k^\mu\left\{\left[\nabla_a X^\nu\nabla^a e^{ik\cdot X}\right]_{\mathrm{w}} + \frac{i\alpha'\gamma k^\nu}{4}R\left[e^{ik\cdot X}\right]_{\mathrm{w}}\right\}\\ &= -ig_c\int d^2\sigma\sqrt{g}\delta\omega s_{\mu\nu}k^\mu\left\{ik_\lambda\left[\partial_a X^\nu\partial^a X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}} + \frac{i\alpha'\gamma k^\nu}{4}R\left[e^{ik\cdot X}\right]_{\mathrm{w}}\right\} \qquad [3.300]\end{aligned}$$

Note that we can simplify the first contribution as, ignoring pre-factors,

$$\begin{aligned}s_{\mu\nu}k^\mu k_\lambda\left[\partial_a X^\nu\partial^a X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}} &= g^{ab}s_{\mu\nu}k^\mu k_\lambda\left[\partial_a X^\nu\partial^b X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}}\\ &= \frac{1}{2}g^{ab}\left(s_{\lambda\nu}k^\lambda k_\mu + s_{\mu\lambda}k^\lambda k^\nu\right)\left[\partial_a X^\mu\partial^b X^\nu e^{ik\cdot X}\right]_{\mathrm{w}} \qquad [3.301]\end{aligned}$$

We thus find for the symmetric part

$$\begin{aligned}\mathcal{J}_{2b}^{s} = \frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega\Bigg\{&\alpha' R\left(\frac{\gamma}{2}s_{\mu\nu}k^\mu k^\nu\right)\left[e^{ik\cdot X}\right]_{\mathrm{w}}\\ &+ g^{ab}\left(s_{\lambda\nu}k^\lambda k_\mu + s_{\mu\lambda}k^\lambda k^\nu\right)\left[\partial_a X^\mu\partial^b X^\nu e^{ik\cdot X}\right]_{\mathrm{w}}\Bigg\} \qquad [3.302]\end{aligned}$$

Let us now focus on the antisymmetric part

$$\mathcal{J}_{2b}^{a} = -\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{ab}a_{\mu\nu}\left\{k^\mu\nabla_a\left[\partial_b X^\nu e^{ik\cdot X}\right]_{\mathrm{w}} + k^\nu\nabla_b\left[\partial_a X^\mu e^{ik\cdot X}\right]_{\mathrm{w}}\right\} \qquad [3.303]$$

The only non-vanishing component of ϵ^{ab} is $\epsilon^{z\bar{z}}$ and so we have

$$\mathcal{J}_{2b}^{a} = -\frac{ig_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{z\bar{z}}a_{\mu\nu}\left\{k^\mu\nabla_z\left[\partial_{\bar{z}} X^\nu e^{ik\cdot X}\right]_{\mathrm{w}} + k^\nu\nabla_{\bar{z}}\left[\partial_z X^\mu e^{ik\cdot X}\right]_{\mathrm{w}}\right\} \qquad [3.304]$$

We can again just take over [3.260], without a factor of four. Note that the second term in [3.260] will bring down an extra factor of k^μ resulting in a combination $a_{\mu\nu}k^\mu k^\nu$ which is zero by antisymmetry of $a_{\mu\nu}$. So we get

$$\begin{aligned}\mathcal{J}_{2b}^{a} &= \frac{g_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{z\bar{z}}a_{\mu\nu}\left\{k^\mu\left[\partial_{\bar{z}} X^\nu\partial_z e^{ik\cdot X}\right]_{\mathrm{w}} + k^\nu\left[\partial_z X^\mu\partial_{\bar{z}} e^{ik\cdot X}\right]_{\mathrm{w}}\right\}\\ &= \frac{g_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{z\bar{z}}a_{\mu\nu}\left\{k^\mu\left[\partial_{\bar{z}} X^\nu ik_\lambda\partial_z X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}} + k^\nu\left[\partial_z X^\mu ik^\lambda\partial_{\bar{z}} X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}}\right\}\\ &= \frac{g_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{ab}a_{\mu\nu}\left\{k^\mu\left[\partial_b X^\nu ik_\lambda\partial_a X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}} + k^\nu\left[\partial_a X^\mu ik^\lambda\partial_b X^\lambda e^{ik\cdot X}\right]_{\mathrm{w}}\right\}\\ &= \frac{g_c}{2}\int d^2\sigma\delta\omega\sqrt{g}i\epsilon^{ab}\left(s_{\lambda\nu}k^\lambda k_\mu + s_{\mu\lambda}k^\lambda k^\nu\right)\left[\partial_a X^\mu\partial^b X^\nu e^{ik\cdot X}\right]_{\mathrm{w}} \qquad [3.305]\end{aligned}$$

Adding the symmetric and antisymmetric part we find

$$\mathcal{J}_{2b} = \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \Big\{ \alpha' R \left(\frac{\gamma}{2} s_{\mu\nu} k^\mu k^\nu \right) \left[e^{ik\cdot X(\sigma)} \right]_\mathrm{r} + (g^{ab} + i\epsilon^{ab}) \left(s_{\mu\lambda} k_\nu k^\lambda + s_{\lambda\nu} k_\mu k^\lambda \right) \left[\partial_b X^\nu \partial_a X^\mu e^{ik\cdot X(\sigma)} \right]_\mathrm{r} \Big\} \quad [3.306]$$

This is the same result [3.198] as in our previous calculation, if we set again $\gamma = -2/3$ there.

Let us finally focus on the term where both functional derivatives act on the $\partial_a X^\mu \partial_b X^\nu$. Once again, we can follow the calculation from the past as long as we don't mess around with derivatives of operators. This means we can go straight to [3.202]

$$\mathcal{J}_{2c} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} g^{ab} s^\mu_\mu \partial'_a \partial_b \delta_\mathrm{W} \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} \left[e^{ik\cdot X(\sigma)} \right]_\mathrm{r} \quad [3.307]$$

But the only non-vanishing metric components are $g^{z\bar{z}}$ and we have shown in [**??**] that $\partial_{z'} \partial_{\bar{z}} \delta_\mathrm{W} \Delta(\sigma', \sigma)) \Big|_{\sigma'=\sigma=\sigma_1} = 0$ and so in Weyl normal ordering we simply have

$$\mathcal{J}_{2c} = 0 \quad [3.308]$$

Bringing the results together, we find We can now bring all the contributions together. They are of the form

$$\delta_\mathrm{W} V_1 = \frac{g_c}{2} \int d^2\sigma \sqrt{g}\, \delta\omega \Big\{ (g^{ab} \mathbb{s}_{\mu\mu} + i\epsilon^{ab} \mathbb{a}_{\mu\nu}) \left[\partial_a X^\mu \partial_a X^\mu e^{ik\cdot X(\sigma)} \right]_\mathrm{r} + \alpha' R\, \mathbb{f} \left[e^{ik\cdot X(\sigma)} \right]_\mathrm{r} \Big\} \quad [3.309]$$

The contributions form the different parts to $\mathbb{s}$, $\mathbb{a}$ and $\mathbb{f}$ are summarised in the table below.

	$\mathbb{s}_{\mu\nu}$	$\mathbb{a}_{\mu\nu}$	$\mathbb{f}$
$\mathcal{J}_1$	$4k_\mu k_\nu \tilde{\phi}$	–	$\gamma k^2 \tilde{\phi}$
$\mathcal{J}_{2a}$	$-k^2 s_{\mu\nu}$	$-k^2 a_{\mu\nu}$	–
$\mathcal{J}_{2b}$	$k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\nu\lambda}$	$k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda}$	$\frac{\gamma}{2} k^\mu k^\nu s_{\mu\nu}$
$\mathcal{J}_{2c}$	–	–	–
$\mathcal{J}_3$	–	–	$-k^2 \tilde{\phi}$

Table 3.3: Weyl transformation of the massless vertex operator; contributions to [3.205]

Thus

$$\begin{aligned}
\mathbb{s}_{\mu\nu} &= -k^2 s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} + 4k_\mu k_\nu \tilde{\phi} \\
\mathbb{a}_{\mu\nu} &= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda} \\
&= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda} \\
\mathbb{f} &= (\gamma - 1)k^2 \tilde{\phi} + \frac{\gamma}{2} k^\mu k^\nu s_{\mu\nu}
\end{aligned} \tag{3.310}$$

Requiring Weyl invariance of the vertex operator V_1 requires that $\mathbb{s}_{\mu\nu} = \mathbb{a}_{\mu\nu} = \mathbb{f} = 0$. We can now go through the same reasoning as for (3.6.22), which we briefly repeat here. First we take an n satisfying $n^2 = 0$ and $k \cdot n = 1$. Also $n^\mu s_{\mu\nu} = n^\mu a_{\mu\nu} = 0$. Start by requiring $\mathbb{s}_{\mu\nu} n^\mu n^n u = 0$. This means

$$0 = -k^2 s_{\mu\nu} n^\mu n^\nu + k_\nu k^\omega s_{\mu\omega} n^\mu n^\nu + k_\mu k^\omega s_{\nu\omega} n^\mu n^\nu + 4k_\mu k_\nu n^\mu n^\nu \tilde{\phi} = 4\tilde{\phi} \tag{3.311}$$

so that

$$\tilde{\phi} = 0 \tag{3.312}$$

Now require $\mathbb{s}_{\mu\nu} n^\mu = 0$. This gives

$$0 = -k^2 s_{\mu\nu} n^\mu + k_\nu k^\omega s_{\mu\omega} n^\mu + k_\mu k^\omega s_{\nu\omega} n^\mu = (k \cdot n) k^\omega s_{\nu\omega} \tag{3.313}$$

so that

$$k^\mu s_{\mu\nu} = 0 \tag{3.314}$$

Similarly, requiring $A_{\mu\nu} n^\mu = 0$ leads to

$$\begin{aligned}
0 &= -k^2 a_{\mu\nu} n^\mu + k_\nu k^\omega a_{\mu\omega} n^\mu - k_\mu k^\omega a_{\nu\omega} n^\mu \\
&= (k \cdot n) k^\mu a_{\nu\mu}
\end{aligned} \tag{3.315}$$

so that

$$k^\mu a_{\mu\nu} = 0 \tag{3.316}$$

Finally, requiring $\mathbb{s}_{\mu\nu} = 0$ gives

$$0 = -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} = -k^2 s_{\mu\nu} \tag{3.317}$$

and so also

$$k^2 = 0 \tag{3.318}$$

Note that $s_{\mu\nu} = 0$ just means that there would be no symmetric part. We now see that $\mathbb{a}_{\mu\nu} = 0$ is also satisfied and so is $\mathbb{f} = 0$. We have exactly the same on-shell conditions as from the renormalisation scheme in Joe's book.

We are now, finally, in a position to argue (3.6.18). And it is now pretty simple. Using a the Weyl normal ordering renormalisation scheme we have found the above mass-shell conditions as in Joe's book. Going back through the calculation of the on-shell conditions in Joe's book we see that these can only be satisfied if (3.6.18) is valid. As physical results should be independent of the renormalisation scheme and mass-shell conditions certainly are physical results, this shows that (3.6.18) is indeed valid.

3.42 p 106: Eq (3.6.20) The Independent Parameters of the Massless Vertex Operator, I

Let us first consider the change in $a_{\mu\nu}$. This change will only impact $A_{\mu\nu}$:

$$\begin{aligned} A_{\mu\nu} \to & A_{\mu\nu} - k^2(\zeta_\mu k_\nu - k_\mu \zeta_\nu) + k_\nu k^\lambda(\zeta_\mu k_\lambda - k_\mu\zeta_\lambda) - k_\mu k^\lambda(\zeta_\nu k_\lambda - k_\nu\zeta_\lambda) \\ &= A_{\mu\nu} - k^2\zeta_\mu k_\nu + k^2 k_\mu\zeta_\nu + k^2\zeta_\mu k_\nu - (k\cdot\zeta)k_\mu k_n u - k^2 k_\mu\zeta_\nu + (k\cdot\zeta)k_\mu k_\nu \\ &= A_{\mu\nu} \end{aligned} \quad [3.319]$$

and so

$$\delta_{\mathrm{W}} V_1(s_{\mu\nu}, a_{\mu\nu} + \zeta_\mu k_\nu - k_\mu\zeta_\nu, \phi) = \delta_{\mathrm{W}} V_1(s_{\mu\nu}, a_{\mu\nu}, \phi) \quad [3.320]$$

Similarly, we find

$$\begin{aligned} S_{\mu\nu} \to & S_{\mu\nu} - k^2(\xi_\mu k_\nu + k_\mu\xi_n u) + k_\nu k^\lambda(\xi_\mu k_\lambda + k_\mu\xi_\lambda) + k_\mu k^\lambda(\xi_\nu k_\lambda + k_\nu\xi_\lambda) \\ & - \frac{1}{3}k_\mu k_\nu 2(k\cdot\xi) - \frac{4}{3}k_\mu k_\nu 2(k\cdot\xi) \\ = & S_{\mu\nu} - k^2\xi_\mu k_\nu - k^2 k_\mu\xi_n u + k^2 k_\nu\xi_\mu + k_\mu k_\nu(k\cdot\xi) + k^2 k_\mu x_\nu + k_\mu k_\nu(k\cdot\xi) \\ & - 2k_\mu k_\nu(k\cdot\xi) = S_{\mu\nu} \end{aligned} \quad [3.321]$$

We also have

$$\begin{aligned} F \to F - \frac{5}{3}k^2\left(-\frac{1}{3}(k\cdot\xi)\right) - \frac{1}{3}k^\mu k^\nu(\xi_\mu k_\nu + k_\mu\xi_\nu) + \frac{1}{18}k^2 2(k\cdot\xi) \\ = F + \left[\frac{5}{9} - \frac{2}{3} + \frac{1}{9}\right]k^2(k\cdot\xi) = F \end{aligned} \quad [3.322]$$

and so

$$\delta_{\mathrm{W}} V_1(s_{\mu\nu} + \xi_\mu k_\nu + k_\mu\xi_\nu, a_{\mu\nu}, \phi - \frac{1}{3}k\cdot\xi) = \delta_{\mathrm{W}} V_1(s_{\mu\nu}, a_{\mu\nu}, \phi) \quad [3.323]$$

3.43 p 106: Eq (3.6.21) The Independent Parameters of the Massless Vertex Operator, II

Let us chose our axes so that $k = (1, 1, 0, \cdots, 0)$. It satisfies $k^2 = 0$ as it should. n is then necessarily of the form $n = (-\frac{1}{2}, \frac{1}{2}, \vec{n})$ in order to satisfy $k \cdot n = 1$. Now $n^2 = 0$ implies $\vec{n}^2 = 0$ and so $\vec{n} = 0$. We can thus always chose

$$\begin{aligned} k &= (1, 1, 0, \cdots, 0) \\ n &= (-\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0) \end{aligned} \qquad [3.324]$$

Let us now start from a general $s_{\mu\nu}$ we allow a transformation (3.6.20a) but require that the transformed field satisfies (3.6.61), i.e.

$$\begin{aligned} 0 &= n^\mu(s_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi \nu) \\ &= n^\mu s_{\mu\nu} + (n \cdot \xi) k^\nu + \xi^\nu \end{aligned} \qquad [3.325]$$

Let us write this out explicitly. For $\nu = 0, 1$ we have

$$\begin{aligned} 0 &= \frac{1}{2}(-s_{00} + s_{10}) + \frac{1}{2}(\xi_0 + \xi_1) + \xi_0 \\ 0 &= \frac{1}{2}(-s_{01} + s_{11}) + \frac{1}{2}(\xi_0 + \xi_1) + \xi_1 \end{aligned} \qquad [3.326]$$

This is easily solved for ξ_0 and ξ_1:

$$\begin{aligned} \xi_0 &= \frac{1}{8}\left(3s_{00} - 3s_{10} - s_{01} + s_{11}\right) \\ \xi_1 &= \frac{1}{8}\left(-s_{00} + s_{10} + 3s_{01} - 3s_{11}\right) \end{aligned} \qquad [3.327]$$

The other equations are, for $\nu = 2, \cdots, D-1$

$$0 = \frac{1}{2}(-s_{0\nu} + s_{1\nu}) + \xi_\nu \qquad [3.328]$$

which is solved by

$$\xi_\nu = \frac{1}{2}(s_{0\nu} - s_{1\nu}) \qquad [3.329]$$

In other words, given an $s_{\mu\nu}$ we can always transform it to a new form that satisfies $n^\mu s_{\mu\nu} = 0$ and still has the same Weyl transformation. A similar argument holds obviously for $a_{\mu\nu}$.

3.44 p 106: Eq (3.6.22) The Independent Parameters of the Massless Vertex Operator, III

Start by requiring $S_{\mu\nu} n^\mu n^n u = 0$. This means

$$\begin{aligned} 0 =& -k^2 s_{\mu\nu} n^\mu n^\nu + k_\nu k^\omega s_{\mu\omega} n^\mu n^\nu + k_\mu k^\omega s_{\nu\omega} n^\mu n^\nu - (1+\gamma) k_\mu k_\nu s^\omega_\omega n^\mu n^\nu + 4 k_\mu k_\nu n^\mu n^\nu \phi \\ =& \left[-(1+\gamma) s^\omega_\omega + 4\phi\right] k_\mu k_\nu n^\mu n^\nu = \left[-(1+\gamma) s^\omega_\omega + 4\phi\right] (k\cdot n)^2 = -(1+\gamma) s^\omega_\omega + 4\phi \end{aligned} \quad [3.330]$$

so that

$$\phi = \frac{1+\gamma}{4} s^\omega_\omega = \frac{1}{12} s^\omega_\omega \quad [3.331]$$

Now require $S_{\mu\nu} n^\mu = 0$. This gives

$$\begin{aligned} 0 =& -k^2 s_{\mu\nu} n^\mu + k_\nu k^\omega s_{\mu\omega} n^\mu + k_\mu k^\omega s_{\nu\omega} n^\mu - (1+\gamma) k_\mu k_\nu s^\omega_\omega n^\mu + 4 k_\mu k_\nu n^\mu \phi \\ =& (k\cdot n) k^\omega s_{\nu\omega} \end{aligned} \quad [3.332]$$

so that

$$k^\mu s_{\mu\nu} = 0 \quad [3.333]$$

Similarly, requiring $A_{\mu\nu} n^\mu = 0$ leads to

$$\begin{aligned} 0 =& -k^2 a_{\mu\nu} n^\mu + k_\nu k^\omega a_{\mu\omega} n^\mu - k_\mu k^\omega a_{\nu\omega} n^\mu \\ =& (k\cdot n) k^\mu a_{\nu\mu} \end{aligned} \quad [3.334]$$

so that

$$k^\mu a_{\mu\nu} = 0 \quad [3.335]$$

Finally, requiring $S_{\mu\nu} = 0$ gives

$$\begin{aligned} 0 =& -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} - (1+\gamma) k_\mu k_\nu s^\omega_\omega + 4 k_\mu k_\nu \phi \\ =& -k^2 s_{\mu\nu} \end{aligned} \quad [3.336]$$

and so also

$$k^2 = 0 \quad [3.337]$$

Note that $s_{\mu\nu} = 0$ just means that there would be no symmetric part. We now see that $A_{\mu\nu} = 0$ is also satisfied and so is $F = 0$.

3.45 p 108: Eq (3.7.5) The Graviton from the Background Field

Expand the exponential of (3.7.2) using (3.7.3) to linear order

$$\begin{aligned}\exp[-S_\sigma] &= \exp\left[-\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}\big(\eta_{\mu\nu}+\chi_{\mu\nu}(X)\big)\partial_a X^\mu\partial_b X^\nu\right]\\ &= \exp\left[-\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\right]\\ &\quad\times\left[1-\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}\chi_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu+\cdots\right]\end{aligned} \qquad [3.338]$$

If we now set

$$\chi_{\mu\nu}(X) = -4\pi g_c e^{ik\cdot X}s_{\mu\nu} \qquad [3.339]$$

then we find

$$\exp[-S_\sigma] = \exp[-S_p]\left[1+\frac{g_c}{\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}s_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X}+\cdots\right] \qquad [3.340]$$

with S_p the Polyakov action. Using the formula for the massless closed string vertex operator

$$\exp[-S_\sigma] = \exp[-S_p]\big[1+V_1(s_{\mu\nu},a_{\mu\nu}=0,\phi=0;X)+\cdots\big] \qquad [3.341]$$

and so the linear term in $G_{\mu\nu}(X)$ gives an interaction that comes from the insertion of a vertex for a symmetric 2-tensor, i.e. a graviton, with spacetime momentum k^μ.

3.46 p 109: Eq (3.7.7) The Spacetime Gauge Invariance of the Antisymmetric Tensor

We have, renaming dummy indices and using the antisymmetry of ϵ^{ab}

$$\epsilon^{ab}(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu)\partial_a X^\mu\partial_b X^\nu = 2\epsilon^{ab}\partial_\mu\xi_\nu\partial_a X^\mu\partial_b X^\nu \qquad [3.342]$$

Using partial integration

$$\begin{aligned}\int d^2\sigma\,\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu\partial_a X^\mu\partial_b X^\nu &= \int d^2\Big\{\sigma\,\partial_a\left[\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu X^\mu\partial_b X^\nu\right]-\partial_a\left[\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu\partial_b X^\nu\right]X^\mu\Big\}\\ &= \int d^2\sigma\,\Big\{\partial_a\left[\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu X^\mu\partial_b X^\nu\right]-\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu\partial_a\partial_b X^\nu X^\mu\Big\}\\ &= \int d^2\sigma\,\partial_a\left[\sqrt{\sigma}\,\epsilon^{ab}\partial_\mu\xi_\nu X^\mu\partial_b X^\nu\right]\end{aligned} \qquad [3.343]$$

We have used the fact that by definition $\sqrt{g}\epsilon^{12}=1$ so that $\partial_a(\sqrt{g}\epsilon^{ab})=0$ and that $\epsilon^{ab}\partial_a\partial b X^\nu=0$ by symmetry. We thus see that under such a transformation S_σ is indeed a total derivative.

3.47 p 109: Eq (3.7.7) The Spacetime Gauge Invariance of the Three-Tensor $H_{\omega\mu\nu}$

$$\begin{aligned}\delta H_{\omega\mu\nu} &= \partial_\omega(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu) + \partial_\mu(\partial_\nu\xi_\omega - \partial_\omega\xi_\nu) + \partial_\nu(\partial_\omega\xi_\mu - \partial_\mu\xi_\omega)\\ &= \partial_\omega\partial_\mu\xi_\nu - \partial_\omega\partial_\nu\xi_\mu + \partial_\mu\partial_\nu\xi_\omega - \partial_\mu\partial_\omega\xi_\nu + \partial_\nu\partial_\omega\xi_\mu - \partial_\nu\partial_\mu\xi_\omega\\ &= 0 \end{aligned} \qquad [3.344]$$

3.48 p 110: The Most General Classical Action Invariant under a Rigid Weyl Transformation

The point is that adding additional derivatives can only come in pairs $\partial_a X^\mu \partial_b X^\nu$ combined with a worldsheet metric g^{ab}, otherwise we can't have a worldsheet diffeomorphism invariant action. If we have $n = 2k$ such derivatives then the action is of the form $\int d^2\sigma\sqrt{g}g^{a_1b_1}\cdots g^{a_kb_k}\cdots$, where we have only written the terms that change under a Weyl transformation. But under $g_{ab} \to e^{2\omega}$, with ω constant, then this changes into $e^{(2-2k)\omega}\int d^2\sigma\sqrt{g}g^{a_1b_1}\cdots g^{a_kb_k}\cdots$. We can thus only have invariance under rigid Weyl transformations if we have $n = 2k = 2$ derivatives.

3.49 p 110: Eq (3.7.11) The Linear Approximation of the Non-linear Sigma Model

Using the definition of the Polyakov action and the Vertex operator V_1, one find to linear order

$$\begin{aligned} S_\sigma = S_p - V_1 &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\\ &\quad - \frac{g_c}{\alpha'}\int d^2\sigma\,\sqrt{g}\,\left[(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} + \alpha'\phi R e^{ik\cdot X}\right]\\ &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\left\{\left[g^{ab}\left(\eta_{\mu\nu} - 4\pi g_c s_{\mu\nu}e^{ik\cdot X}\right) + i\epsilon^{ab}\left(-4\pi g_c a_{\mu\nu}e^{ik\cdot X}\right)\right]\right.\\ &\qquad\qquad \left.\times\,\partial_a X^\mu\partial_b X^\nu + \alpha' R\left(-4\pi g_c\phi e^{ik\cdot X}\right)\right] \end{aligned} \qquad [3.345]$$

Comparing this with the no nonlinear sigma model, we indeed find to linear order,

$$\begin{aligned} G_{\mu\nu}(X) &= \eta_{\mu\nu} - 4\pi g_c s_{\mu\nu}e^{ik\cdot X}\\ B_{\mu\nu}(X) &= -4\pi g_c a_{\mu\nu}e^{ik\cdot X}\\ \Phi(X) &= -4\pi g_c\phi e^{ik\cdot X} \end{aligned} \qquad [3.346]$$

3.50 p 111: Eq (3.7.13) The β Functions to First Order

Recall first that all these equations are valid as operator equations, so we can use (3.4.6)

$$\delta_{\mathrm{W}}\langle \cdots \rangle = -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\langle T^a_a(\sigma)\cdots\rangle \qquad [3.347]$$

We split the S_σ again in the Polyakov and vertex part. The Weyl transformation of the Polyakov part is simply the Weyl anomaly (3.4.15)

$$(T_p)^a_a = -\frac{c}{12}R = -\frac{D-26}{12}R \qquad [3.348]$$

The Weyl variation of the vertex part is given by (3.6.16)

$$\begin{aligned}\delta_{\mathrm{W}}V_1 &= \frac{g_c}{2}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\Big\{(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + \alpha' FR\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\}\\ &= -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)(T_\sigma)^a_a(\sigma) \qquad [3.349]\end{aligned}$$

with $S_{\mu\nu}, A_{\mu\nu}$ and F given by (3.6.17). We have written it is the form of (3.4.6) in the understanding that this is valid as an operator equation. We can thus write

$$(T_\sigma)^a_a = (-2\pi)\frac{g_c}{2}\Big\{(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + \alpha' FR\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\} \qquad [3.350]$$

We thus have, combining both parts

$$T^a_a = -\,g_c\pi(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} - g_c\pi\alpha' FRe^{ik\cdot X} - \frac{D-26}{12}R \qquad [3.351]$$

We have dropped the renormalisation symbols for convenience. Let us write this in the form (3.7.12)

$$T^a_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{1}{2}\beta^\Phi R \qquad [3.352]$$

Let us break this down in parts top avoid long formula. We start with the last term

$$-\frac{1}{2}\beta^\Phi R = -g_c\pi\alpha' FRe^{ik\cdot X} - \frac{D-26}{12}R \qquad [3.353]$$

Using (3.6.17c) for F and taking from here on $\gamma = 0$ we find

$$\beta^\Phi = 2g_c\pi\alpha'(-k^2\phi)e^{ik\cdot X} + \frac{D-26}{6} = 2g_c\pi\alpha'\phi\,\partial^2 e^{ik\cdot X} + \frac{D-26}{6} \qquad [3.354]$$

Note that here ∂^2 and later ∂_μ denote derivatives w.r.t. the spacetime field X^μ and not the worldsheet coordinate σ^a. We now use the definition of the dilaton field $\Phi(X)$ in (3.7.11c)

$$\begin{aligned}\beta^\Phi &= 2g_c\pi\alpha'\left(-\frac{1}{4\pi g_c}\partial^2\Phi\right)+\frac{D-26}{6}\\ &= \frac{D-26}{6}-\frac{\alpha'}{2}\partial^2\Phi\end{aligned} \qquad [3.355]$$

Consider now the second term

$$-\frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu = -g_c\pi i\epsilon^{ab}A_{\mu\nu}\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} \qquad [3.356]$$

Using $A_{\mu\nu}$ from (3.6.16b) we find

$$\begin{aligned}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu &= 2\alpha' g_c\pi\epsilon^{ab}(-k^2 a_{\mu\nu}+k_\nu k^\omega a_{\mu\omega}-k_\mu k^\omega a_{\nu\omega})\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X}\\ &= 2\alpha' g_c\pi\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu(a_{\mu\nu}\partial^2-a_{\mu\omega}\partial_\nu\partial^\omega+a_{\nu\omega}\partial_\mu\partial^\omega)e^{ik\cdot X}\end{aligned} \qquad [3.357]$$

We use the expression for $B_{\mu\nu}(X)$ in (3.7.11b)

$$\begin{aligned}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu &= 2\alpha' g_c\pi\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\left(-\frac{1}{4\pi g_c}\right)\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\\ &= -\frac{\alpha'}{2}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\end{aligned} \qquad [3.358]$$

from which we get

$$\begin{aligned}\beta^B_{\mu\nu} &= -\frac{\alpha'}{2}\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\\ &= -\frac{\alpha'}{2}\partial^\omega H_{\omega\mu\nu}\end{aligned} \qquad [3.359]$$

where we have used the definition of the field strength (3.7.8). Finally we take the first term

$$-\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu\partial_b X^\nu = -g_c\pi g^{ab}S_{\mu\nu}\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} \qquad [3.360]$$

Using the definition of $S_{\mu\nu}$ in (3.6.17b), setting $\gamma=0$ and ignoring the $g^{ab}\partial_a X^\mu\partial_b X^\nu$ we have

$$\begin{aligned}\beta^G_{\mu\nu} &= 2\alpha' g_c\pi(-k^2 s_{\mu\nu}+k_\nu k^\omega s_{\mu\omega}+k_\mu k^\omega s_{\nu\omega}-k_\mu k_\nu s^\omega_\omega+4k_\mu k_\nu\phi)e^{ik\cdot X}\\ &= 2\alpha' g_c\pi(s_{\mu\nu}\partial^2-s_{\mu\omega}\partial_\nu\partial^\omega-s_{\nu\omega}\partial_\mu\partial^\omega+s^\omega_\omega\partial_\mu\partial_\nu-4\phi\partial_\mu\partial_\nu)e^{ik\cdot X}\\ &= -\frac{2\alpha' g_c\pi}{4\pi g_c}(\partial^2\chi_{\mu\nu}-\partial_\nu\partial^\omega\chi_{\mu\omega}-\partial_\mu\partial^\omega\chi_{\nu\omega}+\partial_\mu\partial_\nu\chi^\omega_\omega-4\partial_\mu\partial_\nu\Phi)\\ &= -\frac{\alpha'}{2}(\partial^2\chi_{\mu\nu}-\partial_\nu\partial^\omega\chi_{\mu\omega}-\partial_\mu\partial^\omega\chi_{\nu\omega}+\partial_\mu\partial_\nu\chi^\omega_\omega)+2\alpha'\partial_\mu\partial_\nu\Phi)\end{aligned} \qquad [3.361]$$

where we have used (3.7.11) and the fact that $G_{\mu\nu}=\eta_{\mu\nu}+\chi_{\mu\nu}$ to that order.

3.51 p 111: Eq (3.7.14) The β Functions with two Spacetime Derivatives

We will not derive all these equations in detail. But let us remember that it was already argued before that the string action leads to spacetime general coordinate symmetry. We should thus not be surprised that if we add higher order terms, (part of) these will organise themselves in spacetime covariant derivatives. What we will do is two things and both are related to $\beta^G_{\mu\nu}$.

The first thing is just point out that if we would work out the spacetime Ricci tensor $\boldsymbol{R}_{\mu\nu}$ to first order in $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$ one indeed recovers the relevant part of (3.7.13a). This is a straightforward calculation which we now sketch. We expand the spacetime field $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$, use it to raise and lower indices and work to linear order. We then have $G^{\mu\nu} = \eta_{\mu\nu} - \chi_{\mu\nu}$. The spacetime connection is then given by

$$\boldsymbol{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}(\partial_\mu \chi^\sigma_\nu + \partial_\nu \chi^\sigma_\mu - \partial^\sigma \chi_{\mu\nu}) \qquad [3.362]$$

The spacetime curvature tensor is

$$\begin{aligned}\boldsymbol{R}^\mu_{\nu\sigma\lambda} &= \partial_\sigma \Gamma^\mu_{\lambda\nu} - \partial_\lambda \Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\sigma\rho}\Gamma^\rho_{\lambda_\nu} - \Gamma^\mu_{\lambda\rho}\Gamma^\rho_{\nu\sigma} \\ &= \frac{1}{2}(\partial_\sigma\partial_\nu\chi^\mu_\lambda - \partial_\lambda\partial_\nu\chi^\mu_\sigma - \partial_\sigma\partial^\mu\chi_{\lambda\nu} + \partial_\lambda\partial^\mu\chi_{\sigma\nu})\end{aligned} \qquad [3.363]$$

and the spacetime Ricci tensor is

$$\boldsymbol{R}_{\nu\lambda} = R^\mu_{\nu\mu\lambda} = \frac{1}{2}(\partial_\mu\partial_\nu\chi^\mu_\lambda - \partial_\lambda\partial_\nu\chi^\mu_\mu - \partial_\mu\partial^\mu\chi_{\lambda\nu} + \partial_\lambda\partial^\mu\chi_{\mu\nu}) \qquad [3.364]$$

We see form this that $\alpha'\boldsymbol{R}_{\mu\nu}$ gives exactly the χ dependent part in (3.7.13a)

The second part is more involved. We will show how the H^2 term appears in $\beta^G_{\mu\nu}$. This is actually the subject of exercise 3.11 and is quite instructive to show here. This solution is taken from Matthew Headrick who has published the solutions to about half of the exercises of Polchinki's book in arXiv:0812.4408 [hep-th].

We start by a simplification. Since our interest is in the H^2 term we might as well take $G_{\mu\nu}$ to be constant, Φ to vanish and $B_{\mu\nu}$ to be linear in X. This means that we can write $B_{\mu\nu} = b_{\omega\mu\nu}X^\omega$ for some constant $b_{\omega\mu\nu}$. The antisymmetric part of the non-linear sigma model action then becomes

$$S^{\mathrm{A}}_\sigma = \int d^2\sigma\,\sqrt{g}\,i\epsilon^{ab} b_{\omega\mu\nu} X^\omega \partial_a X^\mu \partial_b X^\nu \qquad [3.365]$$

At the end of this section, we will argue that it is only this linear contribution that we need to be concerned about in checking the Weyl invariance. Let us now consider this, partially integrate and use the fact that $\partial_a(\sqrt{g}\epsilon^{bc}) = 0$,

$$\sqrt{g}\,\epsilon^{ab} X^\mu \partial_a X^\nu \partial_b X^\omega = \partial^a(\epsilon^{ab}\sqrt{g}\,X^\mu\partial_a X^\nu\partial_b X^\omega) - X^\nu\partial_a(\epsilon^{ab}\sqrt{g}\,X^\mu\partial_b X^\omega) \qquad [3.366]$$

The first term vanishes as it is a boundary term and in the second term we use $\partial_a(\epsilon^{ab}\sqrt{g}) = 0$ and the fact that the $\partial_a\partial_b X^\omega$ vanishes when contracted with ϵ^{ab}. We do the same trick again

$$\epsilon^{ab}\sqrt{g}\,X^\mu\partial_a X^\nu\partial_b X^\omega = -\,\epsilon^{ab}\sqrt{g}\,X^\nu\partial_a X^\mu\partial_b X^\omega = +\epsilon^{ab}\sqrt{g}\,\partial_b X^\nu\partial_a X^\mu X^\omega \qquad [3.367]$$

We can therefore write

$$\begin{aligned} S_\sigma^{\mathrm{A}} &= \int d^2\sigma\,\sqrt{g}\,i\epsilon^{ab}b_{\omega\mu\nu}\frac{1}{3}(X^\omega\partial_a X^\mu\partial_b X^\nu + X^\mu\partial_a X^\nu\partial_b X^\omega + X^\nu\partial_a X^\omega\partial_b X^\mu) \\ &= \int d^2\sigma\,\sqrt{g}\,i\epsilon^{ab}\frac{1}{3}(b_{\omega\mu\nu} + b_{\mu\nu\omega} + b_{\nu\omega\mu})X^\omega\partial_a X^\mu\partial_b X^\nu \end{aligned} \qquad [3.368]$$

In the linear approximation, the field strength is then given by

$$\begin{aligned} H_{\omega\mu\nu} &= \partial_\omega(b_{\sigma\mu\nu}X^\sigma) + \partial_\mu(b_{\sigma\nu\omega}X^\sigma) + \partial_\nu(b_{\sigma\omega\mu}X^\sigma) \\ &= b_{\sigma\mu\nu}\delta^\sigma_\omega + b_{\sigma\nu\omega}\delta^\sigma_\mu + b_{\sigma\omega\mu}\delta^\sigma_\nu = b_{\omega\mu\nu} + b_{\mu\nu\omega} + b_{\nu\omega\mu} \end{aligned} \qquad [3.369]$$

Therefore

$$S_\sigma^{\mathrm{A}} = \int d^2\sigma\,\sqrt{g}\,i\epsilon^{ab}\frac{1}{3}H_{\omega\mu\nu}X^\omega\partial_a X^\mu\partial_b X^\nu \qquad [3.370]$$

In this approximation the non-linear sigma model becomes

$$S_\sigma = \int d^2\sigma\,\sqrt{g}\,\left(G_{\mu\nu}g^{ab}\partial_a X^\mu\partial_b X^\nu + \frac{i}{3}\epsilon^{ab}H_{\omega\mu\nu}X^\omega\partial_a X^\mu\partial_b X^\nu\right) \qquad [3.371]$$

Now go to the conformal gauge and use complex coordinates. Remember that

$$\begin{aligned} \sqrt{g}\,g^{ab}\partial_a X^\mu\partial_b X^\nu &= 4(\partial X^\mu\bar\partial X^\nu + \bar\partial X^\mu\partial X^\nu) \\ \sqrt{g}\,\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu &= -\,4i(\partial X^\mu\bar\partial X^\nu - \bar\partial X^\mu\partial X^\nu) \end{aligned} \qquad [3.372]$$

plus $d^2z = 2d^2\sigma$ and we can write the action as $S_\sigma = S_0 +_i$ with

$$\begin{aligned} S_0 &= \frac{1}{2\pi\alpha'}G_{\mu\nu}\int d^2z\,\partial X^\mu\bar\partial X^\nu \\ S_i &= \frac{1}{6\pi\alpha'}H_{\omega\mu\nu}\int d^2z\,X^\omega\partial X^\mu\bar\partial X^\nu \end{aligned} \qquad [3.373]$$

S_0 is the the action for the free theory and S_i is the action for the interaction, which we will treat as a perturbation. We can expand the expectation value of some operator $\mathcal{O}(X)$ in the path integral formalism as

$$\begin{aligned} \langle\mathcal{O}(X)\rangle &= \int[dX]\,\mathcal{O}(X)e^{-S_0-S_i} = \int[dX]\,\mathcal{O}(X)e^{-S_0}\left(1 - S_i + \frac{1}{2}S_i^2 + \cdots\right) \\ &= \langle\mathcal{O}(X)\rangle_0 - \langle S_i\mathcal{O}(X)\rangle_0 + \frac{1}{2}\langle S_i^2\mathcal{O}(X)\rangle_0 \end{aligned} \qquad [3.374]$$

where $\langle\cdots\rangle_0$ denotes the expectation value taken in the free theory. We have already calculated the Weyl variation of the first two terms. The Weyl variation of the first term leads to the Weyl anomaly term proportional to $D-26$, whilst the Weyl variation of the second term gives a term linear in $H_{\omega\mu\nu}$, precisely the term that we find in $\beta^B_{\mu\nu}$ in (3.7.13b). It is the Weyl variation of the third term, that is quadratic in H, that interests us and in particular the part proportional to $\int d^2z\,\langle\partial X^\mu\bar{\partial}X^\nu\cdots\rangle_0$ as that is the part that will contribute to $\beta^G_{\mu\nu}$. Focussing on this term

$$\begin{aligned}\frac{1}{2}\langle S_i^2\cdots)\rangle_0 = &\frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\,d^2z' \\ &\times\left\langle : X^\omega(z,\bar{z})\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z}) : : X^{\omega'}(z',\bar{z}')\partial' X^{\mu'}(z')\bar{\partial}' X^{\nu'}(\bar{z'})\cdots\right\rangle_0\end{aligned} \quad [3.375]$$

We have written out the normal product signs explicitly and also notice that X is neither holomorphic nor antiholomorphic, but depends on z and $\bar{z}$. To calculate the Weyl variation of this, we will only need the singular parts of the OPE. Indeed the nonsingular parts becomes zero after the integration. In working out the OPE we need to identify the singular term with in the numerator a $\partial X\bar{\partial}X$ as that is the structure we are looking for. Hence we need to perform two cross-contractions. To calculate how many such terms there are, let us first identify how many possibilities there are to keep one uncontracted factor in each normal order product. This is clearly $3\times 3 = 9$. We then have two contractions to make between twice two factors and this can be done in two ways. In total there are thus $9\times 2 = 18$ possible terms.

Let us work out one such term. The contraction we need to use is from the free theory, i.e.

$$\overbrace{X^\mu(z,\bar{z})X^\nu}(z',\bar{z}') = -\frac{\alpha'}{2}G^{\mu\nu}\ln|z-z'|^2 \quad [3.376]$$

consider the case where we contract $X^\omega(z,\bar{z})$ with $\partial' X^{\mu'}(z')$ and $\bar{\partial}X^\nu(\bar{z})$ with $X^{\omega'}(z',\bar{z}')$:

This term gives

$$
\begin{aligned}
\frac{1}{2}\langle S_i^2 \cdots\rangle_0^{(1)} =& \frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\, d^2z' \\
&\times X^{\omega}(z,\bar{z})\partial' X^{\mu'}(z')\;\bar{\partial}X^{\nu}(\bar{z})X^{\omega'}(z',\bar{z}')\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu'}(\bar{z}') : \cdots\right\rangle_0 \\
=& \frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\, d^2z' \left(-\frac{\alpha'}{2}G^{\omega\mu'}\partial'\ln|z-z'|^2\right) \\
&\times\left(-\frac{\alpha'}{2}G^{\nu\omega'}\bar{\partial}\ln|z-z'|^2\right)\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu'}(\bar{z}') :\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H_{\omega\mu\nu}H_{\omega'\mu'\nu'}G^{\omega\mu'}G^{\nu\omega'}\int d^2z\, d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu'}(\bar{z}') : \cdots\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H^{\mu'}{}_{\mu\nu}H^{\nu}{}_{\mu'\nu'}\int d^2z\, d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu'}(\bar{z}') : \cdots\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\int d^2z\, d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu}(\bar{z}') : \cdots\right\rangle_0 \quad [3.377]
\end{aligned}
$$

In the last line we have renamed indices and used the symmetry of $H_{\omega\mu\nu}$. It so happens that the 17 other terms give the same contribution. Therefore

$$
\frac{1}{2}\langle S_i^2 \cdots\rangle_0 = -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\int d^2z\, d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^{\mu}(z)\bar{\partial}' X^{\nu}(\bar{z}') : \cdots\right\rangle_0 \quad [3.378]
$$

This integral is logarithmically divergent at $z = z'$. The expectation value has no other singularities as it is normal ordered. We can thus also expand z' around z in the expectation value and find

$$
\frac{1}{2}\langle S_i^2 \cdots\rangle_0 = -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_{\nu}{}^{\lambda\omega}\int d^2z\,\left\langle :\partial X^{\mu}(z)\bar{\partial} X^{\nu}(\bar{z}) : \cdots\right\rangle_0 \int d^2z'\,\frac{1}{|z-z'|^2} \quad [3.379]
$$

We need to regularise the last integral in such a way that it remains diffeomorphism invariant (as we don't want to introduce an anomaly in that symmetry!). The diffeomorphism invariant distance at short distance is given by (3.6.9), i.e $d(z,z') = |z-z'|e^{\omega(z)}$. We can thus introduce a diffeomorphism invariant cut-off in the integral at $|z-z'| = \varepsilon e^{-\omega}$. In polar coordinates the last integral then becomes

$$
\begin{aligned}
\int d^2z'\,\frac{1}{|z-z'|^2} &= \int_{\varepsilon e^{-\omega}}^{\infty} r\,dr\int_0^{2\pi}d\theta\,\frac{1}{r^2} = 2\pi\int_{\varepsilon e^{-\omega}}^{\infty}\frac{dr}{r} \\
&= -2\pi\ln\varepsilon e^{-\omega} + \text{terms independent of }\omega \\
&= -2\pi\ln\varepsilon + 2\pi\omega + \text{terms independent of }\omega \quad [3.380]
\end{aligned}
$$

We need the Weyl variation of the expectation value, i.e. the transformation under a change $\omega \to \omega + \delta\omega$. Thus

$$\begin{aligned}\delta_{\mathrm{W}}\frac{1}{2}\langle S_i^2\cdots\rangle_0 &= -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,\big\langle :\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z}):\cdots\big\rangle_0 \times 2\pi\delta\omega(z)\\ &= -\frac{1}{8\pi}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,\delta\omega(z)\,\big\langle :\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z}):\cdots\big\rangle_0 \qquad [3.381]\end{aligned}$$

Rewriting this in a worldsheet covariant way and taking into account that an expectation value in the free theory will only differ by higher order terms to an expectation in the full theory, we can write

$$\begin{aligned}\delta_{\mathrm{W}}\frac{1}{2}\langle S_i^2\cdots\rangle_0 &= -\frac{1}{16\pi}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)g^{ab}\,\langle :\partial_aX^\mu\partial_bX^\nu:\cdots\rangle\\ &= -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\langle T^{[H^2]}{}_a{}^a\cdots\rangle \qquad [3.382]\end{aligned}$$

with

$$T^{[H^2]}{}_a{}^a = \frac{1}{8}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} \qquad [3.383]$$

According to the definition (3.7.12) this gives a contribution to $\beta^G_{\mu\nu}$ of the form

$$\beta^G_{\mu\nu} = \cdots - \frac{\alpha'}{4}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} \qquad [3.384]$$

Adding to this result the fact that the we already showed how the spacetime Ricci scalar appears and that we can replace the partial derivatives by covariant derivatives, we recover (3.7.14a).

In the above derivation we assumed that $B_{\mu\nu}$ was linear in X. Let us discuss what happens if we drop this requirement. Clearly a constant term will not impact the discussion as this reverts us back to the original antisymmetric part of the vertex operator. Recall what happened with our linear term. We basically had to work out an OPE of the symbolic form $(X\partial X\bar{\partial}X)(X\partial X\bar{\partial}X)$. In order to be left with a product $\partial X\bar{\partial}X$ we need the contractions $\overbracket{X\bar{\partial}X}\ \partial\overbracket{XX}$. This gives us a $z^{-1}\bar{z}^{-1} = |z|^{-2} = r^{-2}$ which led to the integral $\int dr/r$ giving us the logarithmic divergence we regularised.

Let us now assume that $B_{\mu\nu}$ depends on ∂X. Note that this is now a worldsheet derivative. We start with just one ∂X. We then have to take two contractions between $(\partial X\partial X\bar{\partial}X)(\partial X\partial X\bar{\partial}X)$. But if we are to be left with a product $\partial X\bar{\partial}X$, this means that we need to take two contractions between $\partial X\bar{\partial}X$ and $\partial X\partial X$. Clearly one of them needs to be $\partial\overbracket{X\bar{\partial}X}$ which vanishes. If $B_{\mu\nu}$ contains a mixture of $\partial^m X\,\bar{\partial}^n X$ we have the same issue. Leaving out a $\partial X\bar{\partial}X$ from the product $(\partial^m X\,\bar{\partial}^n X\partial X\bar{\partial}X)(\partial^m X\,\bar{\partial}^n X\partial X\bar{\partial}X)$ leaves a different number of ∂X and $\bar{\partial}X$ on both sides to be contracted, and so we are necessarily left with a contraction that vanishes. Thus, terms of the form $\partial^m X\,\bar{\partial}^n X$ do not contribute and cannot cause an anomaly.

Let us now assume that $B_{\mu\nu}$ is of the $(n+1)$-th order in X. We then need to have contractions between $(X^{n+1}\partial X\bar{\partial}X)(X^{n+1}\partial X\bar{\partial}X)$. Leaving out a product $\partial X\bar{\partial}X$ we are left with one contraction $\overbracket{X\bar{\partial}X}$, one contraction $\partial\overbracket{XX}$ and n contractions $\overbracket{XX}$, thus an integral of the form

$$\begin{aligned}\int d^2z\,\frac{1}{|z|^2}\left(\ln|z|^2\right)^n &= 2\pi\int_{\varepsilon e^{-\omega}} dr\,\frac{\ln^n r^2}{r} = 2\pi\times\frac{\ln^{n+1} r^2}{2(n+1)}\Big|_{\varepsilon e^{-\omega}} \\ &= -\frac{\pi}{n+1}\left[\ln\left(\varepsilon e^{-\omega}\right)^2\right]^{n+1} + \text{terms independent of } \omega \\ &= -\frac{2^{n+1}\pi}{n+1}(\ln\varepsilon-\omega)^{n+1} + \text{terms independent of } \omega \end{aligned} \qquad [3.385]$$

This will result in a contribution to the Weyl variation of the form

$$2^{n+1}\pi(\ln\varepsilon-\omega)^2\delta\omega \sim 2^{n+1}\pi\delta\omega\ln^n\varepsilon \qquad [3.386]$$

In the last equation we have pushed the curvature to infinity, by setting $\omega = 0$. We see that we recover the previous, linear, result if we set $n = 0$, as we should. The upshot of all this is that that if $n \geq 1$, i.e. if we have more than one factor of X, then the possible Weyl anomaly terms comes coupled with the divergence and can be removed a the counterterm. We will leave it as an exercise to the reader to show that a general term of the form $X^k(\partial^m X)^{k_m}(\bar{\partial}^n X)^{k_n}$ can similarly be removed by a counterterm, unless it is only linear in X.

3.52 p 113: Eq (3.7.19) The β Function for the Linear Dilaton Model

Setting $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\Phi = V_\mu X^\mu$, (3.7.14c) becomes

$$\beta^\Phi = \frac{D-26}{6} - 0 + \alpha'\boldsymbol{\nabla}_\mu\Phi\boldsymbol{\nabla}^\mu\Phi + 0 \qquad [3.387]$$

As Φ is a scalar field w.r.t. spacetime we have $\boldsymbol{\nabla}_\mu\Phi = \partial_\mu\Phi = V_\mu$ and thus we have

$$\beta^\Phi = \frac{D-26}{6} - 0 + \alpha' V_\mu V^\mu \qquad [3.388]$$

Requiring the β function to vanish give (3.7.19).

3.53 p 114: Eq (3.7.20) The Effective Spacetime Action

We do this in separate steps, starting with the variation of the dilaton field

$$
\begin{aligned}
\delta_\Phi \boldsymbol{S} &= \frac{1}{2\kappa_0^2}\int d^D x\, \delta_\Phi \left\{ \sqrt{-G}\, e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\Big]\right\} \\
&= \frac{1}{2\kappa_0^2}\int d^D x \left\{ e^{-2\Phi}(-2\delta\Phi)\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\Big] \right. \\
&\qquad\qquad \left. + \sqrt{-G}\, e^{-2\Phi} 8\partial_\mu\Phi\partial^\mu\delta\Phi \right\} \\
&= \frac{1}{2\kappa_0^2}\int d^D x \left\{ -2e^{-2\Phi}\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\Big] \right. \\
&\qquad\qquad \left. - 8\partial^\mu\Big[\sqrt{-G}\, e^{-2\Phi}\partial_\mu\Phi\Big]\right\}\delta\Phi
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\kappa_0^2}\int d^D x\, e^{-2\Phi}\sqrt{-G}\,\bigg[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi \\
&\qquad + 4(-G)^{-1/2}\big(\partial^\mu\sqrt{-G}\big)\partial_\mu\Phi + 4\big(-2\partial^\mu\Phi\big)\partial_\mu\Phi + 4\partial^\mu\partial_\mu\Phi\bigg]\delta\Phi \\
&= -\frac{1}{\kappa_0^2}\int d^D x\, e^{-2\Phi}\sqrt{-G}\,\bigg[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - 4\partial_\mu\Phi\partial^\mu\Phi \\
&\qquad + 2G^{\nu\sigma}\partial^\mu G_{\nu\sigma}\partial_\mu\Phi + 4\partial_\mu\partial^\mu\Phi\bigg]\delta\Phi \\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^D x\, e^{-2\Phi}\sqrt{-G}\, 2\delta\Phi \left\{ -4\bigg[\frac{D-26}{6} + \alpha'\partial_\mu\Phi\partial^\mu\Phi - \frac{\alpha'}{2}\partial_\mu\partial^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\bigg] \right. \\
&\qquad \left. + \alpha'\boldsymbol{R} + 2\alpha'\partial^\mu\partial_\mu\Phi - \frac{\alpha'}{4}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right\} \\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^D x\, e^{-2\Phi}\sqrt{-G}\, 2\delta\Phi\Big(-4\beta^\Phi + \beta^{G\,\mu}_{\ \ \mu}\Big) \qquad [3.389]
\end{aligned}
$$

In the last line we have replaced the ordinary derivative by the covariant derivative, used the fact that the spacetime metric is covariantly constant, $\nabla_\mu G^{\mu\nu} = 0$ and used the definition in (3.7.14).

Let us now consider

$$\begin{aligned}
\delta_B \boldsymbol{S} &= \frac{1}{2\kappa_0^2}\int d^Dx\,\delta_B\left\{\sqrt{-G}\,e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\right\}\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\left(-\frac{1}{6}H^{\mu\nu\lambda}\delta_B H_{\mu\nu\lambda}\right) = -\frac{1}{4\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}H^{\mu\nu\lambda}\delta_B\partial_\mu B_{\nu\lambda}\\
&= \frac{1}{4\kappa_0^2}\int d^Dx\,\partial_\mu\left(\sqrt{-G}\,e^{-2\Phi}H^{\mu\nu\lambda}\right)\delta B_{\nu\lambda}\\
&= \frac{1}{4\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\Big(+\frac{1}{2}G^{\nu\sigma}\partial_\mu G_{\nu\sigma}H^{\mu\nu\lambda}-2\partial_\mu\Phi H^{\mu\nu\lambda}+\partial_\mu H^{\mu\nu\lambda}\Big)\delta B_{\nu\lambda}\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\Big(-\frac{\alpha'}{4}G^{\nu\sigma}\partial_\mu G_{\nu\sigma}H^{\mu\nu\lambda}+\alpha'\partial_\mu\Phi H^{\mu\nu\lambda}-\frac{\alpha'}{2}\partial_\mu H^{\mu\nu\lambda}\Big)\delta B_{\nu\lambda}\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\beta^B_{\mu\nu}\delta B_{\nu\lambda}
\end{aligned} \qquad [3.390]$$

Here also, we have replaced the ordinary derivative by the covariant derivative, used the fact that the spacetime metric is covariantly constant, $\nabla_\mu G^{\mu\nu}=0$ and used the definition in (3.7.14).

Finally we consider

$$\delta_G \boldsymbol{S} = \frac{1}{2\kappa_0^2}\int d^Dx\,\delta_G\left\{\sqrt{-G}\,e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\right\} \qquad [3.391]$$

Recall that we already worked out the variation of the Einstein-Hilbert action, see [1.35]

$$\delta_G\int d^Dx\,\sqrt{-G}\,\boldsymbol{R} = \int d^Dx\,\sqrt{-G}\left(\boldsymbol{R}_{\mu\nu}-\frac{1}{2}G_{\mu\nu}\boldsymbol{R}\right)\delta G^{\mu\nu} \qquad [3.392]$$

But we need to be careful as we have an extra factor $e^{-2\Phi}$. Just as for [1.35] we split the calculation

$$\delta_G\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} = \sum_{a=1}^{4}\delta_G I_a \qquad [3.393]$$

with

$$\begin{aligned}
\delta_G I_1 &= \int d^Dx\,(\delta_G\sqrt{-G})\,e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} = \int d^Dx\,\frac{1}{2}\sqrt{-G}G^{\mu\nu}\delta G_{\mu\nu}e^{-2\Phi}G^{\rho\sigma}\boldsymbol{R}_{\rho\sigma}\\
&= \int d^Dx\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{2}G^{\mu\nu}\boldsymbol{R}\Big]\delta G_{\mu\nu}
\end{aligned} \qquad [3.394]$$

Next,

$$\delta_G I_2 = \int d^D x \sqrt{-G}\,(\delta_G e^{-2\Phi}) G^{\mu\nu} \boldsymbol{R}_{\mu\nu} = 0 \qquad [3.395]$$

Third,

$$\begin{aligned}\delta_G I_3 &= \int d^D x \sqrt{-G}\, e^{-2\Phi} (\delta_G G^{\mu\nu}) \boldsymbol{R}_{\mu\nu} = \int d^D x \sqrt{-G}\, e^{-2\Phi} (-G^{\mu\rho} G^{\nu\sigma} \delta G_{\rho\sigma}) \boldsymbol{R}_{\mu\nu} \\ &= \int d^D x \sqrt{-G} e^{-2\Phi} \Big[- \boldsymbol{R}^{\mu\nu} \Big] \delta G_{\mu\nu} \end{aligned} \qquad [3.396]$$

Finally, and this is where the change occurs compared to the Einstein-Hilbert action

$$\delta_G I_4 = \int d^D x \sqrt{-G}\, e^{-2\Phi} G^{\mu\nu} \delta_G \boldsymbol{R}_{\mu\nu} = \qquad [3.397]$$

We already calculated the variation of the Ricci tensor in [1.31] $\delta \boldsymbol{R}_{\mu\nu} = \delta \boldsymbol{R}^{\rho}_{\mu\rho\nu} = \nabla_\rho \delta \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho_{\mu\rho}$. Thus

$$\begin{aligned}\delta_G I_4 &= \int d^D x \sqrt{-G}\, e^{-2\Phi} G^{\mu\nu} (\nabla_\rho \delta_G \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta_G \Gamma^\rho_{\mu\rho}) \\ &= \int d^D x\, e^{-2\Phi} \big[\sqrt{-G}\, \nabla_\rho (G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\nu}) - \sqrt{-G}\, \nabla_\nu (G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\rho}) \big] \\ &= \int d^D x\, e^{-2\Phi} \big[\partial_\rho (\sqrt{-G}\, G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\nu}) - \partial_\nu (\sqrt{-G}\, G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\rho}) \big] \end{aligned} \qquad [3.398]$$

In the case of the Einstein-Hilbert action, i.e. $\Phi = 0$, this is a total derivative and vanishes. This times this is not the case as we get a contribution from the dilaton field upon partial integration

$$\delta_G I_4 = 2 \int d^D x\, e^{-2\Phi} \sqrt{-G}\, G^{\mu\nu} \big[\partial_\rho \Phi \delta_G \Gamma^\rho_{\mu\nu} - \partial_\nu \Phi \delta_G \Gamma^\rho_{\mu\rho} \big] \qquad [3.399]$$

Unfortunately, this time we need to work out the variations of the connections

$$\begin{aligned}\delta_G \Gamma^\rho_{\mu\nu} &= \delta_G \frac{1}{2} G^{\rho\sigma} (\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu}) \\ &= \frac{1}{2} \Big[- G^{\rho\kappa} G^{\sigma\tau} \delta G_{\kappa\tau} (\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu}) \\ &\qquad + G^{\rho\sigma} (\partial_\mu \delta G_{\sigma\nu} + \partial_\nu \delta G_{\sigma\mu} - \partial_\sigma \delta G_{\mu\nu}) \Big] \end{aligned} \qquad [3.400]$$

Recall that by covariance we can, to that order, replace all partial derivatives by covariant derivatives. This means that we can ignore the first line. We then split the calculation in two

$$\begin{aligned}\delta_G I_{4a} =& 2\int d^Dx\, e^{-2\Phi}\sqrt{-G}\, G^{\mu\nu}\partial_\rho\Phi\frac{1}{2}G^{\rho\sigma}(\partial_\mu\delta G_{\sigma\nu}+\partial_\nu\delta G_{\sigma\mu}-\partial_\sigma\delta G_{\mu\nu})\\ =& -\int d^Dx\,\sqrt{-G}\,G^{\mu\nu}G^{\rho\sigma}\Big[\partial_\mu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\nu}+\partial_\nu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\mu}\\ &\qquad -\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\mu\nu}\Big]\\ =& -\int d^Dx\,\sqrt{-G}\,G^{\mu\nu}G^{\rho\sigma}\Big[2\partial_\mu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\nu}-\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\mu\nu}\Big]\\ =& -\int d^Dx\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)(2G^{\sigma\nu}G^{\rho\mu}-G^{\mu\nu}G^{\rho\sigma})\delta G_{\mu\nu}\end{aligned} \qquad [3.401]$$

The second part is

$$\begin{aligned}\delta_G I_{4b} =& -2\int d^Dx\, e^{-2\Phi}\sqrt{-G}\, G^{\mu\nu}\partial_\nu\Phi\frac{1}{2}G^{\rho\sigma}(\partial_\mu\delta G_{\sigma\rho}+\partial_\rho\delta G_{\sigma\mu}-\partial_\sigma\delta G_{\rho\mu})\\ =& \int d^Dx\,\sqrt{-G}\,G^{\mu\nu}G^{\rho\sigma}\Big[\partial_\mu\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\sigma}+\partial_\rho\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\sigma\mu}\\ &\qquad -\partial_\sigma\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\mu}\Big]\\ =& \int d^Dx\,\sqrt{-G}\,G^{\mu\nu}G^{\rho\sigma}\partial_\mu\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\sigma}\\ =& \int d^Dx\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)G^{\mu\nu}G^{\rho\sigma}\delta G_{\mu\nu}\end{aligned} \qquad [3.402]$$

Therefore

$$\begin{aligned}\delta_G I_4 =& \int d^Dx\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)(-2G^{\sigma\nu}G^{\rho\mu}+G^{\mu\nu}G^{\rho\sigma}+G^{\mu\nu}G^{\rho\sigma})\delta G_{\mu\nu}\\ =& \int d^Dx\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)2(G^{\mu\nu}G^{\rho\sigma}-G^{\sigma\nu}G^{\rho\mu})\delta G_{\mu\nu}\\ =& \int d^Dx\,\sqrt{-G}\,e^{-2\Phi}(-2\partial_\sigma\Phi\partial_\rho\Phi+\partial_\rho\partial_\sigma\Phi)\,2(G^{\mu\nu}G^{\rho\sigma}-G^{\sigma\nu}G^{\rho\mu})\delta G_{\mu\nu}\\ =& \int d^Dx\,\sqrt{-G}\,e^{-2\Phi}(-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+4\partial^\mu\Phi\partial^\nu\Phi+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi)\delta G_{\mu\nu}\end{aligned} \qquad [3.403]$$

Adding the four pieces together we find

$$\begin{aligned}\delta_G\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} =& \int d^Dx\sqrt{-G}e^{-2\Phi}\Big(\frac{1}{2}G^{\mu\nu}\boldsymbol{R}-\boldsymbol{R}^{\mu\nu}\\ &-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+4\partial^\mu\Phi\partial^\nu\Phi+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi\Big)\delta G_{\mu\nu}\end{aligned} \qquad [3.404]$$

As expected this gives back the Einstein equations when we have a constant Φ, but we see that the dilaton field gives a correction that includes its derivative only.

There is another thing that we need to be careful about as well. Any indices upstairs have been raised via the spacetime "metric" $G^{\mu\nu}$ so they also carry a metric dependence. For example

$$\begin{aligned}\delta_G H_{\mu\nu\lambda}H^{\mu\nu\lambda} &= H_{\mu\nu\lambda}\delta_G(G^{\mu\sigma}G^{\nu\rho}G^{\lambda\kappa}H_{\sigma\rho\kappa}) = 3H_{\mu\nu\lambda}G^{\mu\sigma}G^{\nu\rho}H_{\sigma\rho\kappa}\delta G^{\lambda\kappa}\\ &= -3H_{\mu\nu\lambda}G^{\mu\sigma}G^{\nu\rho}H_{\sigma\rho\kappa}G^{\lambda\tau}G^{\kappa\eta}\delta G_{\tau\eta}\\ &= -3H^{\sigma\rho\tau}H^{\eta}{}_{\rho\sigma}\delta G_{\tau\eta} = -3H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\delta G_{\mu\nu}\end{aligned} \tag{3.405}$$

Similarly

$$\delta_G\partial_\mu\Phi\partial^\mu\Phi = \delta_G G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = -G^{\mu\rho}G^{\nu\sigma}\partial_\mu\Phi\partial_\nu\Phi\delta G_{\rho\sigma} = -\partial^\mu\Phi\partial^\nu\Phi\delta G_{\mu\nu} \tag{3.406}$$

So we have

$$\begin{aligned}\delta_G\boldsymbol{S} = \frac{1}{2\kappa_0^2}\int d^Dx\, e^{-2\Phi}\Bigg\{&\delta_G\big(\sqrt{-G}\big)\Big[-\frac{2(D-26)}{3\alpha'} - \frac{1}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda} + 4\partial_\sigma\Phi\partial^\sigma\Phi\Big]\\ &+\sqrt{-G}\Big(-\boldsymbol{R}^{\mu\nu} + \frac{1}{2}G^{\mu\nu}\boldsymbol{R} - 4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu} + 4\partial^\mu\Phi\partial^\nu\Phi + 2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu} - 2\partial^\mu\partial^\nu\Phi\\ &\qquad + \frac{1}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma} - 4\partial^\mu\Phi\partial^\nu\Phi\Big)\delta G_{\mu\nu}\Bigg\}\\ = \frac{1}{2\kappa_0^2}\int d^Dx\sqrt{-G}\,e^{-2\Phi}\Bigg\{&\frac{1}{2}G^{\mu\nu}\delta G_{\mu\nu}\Big[-\frac{2(D-26)}{3\alpha'} - \frac{1}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda} + 4\partial_\sigma\Phi\partial^\sigma\Phi\Big]\\ &+\Big(-\boldsymbol{R}^{\mu\nu} + \frac{1}{2}G^{\mu\nu}\boldsymbol{R} - 4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu} + 2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu} - 2\partial^\mu\partial^\nu\Phi + \frac{1}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\Big)\delta G_{\mu\nu}\Bigg\}\\ = -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\sqrt{-G}\,e^{-2\Phi}&\Big[\alpha'\boldsymbol{R}^{\mu\nu} + 2\alpha'\partial^\mu\partial^\nu\Phi - \frac{\alpha'}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\\ &-\frac{1}{2}G^{\mu\nu}\Big(\alpha' G^{\mu\nu}\boldsymbol{R} + 4\alpha'\partial_\sigma\partial^\sigma\Phi - \frac{\alpha'}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda} - \frac{2(D-26)}{3} - 4\alpha'\partial_\sigma\Phi\partial^\sigma\Phi\Big)\Big]\end{aligned} \tag{3.407}$$

We can rewrite the term between brackets as

$$\mathfrak{B}^{\mu\nu} = \beta^{G\,\mu\nu} - \frac{1}{2}G^{\mu\nu}(\beta^{G\,\sigma}{}_{\sigma} - 4\beta^{\Phi}) \tag{3.408}$$

Indeed

$$\begin{aligned}\mathfrak{B}^{\mu\nu} &= \alpha'\boldsymbol{R}^{\mu\nu} + 2\alpha'\nabla^\mu\nabla^\nu\Phi - \frac{\alpha'}{4}H^{\mu\rho\sigma}H^\nu{}_{\rho\sigma} - \frac{1}{2}G^{\mu\nu}\Big(\alpha'\boldsymbol{R} + 2\alpha'\nabla^2\Phi - \frac{\alpha'}{4}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma} \\ &\qquad - \frac{2(D-26)}{3} + 2\alpha'\nabla^2\Phi - 4\alpha'\nabla_\sigma\Phi\nabla^\sigma\Phi + \frac{\alpha'}{6}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma}\Big) \\ &= \alpha'\boldsymbol{R}^{\mu\nu} + 2\alpha'\nabla^\mu\nabla^\nu\Phi - \frac{\alpha'}{4}H^{\mu\rho\sigma}H^\nu{}_{\rho\sigma} - \frac{1}{2}G^{\mu\nu}\Big(\alpha'\boldsymbol{R} + 4\alpha'\nabla^2\Phi - \frac{\alpha'}{12}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma} \\ &\qquad - \frac{2(D-26)}{3}i - 4\alpha'\nabla_\sigma\Phi\nabla^\sigma\Phi\Big)\end{aligned} \tag{3.409}$$

and we see that if we replace the covariant derivatives by partial derivatives we do indeed find the expression between brackets in [3.407].

3.54 p 114: Eq (3.7.23) The Ricci Scalar after a Weyl Transformation

I expect most of you are expecting yet another long and tedious calculation. But if that is the case then you have a short memory. Indeed we have used this formula many times for the specific case of two dimensions, but when we originally derived it in chapter one we did so for general dimensions D, see [1.21]. Happy and light-footed we move on to the next challenge.

3.55 p 114: Eq (3.7.25) The Space Time Action with Einstein Metric

Let us first work out the pre-factor in (3.7.20). We have

$$G_{\mu\nu} = e^{-2\omega}\tilde{G}_{\mu\nu} = e^{-4(\Phi_0-\Phi)/(D-2)}\tilde{G}_{\mu\nu} = e^{4\tilde{\Phi}/(D-2)}\tilde{G}_{\mu\nu} \tag{3.410}$$

implying that

$$\sqrt{-G} = e^{2D\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}} \tag{3.411}$$

so that

$$\sqrt{-G}e^{-2\Phi} = e^{2D\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,e^{-2(\Phi_0+\tilde{\Phi})} = e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}} \tag{3.412}$$

We will absorb the $e^{-2\Phi_0}$ into the κ_0. Let us now do term by term in (3.7.20)

$$\begin{aligned}\mathfrak{T}_1 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'}\right] \\ &= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\,\left[-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)}\right]\end{aligned} \tag{3.413}$$

Next, consider the term with the curvature. Using (3.7.23)

$$\begin{aligned}\mathfrak{T}_2 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\boldsymbol{R}\\ &= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\left[e^{2\omega}\tilde{\boldsymbol{R}}+2(D-1)\nabla^2\omega+(D-2)(D-1)\partial\omega\cdot\partial\omega\right]\\ &= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\left[e^{-4\tilde{\Phi}/(D-2)}\tilde{\boldsymbol{R}}-\frac{4(D-1)}{D-2}\nabla^2\tilde{\Phi}+\frac{4(D-1)}{D-2}\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}\right]\\ &= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left\{\tilde{\boldsymbol{R}}+4e^{4\tilde{\Phi}/(D-2)}\left[-\frac{D-1}{D-2}\nabla^2\tilde{\Phi}+\frac{D-1}{D-2}\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}\right]\right\}\end{aligned} \quad [3.414]$$

We need one more step. We write

$$\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}=G^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}=e^{-4\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}=e^{-4\tilde{\Phi}/(D-2)}\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi} \quad [3.415]$$

Thus

$$\mathfrak{T}_2=\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[\tilde{\boldsymbol{R}}+4\frac{(D-1)}{D-2}\left(-e^{4\tilde{\Phi}/(D-2)}\nabla^2\tilde{\Phi}+\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right)\right] \quad [3.416]$$

Next, the term with the antisymmetric field strength

$$\begin{aligned}\mathfrak{T}_3 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\left(-\frac{1}{12}H_{\mu\nu\lambda}G^{\mu\rho}G^{\nu\sigma}G^{\lambda\kappa}H_{\rho\sigma\kappa}\right)\\ &= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,\left(-\frac{1}{12}H_{\mu\nu\lambda}e^{-12\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\rho}\tilde{G}^{\nu\sigma}\tilde{G}^{\lambda\kappa}\tilde{H}_{\rho\sigma\kappa}\right)\\ &= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\,\left[-\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}_{\rho\sigma\kappa}\right]\end{aligned} \quad [3.417]$$

Finally, the term with the dilaton field

$$\begin{aligned}\mathfrak{T}_4 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\\ &= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,4e^{-4\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}\\ &= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[4\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right]\end{aligned} \quad [3.418]$$

Adding the four contributions we find

$$\begin{aligned}\sum_{a=1}^{4}\mathfrak{T}_a=\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\Big[&-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)}+\tilde{\boldsymbol{R}}-\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}_{\rho\sigma\kappa}\\ &+4\frac{(D-1)}{D-2}\left(-e^{4\tilde{\Phi}/(D-2)}\nabla^2\tilde{\Phi}+\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right)+4\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\Big]\end{aligned} \quad [3.419]$$

We seem to be close but not exactly equal to (3.725). What saves us, as usual, is a partial integration. Indeed

$$\begin{aligned}
\mathfrak{X} &= \int d^D x\, \sqrt{-\tilde{G}}\, e^{4\tilde{\Phi}/(D-2)} \nabla^2 \tilde{\Phi} = \int d^D x\, e^{-2D\tilde{\Phi}/(D-2)} \sqrt{-G}\, e^{4\tilde{\Phi}/(D-2)} \nabla_\mu \partial^\mu \tilde{\Phi} \\
&= \int d^D x\, e^{-2\tilde{\Phi}} \partial_\mu \left(\sqrt{-G}\, \partial^\mu \tilde{\Phi} \right) = - \int d^D x\, \partial_\mu \left(e^{-2\tilde{\Phi}} \right) \sqrt{-G}\, \partial^\mu \tilde{\Phi} \\
&= \int d^D x\, e^{-2\tilde{\Phi}} \left(-2 \partial_\mu \tilde{\Phi} \right) \sqrt{-G}\, \partial^\mu \tilde{\Phi} = -2 \int d^D x\, e^{-2\tilde{\Phi}} e^{2D\tilde{\Phi}/(D-2)} \sqrt{-\tilde{G}}\, \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \\
&= 2 \int d^D x\, \sqrt{-\tilde{G}}\, e^{4\tilde{\Phi}/(D-2)} \partial_\mu \tilde{\Phi} G^{\mu\nu} \partial^\nu \tilde{\Phi} = 2 \int d^D x\, \sqrt{-\tilde{G}}\, e^{4\tilde{\Phi}/(D-2)} \partial_\mu \tilde{\Phi} e^{-4\tilde{\Phi}/(D-2)} \tilde{G}^{\mu\nu} \partial_\nu \tilde{\Phi} \\
&= 2 \int d^D x\, \sqrt{-\tilde{G}}\, \partial_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi}
\end{aligned} \qquad [3.420]$$

Note that we had to go back and forth between G and $\tilde{G}$ more than once in this last derivation. This means that we can write the last few terms of [3.419] as

$$\begin{aligned}
&\frac{1}{2\kappa^2} \int d^D x\, \sqrt{-\tilde{G}}\, 4 \left[\frac{D-1}{D-2}(-2+1) + 1 \right] \partial \tilde{\Phi} \cdot \tilde{\partial} \tilde{\Phi} \\
&= \frac{1}{2\kappa^2} \int d^D x\, \sqrt{-\tilde{G}}\, \left[-\frac{4}{D-2} \right] \partial \tilde{\Phi} \cdot \tilde{\partial} \tilde{\Phi}
\end{aligned} \qquad [3.421]$$

Therefore [3.419] becomes, finally,

$$\begin{aligned}
\sum_{a=1}^{4} \mathfrak{T}_a = \frac{1}{2\kappa^2} \int d^D x\, \sqrt{-\tilde{G}} \Big[&- \frac{2(D-26)}{3\alpha'} e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}} \\
&- \frac{1}{12} e^{-8\tilde{\Phi}/(D-2)} H_{\mu\nu\lambda} \tilde{H}_{\rho\sigma\kappa} - \frac{4}{D-2} \partial \tilde{\Phi} \cdot \tilde{\partial} \tilde{\Phi} \Big]
\end{aligned} \qquad [3.422]$$

which is exactly (3.7.25).

3.56 Appendix: Almost Complex Structures, Holomorphic Normal Coordinates, Beltrami Equations and all that Stuff

In this appendix we develop the basics of complex structures and the associated transformation rules. This is almost completely taken over from **[LS]**. We refer the reader for more details to that text, in particular section 2. In fact, we will only select those parts we need.

As a change, most of the calculations in this appendix are elementary algebra or analysis and will not be shown in detail. Once again, all errors are solely due to me.

ALMOST COMPLEX STRUCTURES

We are all familiar with defining complex coordinates $z = x + iy$ and $\bar{z} = x - iy$ on a patch of a two dimensional manifold Σ. These complex coordinates satisfy $\bar{\partial} z = \partial \bar{z} = 0$ and are thus holomorphic and anti-holomorphic coordinates respectively. It turns out to be useful to write this in a different way. Introduce

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.423}$$

This matrix satisfies ${I_a}^b {I_b}^c = -\delta_a^c$, in matrix language $I^2 = -\mathbb{1}$, reminding us of $i^2 = -1$. We can now write the holomorphicity condition $\bar{\partial} z = 0$ as

$$\left(\partial_a + i {I_a}^b \partial_b \right) z(\sigma) = 0 \tag{3.424}$$

Let us make three remarks at this point

1. Given a solution $z(\sigma)$ that satisfies the holomorphicity condition [3.424] we can easily construct another solution $f(\sigma)$. Write $f(\sigma) = u(\sigma) + iv(\sigma)$ with u and v real are require f to satisfy the holomorphicity condition. This leads to the Cauchy-Riemann equations

$$\partial_1 u(\sigma) = \partial_2 v(\sigma) \qquad \text{and} \qquad \partial_2 u(\sigma) = -\partial_1 v(\sigma) \tag{3.425}$$

   Conversely, any solution of the Cauchy-Riemann equations leads to a holomorphic function.

2. Our original definition of the complex variable $z = \sigma^1 + i\sigma^2$ is chosen so that if $\sigma = 0$ then also $z = 0$. We could as put the base point point of the holomorphic coordinate system at any other point σ_1 on the patch of the manifold and define the complex coordinates via

$$z_{\sigma_1}(\sigma) = z(\sigma) - z(\sigma_1) \tag{3.426}$$

   The index $_{\sigma_1}$ refers to the fact that the base point is now at σ_1. Obviously, it follows from this definition that

$$z_{\sigma_1}(\sigma_1) = 0 \tag{3.427}$$

   When we write ∂, ∂_z or $\partial_{z_{\sigma_1}}$ we will always mean differentiation w.r.t. z_{σ_1}, unless it is clear from the context or it is explicitly mentioned that it is not the case. Note that this does not necessarily imply that derivatives of $z_{\sigma_1}(\sigma)$ taken at $\sigma = \sigma_1$ are zero. I.e. in general $\partial^n z_{\sigma_1}(\sigma)\big|_{\sigma=\sigma_1} \neq 0$ for $n \geq 1$.

3. We can view ${I_a}^b$ as mapping tangent vectors into tangent vectors. Indeed let v_a be a tangent vector, i.e. $v \in T(\Sigma)$, then ${I_a}^b v_b = w_a \in T(\Sigma)$. Recall that a derivative ∂_a is an element of the tangent space $T(\Sigma)$ and the exterior derivative $d\sigma^a$ is an element of the dual tangent space $T^*(\Sigma)$. The relevance of this remark will become clear later.

It is natural to generalise the above concept and define a generic almost complex structure J as a local function on the manifold $J(\sigma)$ that satisfies[6]

$$ {J_a}^b {J_b}^c = -\delta_a^c \qquad [3.428] $$

Indeed, by performing a coordinate reparametrisation $\sigma \to \sigma'(\sigma)$ the transformed components would become local anyway, ${I_a}^b \to {\tilde{I}_a}{}^b(\sigma)$. We can then define holomorphic coordinates w w.r.t. the almost complex structure J as satisfying

$$ \left(\partial_a + i{J_a}^b\partial_b\right) w(\sigma) = 0 \qquad [3.429] $$

Because ${J_a}^b$ needs to transform covariantly for the above equation to be well-defined, we are automatically led to define a vector-valued differential one form

$$ \mathsf{J} = d\sigma^a {J_a}^b(\sigma)\partial_b \qquad [3.430] $$

We will denote the exterior derivative on Σ by $\mathsf{d} = d\sigma^a\partial_a$. The differential equation [3.429] then takes the index-free form[7]

$$ (\mathsf{d} + i\mathsf{J})\,(w) = 0 \qquad [3.431] $$

[6]When does an almost complex structure loose its predicate “almost"? This is the case when we can find an atlas such that the almost complex structure is constant on all coordinate patches. We can then define complex coordinates with holomorphic transition functions. The almost complex structure is then integrable and promoted to a complex structure and the manifold is called a complex manifold. A necessary and sufficient condition for the integrability of an almost complex structure is the vanishing of the so-called Nijenhuis tensor.

[7]Because $\mathsf{d}^2 = 0$ this implies $\mathsf{dJ} = 0$, which with the decoration of full indices, and contracting with another J becomes

$$ J^d_{ab} \equiv {J_a}^c\partial_c{J_b}^d - {J_b}^c\partial_c{J_a}^d - {J_c}^d\partial_a{J_b}^c + {J_c}^d\partial_b{J_a}^c = 0 $$

J^d_{ab} is the Nijenhuis tensor mentioned in footnote 6 and its can be shown that its vanishing is a necessary and sufficient condition for the almost complex structure to be integrable and hence to be promoted to a complex structure.

METRICS AND RIEMANNIAN MANIFOLDS

Let us now endow the manifold Σ with complex structure[8] J with a metric

$$\mathrm{g}_J = g_{ab}(\sigma) d\sigma^a d\sigma^b \tag{3.432}$$

The index $_J$ is there to remind us of the almost complex structure. A complex manifold with a metric is called a Riemannian manifold. Here we have assumed that we have first given Σ a complex structure J and then constructed a metric g_J on Σ. But we might as well work the other way around. First we define a metric g on Σ and then we define a complex structure J. It is actually easy to construct a complex structure from a metric

$$J_a{}^b = g^{-1/2} g_{ac} \epsilon^{cb} \tag{3.433}$$

Elementary algebra shows that $J^2 = -\mathbb{1}$ as it should. Note, en passant, that the complex structure is invariant under Weyl rescalings $g_{ab} \to e^{2\omega} g_{ab}$.

BELTRAMI EQUATION

In terms of the complex coordinates $z = \sigma^1 + i\sigma^2$ one can rewrite the metric g_J as

$$\mathrm{g}_J = \frac{1}{4}\left(g_{11} - g_{22} - 2ig_{12}\right) dz^2 + \frac{1}{4}\left(g_{11} - g_{22} + 2ig_{12}\right) d\bar{z}^2 + \frac{1}{2}\left(g_{11} + g_{22}\right) dz d\bar{z} \tag{3.434}$$

It is convenient to write the metric as

$$\mathrm{g}_J = \rho \left| dz + \mu_{\bar{z}}{}^z d\bar{z} \right|^2 \tag{3.435}$$

As we are working in Euclidean space $ds^2 \geq 0$ so that ρ is real and positive. With $\mu_{\bar{z}}{}^z$ complex we have three components defining the metric, the same number as g_{ab}. Straightforward algebra leads to

$$\begin{aligned} g_{11} &= \rho \left(1 + |\mu_{\bar{z}}{}^z|^2 + \mu_{\bar{z}}{}^z + \mu_z{}^{\bar{z}}\right) \\ g_{22} &= \rho \left(1 + |\mu_{\bar{z}}{}^z|^2 - \mu_{\bar{z}}{}^z - \mu_z{}^{\bar{z}}\right) \\ g_{12} &= - i\rho \left(\mu_{\bar{z}}{}^z - \mu_z{}^{\bar{z}}\right) \end{aligned} \tag{3.436}$$

where $\mu_z{}^{\bar{z}} = (\mu_{\bar{z}}{}^z)^*$. We can invert these relations to find[9]

$$\begin{aligned} \rho &= \frac{1}{4}\left(\mathrm{tr}\ g_{ab} + 2\sqrt{g}\right) \\ \mu_{\bar{z}}{}^z &= \frac{g_{11} - g_{22} + 2ig_{12}}{\mathrm{tr}\ g_{ab} + 2\sqrt{g}} \end{aligned} \tag{3.437}$$

[8] From here on we will often use complex structure when we mean almost complex structure. This will not cause any confusion.

[9] There is actually another solution where $\rho = \frac{1}{4}\left(\mathrm{tr}\ g_{ab} - 2\sqrt{g}\right)$ but that root gives a non-orientation preserving coordinate transformation. We are only interested in orientation preserving transformations, i.e. those transformations with positive Jacobians.

From now on, we will often simply write

$$\mu = \mu_{\bar{z}}{}^{z} \qquad \text{and} \qquad \bar{\mu} = \mu_{z}{}^{\bar{z}} \tag{3.438}$$

One can easily show that

$$0 \leq |\mu| < 1 \tag{3.439}$$

(the non-orientation preserving ones have $|\mu| \geq 1$). Note that the $\mu = 0$ bound means $\mathrm{g}_J = \rho\,|dz|^2$ and thus a conformally flat metric. It is also straightforward to work out the almost complex structure in terms of ρ and μ. From [3.433] one finds

$$J = \frac{1}{1-|\mu|^2}\begin{pmatrix} i(\mu-\bar{\mu}) & 1+|\mu|^2+\mu+\bar{\mu} \\ -1-|\mu|^2+\mu+\bar{\mu} & -i(\mu-\bar{\mu}) \end{pmatrix} \tag{3.440}$$

If we substitute this expression for J in the differential equation [3.429] one finds

$$(\bar{\partial} - \mu\partial)w(\sigma) = 0 \tag{3.441}$$

If we remember that μ and J have a nonlinear relationship, then one could be surprised by the simplicity of this equation. It then also deserves its own name, the Beltrami equation. In analogy with the introduction of J as a differential form, we can also introduce a differential form

$$\boldsymbol{\mu} = d\bar{z}\,\mu_z{}^{\bar{z}}\partial_z \tag{3.442}$$

which is called the Beltrami differential. The $\mu = \mu_z{}^{\bar{z}}$ and $\bar{\mu} = \mu_{\bar{z}}{}^{z}$ are its components in the z-coordinate system.

The coordinates z and $\bar{z}$ are holomorphic vs the complex structure I and the coordinates w and $\bar{w}$ are holomorphic vs the complex structure J. Note that z is not holomorphic vs the complex structure J, unless $\mu = 0$ and we have a conformally flat metric. Likewise, w is not holomorphic vs I if $\mu \neq 0$. The Beltrami equation can be viewed as relating holomorphic coordinate systems z and w corresponding to different complex structures.

Using the chain rule and the Beltrami equation we find

$$dw = \partial w dz + \bar{\partial} w d\bar{z} = \partial w(dz + \mu d\bar{z}) \tag{3.443}$$

Using this in [3.435] we find

$$\mathrm{g}_J = g_{ab}d\sigma^a d\sigma^b = \rho(z,\bar{z})\,|\partial w|^{-2} dw d\bar{w} = \rho_0(w,\bar{w})\,dw d\bar{w} \tag{3.444}$$

Note that since coordinate transformations are by definitions invertible, we can write z and $\bar{z}$ as a function of w and $\bar{w}$ which leads to $\rho_0(w,\bar{w}) = \rho(z,\bar{z})\,|\partial w|^{-2}$. This means that it is always possible to go locally to a coordinate system that is conformally flat.

THE TRANSFORMATION OF ρ UNDER A CONFORMAL TRANSFORMATION

Let us now check how ρ transforms under a conformal transformation. We will be a bit cavalier and not mention some of the subtleties, but we refer to **[LS]** for more details. We will derive it from requiring invariance of the line element $\mathrm{g} = ds^2 = \rho(z,\bar{z})dzd\bar{z}$ under an infinitesimal conformal transformation $z \to w(z) = z + \delta z$. This calculation will be performed in detail.

$$
\begin{aligned}
ds^2 &= \rho(w,\bar{w})dwd\bar{w} \\
&= \rho(z+\delta z, \bar{z}+\delta\bar{z})(dz + d\delta z)(d\bar{z} + d\delta\bar{z}) \\
&= [\rho(z,\bar{z}) + \partial_z\rho(z,\bar{z})\delta z + \partial_{\bar{z}}\rho(z,\bar{z})\delta\bar{z}] \times [dzd\bar{z} + d\delta z d\bar{z} + dz d\delta\bar{z}] \\
&= \rho dzd\bar{z} + \rho d\delta z d\bar{z} + \rho dz d\delta\bar{z} + \partial_z\rho\delta z dz d\bar{z} + \partial_{\bar{z}}\rho\delta\bar{z}dzd\bar{z} \\
&= \rho dzd\bar{z} + \rho(\partial_z\delta z dz + \partial_{\bar{z}}\delta z d\bar{z})d\bar{z} + \rho dz(\partial_z\delta\bar{z}dz + \partial_{\bar{z}}\delta\bar{z}d\bar{z}) + (\partial_z\rho\delta z + \partial_{\bar{z}}\rho\delta\bar{z})dzd\bar{z} \\
&= \rho dzd\bar{z} + \rho(\partial_z\delta z + \partial_{\bar{z}}\delta\bar{z} + \rho^{-1}\partial_z\rho\delta z + \rho^{-1}\partial_{\bar{z}}\rho\delta\bar{z})dzd\bar{z} + \rho\partial_z\delta\bar{z}(dz)^2 + \rho\partial_{\bar{z}}\delta z(d\bar{z})^2
\end{aligned}
\quad [3.445]
$$

We now recall that the only non-vanishing connections in the conformal gauge are given by, see [3.216]

$$
\Gamma^z_{zz}(\sigma) = \partial_z \ln\rho(\sigma) \qquad \text{and} \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}}(\sigma) = \partial_{\bar{z}}\ln\rho(\sigma) \quad [3.446]
$$

This means that

$$
\begin{aligned}
\nabla_z\delta z &= \partial_z\delta z + \Gamma^z_{zz}\delta z = \partial_z\delta z + \partial_z\ln\rho\,\delta z = \partial_z\delta z + \rho^{-1}\partial_z\rho\,\delta z \\
\nabla_{\bar{z}}\delta\bar{z} &= \partial_{\bar{z}}\delta\bar{z} + \Gamma^{\bar{z}}_{\bar{z}\bar{z}}\delta\bar{z} = \partial_{\bar{z}}\delta\bar{z} + \partial_{\bar{z}}\ln\rho\,\delta\bar{z} = \partial_{\bar{z}}\delta\bar{z} + \rho^{-1}\partial_{\bar{z}}\rho\,\delta\bar{z} \\
\nabla_z\delta\bar{z} &= \partial_z\delta\bar{z} \\
\nabla_{\bar{z}}\delta z &= \partial_{\bar{z}}\delta z
\end{aligned}
\quad [3.447]
$$

Therefore

$$
ds^2 = \rho dzd\bar{z} + \rho\,(\nabla_z\delta z + \nabla_{\bar{z}}\delta\bar{z})dzd\bar{z} + \rho\,\nabla_z\delta\bar{z}(dz)^2 + \rho\,\nabla_{\bar{z}}\delta z(d\bar{z})^2 \quad [3.448]
$$

Let us now compare this with $\delta\mathrm{g}$:

$$
\delta\mathrm{g} = \delta\left[\rho dzd\bar{z} + g_{zz}(dz)^2 + g_{\bar{z}\bar{z}}(d\bar{z})^2\right] \quad [3.449]
$$

Whilst in the conformal gauge $g_{zz} = g_{\bar{z}\bar{z}} = 0$ the change of coordinates may result in off-diagonal components of the metric. We can thus write

$$
\delta\mathrm{g} = \delta\rho dzd\bar{z} + \delta g_{zz}(dz)^2 + \delta g_{\bar{z}\bar{z}}(d\bar{z})^2 \quad [3.450]
$$

Comparing with [3.448] we find

$$\begin{aligned} \delta\rho &= \rho(\nabla_z \delta z + \nabla_{\bar{z}} \delta \bar{z}) \\ \delta g_{zz} &= \rho \, \nabla_z \delta \bar{z} \\ \delta g_{\bar{z}\bar{z}} &= \rho \, \nabla_{\bar{z}} \delta z \end{aligned} \qquad [3.451]$$

which we can rewrite as

$$\begin{aligned} \delta \ln \rho &= \nabla_z \delta z + \nabla_{\bar{z}} \delta \bar{z} \\ \rho^{-1} \delta g_{zz} &= \nabla_z \delta \bar{z} \\ \rho^{-1} \, \delta g_{\bar{z}\bar{z}} &= \nabla_{\bar{z}} \delta z \end{aligned} \qquad [3.452]$$

The Existence of Holomorphic Normal Coordinates

We wish to show that if we have a coordinate system w that is in the conformal gauge, i.e.

$$ds^2 = \rho_0(w, \bar{w}) dw d\bar{w} \qquad [3.453]$$

then we can always perform a conformal transformation

$$w, \bar{w} \to \zeta(w), \bar{\zeta}(\bar{w}) \qquad [3.454]$$

such that the ζ coordinates are holomorphic normal coordinates, i.e. that at a given point p we have the relations

$$\partial_\zeta^n \rho(\zeta, \bar{\zeta})\Big|_p = \partial_{\bar{\zeta}}^n \rho(\zeta, \bar{\zeta})\Big|_p = \delta_{n,0} \qquad [3.455]$$

We first consider $n \geq 1$ and will show that this is possible by explicit construction of $\zeta(w)$ and $\bar{\zeta}(\bar{w})$. Our starting point is [3.444] which we write as

$$\ln \rho_0(w, \bar{w}) = \ln \rho(\zeta, \bar{\zeta}) + \ln |\partial_w \zeta(w)|^2 \qquad [3.456]$$

We take the ∂_w^n derivative of both sides

$$\partial_w^n \ln \rho_0(w, \bar{w}) = \partial_w^n \ln \rho(\zeta, \bar{\zeta}) + \partial_w^n \ln |\partial_w \zeta(w)|^2 \qquad [3.457]$$

The first term on the RHS vanishes when evaluated at p. Indeed, let us start with $n = 1$ and use the chain rule

$$\partial_w \ln \rho(\zeta, \bar{\zeta}) = \partial_w \zeta \partial_\zeta \ln \rho(\zeta, \bar{\zeta}) + \partial_w \bar{\zeta} \partial_{\bar{\zeta}} \ln \rho(\zeta, \bar{\zeta}) \qquad [3.458]$$

When evaluated at p the first term vanishes because of the holomorphic normal coordinates and the last term vanishes because ζ is a conformal transformations and thus $\partial_{\bar{w}}\zeta(w) = 0$. Thus $\partial_w \ln \rho(\zeta, \bar{\zeta})\big|_p = 0$. Next

$$\begin{aligned}\partial_w^2 \ln \rho(\zeta, \bar{\zeta}) &= \partial_w \left[\partial_w \zeta \partial_\zeta \ln \rho(\zeta, \bar{\zeta})\right] \\ &= \partial_w^2 \zeta \partial_\zeta \ln \rho(\zeta, \bar{\zeta}) + \partial_w \zeta \partial_\zeta^2 \ln \rho(\zeta, \bar{\zeta})\end{aligned} \quad [3.459]$$

We have not written out $\partial_w \bar{\zeta}$ terms as these are zero as explained above. Evaluated at p we see that this vanishes as well. We also see that this holds for higher derivatives as we will, in every term be left with a factor $\partial_\zeta^k \ln \rho(\zeta, \bar{\zeta})$ for $1 \geq k \geq n$. Thus we have indeed shown that $\partial_w^n \ln \rho(\zeta, \bar{\zeta})\big|_p = 0$. We thus have

$$\partial_w^n \ln \rho_0(w, \bar{w}) = \partial_w^n \left[\ln \partial_w \zeta(w) + \ln \partial_{\bar{w}} \bar{\zeta}(\bar{w})\right] = \partial_w^n \ln \partial_w \zeta(w) \quad [3.460]$$

evaluated at p. We will drop the cumbersome $\big|_p$ but it will be understood that in the sequel this is to be assumed. If we can now use this relation to express the $\partial_w^n \zeta$ as functions of ρ_0 then we can use these to build a Taylor series that determines $\zeta(w)$ in a neighbourhood of p such that the holomorphic normal condition is satisfied.

The way to do this is by using the Faà di Bruno's formula. This is just a fancy way to write an expression for $\partial_x^n f\big(g(x)\big)$. It is given by

$$\partial_x^n f\big(g(x)\big) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}\big(g'(x), g''(x), \cdots, g^{(n-k+1)}(x)\big) \quad [3.461]$$

Here $B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ are the so-called Bell polynomials[10]. Let us work out the

[10] The Bell polynomials are defined as

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$$

where the sum is taken over all j_ℓ's subject to the conditions $\sum_\ell^{n-k+1} j_\ell = k$ and that also $\sum_\ell^{n-k+1} \ell j_\ell = n$ There is no need for us to go into a detailed analysis of Bell polynomials. For those interested, check the Wikipedia pages. For our purposes we only need to know that the equation

$$x_n = \sum_{k=1}^{n} (-)^{k+1} (k-1)! B_{n,k}(y_1, \cdots, y_{n-k+1})$$

can be inverted as

$$y_n = B_n(x_1, \cdots, x_n)$$

with

$$B_n(x_1, \cdots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, \cdots, x_{n-k+1})$$

first few cases of $\partial_w^n \ln \partial_w \zeta(w)$

$$\partial_w \ln \partial_w \zeta(w) = \frac{\partial_w^2 \zeta}{\partial_w \zeta} = B_{11}\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}\right)$$

$$\partial_w^2 \ln \partial_w \zeta(w) = -\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}\right)^2 + \frac{\partial_w^3 \zeta}{\partial_w \zeta} = -B_{22}\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}, \frac{\partial_w^3 \zeta}{\partial_w \zeta}\right) + B_{21}\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}, \frac{\partial_w^3 \zeta}{\partial_w \zeta}\right) \qquad [3.462]$$

and so on. In general we have

$$\partial_w^n \ln \partial_w \zeta(w) = \sum_{k=1}^{n} (-)^{k+1}(k-1)! B_{n,k}\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}, \frac{\partial_w^3 \zeta}{\partial_w \zeta}, \cdots, \frac{\partial_w^{n-k+2} \zeta}{\partial_w \zeta}\right)$$

$$= B_n\left(\frac{\partial_w^2 \zeta}{\partial_w \zeta}, \frac{\partial_w^3 \zeta}{\partial_w \zeta}, \cdots, \frac{\partial_w^n \zeta}{\partial_w \zeta}\right) \qquad [3.463]$$

With B_n the complete Bell polynomial, see footnote. As per the footnote we can thus invert this relationship to write

$$\partial_w^{n+1} \zeta(p) =) B_n(\partial_w \ln \rho_0, \cdots, \partial_w^n \ln \rho_0)\, \partial_w \zeta(p) \qquad [3.464]$$

At a point p' close to p we can thus use a Taylor expansion

$$\zeta_p(p') = \sum_{n=0}^{\infty} \frac{1}{n!}\left(w_p(p') - w_p(p)\right) \partial_w^n \zeta_p(p)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(w_p(p') - w_p(p)\right)\; B_{n-1}(\partial_w \ln \rho_0, \cdots, \partial_w^{n-1} \ln \rho_0)\, \partial_w \zeta(p) \qquad [3.465]$$

where we have simplified the notation, writing $\partial_w^n \zeta_p(p)$ for $(\partial_p^n(p')/\partial w_p(p')^2)\big|_{p'=p}$.

We have now imposed all the conditions with $n \geq 1$ for the holomorphic normal coordinates, but we still need to impose the $n = 0$ condition, $\rho\big|_p = 1$. From [3.444] i.e.

the so-called complete Bell polynomials. Let us give the explicit form of some of the first Bell polynomials, $B_{n,k}(x_1, \cdots, x_{n-k+1})$

$$B_{11} = x_1$$
$$B_{21} = x_2; \quad B_{22} = x_1^2$$
$$B_{31} = x_3; \quad B_{32} = 3x_1x_2; \quad B_{33} = x_1^3$$
$$B_{41} = x_4; \quad B_{42} = 3x_2^2 + 4x_1x_3; \quad B_{43} = 6x_1^2x_2; \quad B_{44} = x_1^4$$
$$B_{51} = x_5; \quad B_{52} = 10x_2x_3 + 5x_1x_4; \quad B_{53} = 15x_1x_2^2 + 10x_1^2x_3; \quad B_{54} = 10x_1^3x_2; \quad B_{55} = x_1^4$$
$$\ldots$$

$\rho(w,\bar{w})\,|\partial_w\zeta|^{-2} = \rho_0(\zeta,\bar{\zeta})$, taken at p we see that this determines $|\partial_w\zeta|$ and hence $\partial_w\zeta$ only up to a phase:

$$\partial_w\zeta = e^{i\alpha}|\partial_w\zeta| \tag{3.466}$$

and thus

$$\partial_w\zeta(p) = e^{i\alpha}|\partial_w\zeta|_p = e^{i\alpha(p)}\sqrt{\rho_0(p)} \tag{3.467}$$

where we can take the square root of ρ_0 because it is real and positive. It turns out that this phase is in general not globally defined. There are ways to circumvent this, but this is beyond the scope of what we need. See **[LS]** for more details.

To summarise, given a set of coordinates around a point p in the conformal gauge $\rho_0(w,\bar{w})dwd\bar{w}$ we can always find a conformal transformation $\zeta(w), \bar{\zeta}(\bar{w})$ such that at the point p the new coordinate is "as flat as possible," i.e. is a holomorphic normal coordinate satisfying $\partial_\zeta^n\rho(\zeta,\bar{\zeta})\big|_p = \delta_{n,0}$. At a point p' close to p this conformal transformation is given by

$$\zeta_p(p') = e^{i\alpha(p)}\sqrt{\rho_0(p)}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}B_n(\partial_w\ln\rho_0,\cdots,\partial_w^n\ln\rho_0)w_p(p')^{n+1} \tag{3.468}$$

where the phase factor is undetermined, B_n are the complete Bell polynomials and we have used the fact that at the base point $w_p(p) = 0$. Note that, by construction $\zeta_p(p)$ as well as $w_p(p)$ vanishes.

Chapter 4

The String Spectrum

4.1 p 122: Eq (4.1.8) Spurious States, I

We have for any physical state $|\psi\rangle$

$$\langle\psi|\chi\rangle = \langle\psi| \sum_{k=1}^{\infty} L^{\mathrm{m}}_{-k} |\chi_k\rangle = \sum_{k=1}^{\infty} \left[\left(L^{\mathrm{m}}_{-k}\right)^{\dagger} |\psi\rangle \right]^{\dagger} |\chi_k\rangle = \sum_{k=1}^{\infty} [\, L^{\mathrm{m}}_{k} |\psi\rangle]^{\dagger} |\chi_k\rangle = 0 \qquad [4.1]$$

4.2 p 122: Eq (4.1.9) Spurious States, II

Let $|\psi\rangle$ be physical and $|\chi\rangle$ be null. Then for $n > 1$

$$L^{\mathrm{m}}_{n}(|\psi\rangle + |\chi\rangle) = L^{\mathrm{m}}_{n} |\psi\rangle + L^{\mathrm{m}}_{n} |\chi\rangle = 0 \qquad [4.2]$$

We also have that for any other physical state $|\phi\rangle$

$$\langle\phi|\psi + \chi\rangle = \langle\phi|\psi\rangle + \langle\phi|\chi\rangle = \langle\phi|\psi\rangle \qquad [4.3]$$

4.3 p 123: Eq (4.1.11) The Physical Hilbert Space, I: the Tachyon State

Recall (2.7.25) and (2.7.27) for the open string.

$$\begin{aligned} L_0 &= \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \\ L_m &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\alpha_n: \qquad \text{for } m \neq 0 \\ \alpha_0 &= \sqrt{2\alpha'}p \end{aligned} \qquad [4.4]$$

Here and in the remainder of this chapter we will not write the reference $^{\mathrm{m}}$ to the matter sector and the spacetime indices when they just blur the notation. Consider the tachyon state $|0;k\rangle$. We have for $m \geq 1$

$$\begin{aligned} L_m |0;k\rangle &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} : \alpha_{m-n}\alpha_n : |0;k\rangle \\ &= \left(\frac{1}{2} \sum_{n=-\infty}^{-1} : \alpha_{m-n}\alpha_n : + : \alpha_m \alpha_0 : + \frac{1}{2} \sum_{n=1}^{\infty} : \alpha_{m-n}\alpha_n : \right) |0;k\rangle \\ &= \left(\frac{1}{2} \sum_{n=-\infty}^{-1} \alpha_n \alpha_{m-n} + \alpha_0 \alpha_m + \frac{1}{2} \sum_{n=1}^{\infty} \alpha_{m-n}\alpha_n \right) |0;k\rangle = 0 \end{aligned} \tag{4.5}$$

This is the case because we will always have a $\alpha_k |0;k\rangle$ with $k \geq 1$ and this is zero. For $m = 0$ we have

$$L_0 |0;k\rangle = \left(\alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \right) |0;k\rangle = \alpha' k^2 |0;k\rangle \tag{4.6}$$

Requiring $(L_0 + A)|0;k\rangle = 0$ thus implies that $A = -\alpha' k^2$, or as $-k^2 = m^2$ we have

$$m^2 = \frac{A}{\alpha'} \tag{4.7}$$

4.4 p 123: Eq (4.1.16) The L_0 Condition for the Level One State

$$\begin{aligned} (L_0 + A) e \cdot \alpha_{-1} |0;k\rangle &= \left(\alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{\mu n} + A \right) e \cdot \alpha_{-1} |0;k\rangle \\ &= \left(\alpha' p^2 + \alpha_{-1}^{\mu} \alpha_{\mu 1} + A \right) e_\nu \alpha_{-1}^{\nu} |0;k\rangle \\ &= (\alpha' k^2 + A) e \cdot \alpha_{-1} |0;k\rangle + \alpha_{-1}^{\mu} e_\nu \eta^{\mu\nu} |0;k\rangle \\ &= (\alpha' k^2 + A + 1) e \cdot \alpha_{-1} |0;k\rangle \end{aligned} \tag{4.8}$$

Requiring this to be zero gives $\alpha' k^2 + A + 1 = 0$ or, with $-k^2 = m^2$,

$$m^2 = \frac{1 + A}{\alpha'} \tag{4.9}$$

4.5 p 123: Eq (4.1.17) The $L_{m\geq 1}$ Condition for the Level One State

For $m \geq 1$ we have

$$\begin{aligned} L_m\, e\cdot\alpha_{-1}\,|0;k\rangle &= \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\alpha_n:\; e\cdot\alpha_{-1}\,|0;k\rangle \\ &= \frac{1}{2}\left(\sum_{n=-\infty}^{-1} :\alpha_{m-n}\alpha_n: + :\alpha_m\alpha_0: + \sum_{n=1}^{\infty} :\alpha_{m-n}\alpha_n:\right) e\cdot\alpha_{-1}\,|0;k\rangle \\ &= \frac{1}{2}\left(\sum_{n=-\infty}^{-1} \alpha_n\alpha_{m-n} + \alpha_0\alpha_m + \sum_{n=1}^{\infty} \alpha_{m-n}\alpha_n\right) e\cdot\alpha_{-1}\,|0;k\rangle \qquad [4.10] \end{aligned}$$

Because with $m \geq 1$ and $n \leq -1$, $m-n \geq 2$, all the α_{m-n} of the first infinite sum just commute through the α_{-1} and annihilate $|0;k\rangle$. In the second infinite sum, all terms except $\alpha_{m-1}\alpha_1$ annihilate $|0;k\rangle$. We are thus left with

$$L_m\, e\cdot\alpha_{-1}\,|0;k\rangle = \frac{1}{2}\left(\alpha_0\alpha_m + \alpha_{m-1}\alpha_1\right)\, e\cdot\alpha_{-1}\,|0;k\rangle \qquad [4.11]$$

The term with $\alpha_{m-1}\alpha_1$ only gives a non-zero result if $m=1$. Indeed for $m=2$ we have

$$\alpha_1^\mu\alpha_{\mu 1}\, e_\nu\alpha_{-1}^\nu\,|0;k\rangle = \alpha_1^\mu e_\nu\delta_\mu^\nu\,|0;k\rangle = 0 \qquad [4.12]$$

and for $m \geq 3$ the α_{m-1} just commutes to the right to annihilate $|0;k\rangle$ directly. So the only non-trivial condition is

$$\begin{aligned} L_1\, e\cdot\alpha_{-1}\,|0;k\rangle &= \frac{1}{2}\left(\alpha_0\alpha_1 + \alpha_0\alpha_1\right)\, e\cdot\alpha_{-1}\,|0;k\rangle = \alpha_0^\mu\alpha_{\mu 1}e_\nu\alpha_{-1}^\nu\,|0;k\rangle \\ &= \alpha_0^\mu e_\nu\delta_\mu^\nu e_\nu\,|0;k\rangle = e_\mu\alpha_0^\mu\,|0;k\rangle = e_\mu\sqrt{2\alpha'}p^\mu\,|0;k\rangle = \sqrt{2\alpha'}e_\mu k^\mu\,|0;k\rangle \\ &= \sqrt{2\alpha'}(e\cdot k)\,|0;k\rangle \qquad [4.13] \end{aligned}$$

Requiring this to be zero implies that $e.k = 0$.

4.6 p 124: Eq (4.1.18) The Spurious Level One State

We have

$$\begin{aligned} L_{-1}\,|0;k\rangle &= \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{-1-n}\alpha_n:\,|0;k\rangle = \frac{1}{2}(\alpha_0\alpha_{-1} : + \alpha_{-1}\alpha_0)\,|0;k\rangle \\ &= \alpha_{-1}^\mu\sqrt{2\alpha'}p_\mu\,|0;k\rangle = \sqrt{2\alpha'}k_\mu\alpha_{-1}^\mu\,|0;k\rangle = \sqrt{2\alpha'}k\cdot\alpha_{-1}\,|0;k\rangle \qquad [4.14] \end{aligned}$$

Joe's book has taken $\alpha' = 2$. If $|\psi\rangle$ is any physical state then we have $\langle\psi|\, L_{-1}\,|0;k\rangle = 0$ as $L_1\,|\psi\rangle = 0$. Therefore $L_{-1}\,|0;k\rangle$ is spurious.

Note that $L_{-1}|0;k\rangle$ is not necessarily physical. This is easy to see using the Virasoro algebra:

$$L_m L_{-1}|0;k\rangle = \left(L_{-1}L_m + (m+1)L_{m-1} + \frac{c}{12}(m^3 - m)\delta_{m-1,0}\right)|0;k\rangle \tag{4.15}$$

For $m \geq 2$ this is automatically zero, because $L_k|0;k\rangle = 0$ for $k \geq 0$. For $m = 1$ we have

$$L_1 L_{-1}|0;k\rangle = \left(2L_0 + \frac{c}{12}(1^3 - 1)\right)|0;k\rangle = 2L_0|0;k\rangle = 2\alpha' k^2|0;k\rangle \tag{4.16}$$

where we have used [4.6].

4.7 p 124: Eq (4.1.18) The Level One States for Different Values of A

The value of A clearly refers to that of the level one states (4.1.16), i.e.

$$-k^2 = m^2 = \frac{1+A}{\alpha'} \tag{4.17}$$

The three cases are

- $A > -1$ The level one state have positive mass and the spurious state has $L_1 L_{-1}|0;k\rangle \neq 0$ and is not a physical state. The Hilbert space has no extra null states and hence $D-1$ degrees of freedom.
- $A = -1$ The level one states are massless and the spurious state is physical and hence massless. Because a null state is equivalent to another physical state, we loose the extra null state in the Hilbert space and are left with $D-2$ degrees of freedom, in line with a massless particle. Moreover, if we change the polarisation vector $e^\mu \to e^\mu + k^\mu$ then the condition $e \cdot k$ is also satisfied as $k^2 = 0$. This is indicative of the spacetime gauge symmetry.
- $A < -1$ The level one states are tachyons. We still have $D-1$ degrees of freedom, but one has negative norm and so violates unitarity.

4.8 p 124: Eq (4.1.18) The Level Two States

A generic level two state is of the form

$$|E, e; k\rangle = (E_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu \alpha^\mu_{-2})|0;k\rangle \tag{4.18}$$

Let us check the conditions for this to be a physical state. For $m \geq 3$ we have

$$\begin{aligned} L_m|E, e; k\rangle = &\frac{1}{2}\left(\sum_{n=-\infty}^{0} \alpha^\sigma_n \alpha_{\sigma\, m-n} + \alpha^\sigma_1 \alpha_{\sigma\, m-1} + \alpha^\sigma_2 \alpha_{\sigma\, m-2} + \sum_{n=3}^{\infty} \alpha_{\sigma\, m-n}\alpha^\sigma_n\right) \\ &\times (E_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu \alpha^\mu_{-2})|0;k\rangle \end{aligned} \tag{4.19}$$

The infinite sums are zero and so we are left with

$$L_m \left|E, e; k\right\rangle = (\alpha_1^\sigma \alpha_{\sigma\ m-1} + \alpha_2^\sigma \alpha_{\sigma\ m-2})(E_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu) \left|0; k\right\rangle \qquad [4.20]$$

But as $m \geq 3$, if we commute the annihilation operators to the right we will always have a $\alpha_1 \left|0; k\right\rangle$ left and hence this is also zero.

We thus have to check L_2, L_1 and L_0. We start with

$$\begin{aligned} L_2 \left|E, e; k\right\rangle &= \frac{1}{2}\left(\sum_{n=-\infty}^{-1} \alpha_n^\sigma \alpha_{\sigma\ 2-n} + \alpha_0^\sigma \alpha_{\sigma\ 2} + \alpha_1^\sigma \alpha_{\sigma\ 1} + \alpha_2^\sigma \alpha_{\sigma\ 0} + \sum_{n=3}^{\infty} \alpha_{\sigma\ 2-n}\alpha_n^\sigma\right) \\ &\quad \times (E_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu) \left|0; k\right\rangle \\ &= \frac{1}{2}\left(2\alpha_0^\sigma \alpha_{\sigma\ 2} + \alpha_1^\sigma \alpha_{\sigma\ 1}\right)(E_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu) \left|0; k\right\rangle \end{aligned} \qquad [4.21]$$

Now

$$\begin{aligned} \alpha_0^\sigma \alpha_{\sigma\ 2}\alpha_{-1}^\mu \alpha_{-1}^\nu \left|0; k\right\rangle &= \alpha_0^\sigma \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{\sigma\ 2} \left|0; k\right\rangle = 0 \\ \alpha_0^\sigma \alpha_{\sigma\ 2}\alpha_{-2}^\mu \left|0; k\right\rangle &= \alpha_0^\sigma 2\delta_\sigma^\mu \left|0; k\right\rangle = 2\sqrt{2\alpha'}k^\mu \left|0; k\right\rangle \\ \alpha_1^\sigma \alpha_{\sigma\ 1}\alpha_{-1}^\mu \alpha_{-1}^\nu \left|0; k\right\rangle &= \alpha_1^\sigma(\alpha_{-1}^\mu \alpha_{\sigma\ 1} + \delta_\sigma^\mu)\alpha_{-1}^\nu \left|0; k\right\rangle \\ &= \alpha_1^\sigma \alpha_{-1}^\mu \alpha_{\sigma\ 1}\alpha_{-1}^\nu \left|0; k\right\rangle + \alpha_1^\mu \alpha_{-1}^\nu \left|0; k\right\rangle \\ &= \eta^{\sigma\mu}\delta_\sigma^\nu \left|0; k\right\rangle + \eta^{\mu\nu} \left|0; k\right\rangle = 2\eta^{\mu\nu} \left|0; k\right\rangle \\ \alpha_1^\sigma \alpha_{\sigma\ 1}\alpha_{-2}^\mu \left|0; k\right\rangle &= 0 \end{aligned} \qquad [4.22]$$

Therefore

$$L_2 \left|E, e; k\right\rangle = \frac{1}{2}\left(4\sqrt{2\alpha'}k^\mu e_\mu + 2\eta^{\mu\nu}E_{\mu\nu}\right) \left|0; k\right\rangle \qquad [4.23]$$

Setting this to zero gives the condition

$$2\sqrt{2\alpha'}k \cdot e + E_\mu^\mu = 0 \qquad [4.24]$$

Next, we consider L_1

$$\begin{aligned} L_1 \left|E, e; k\right\rangle &= \frac{1}{2}\left(\sum_{n=-\infty}^{-2} \alpha_n^\sigma \alpha_{\sigma\ 1-n} + \alpha_{-1}^\sigma \alpha_{\sigma\ 2} + \alpha_0^\sigma \alpha_{\sigma\ 1} + \alpha_1^\sigma \alpha_{\sigma\ 0} + \alpha_{-1}^\sigma \alpha_{\sigma\ 2} + \sum_{n=3}^{\infty} \alpha_{\sigma\ 1-n}\alpha_n^\sigma\right) \\ &\quad \times (E_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu) \left|0; k\right\rangle \\ &= \left(\alpha_{-1}^\sigma \alpha_{\sigma\ 2} + \alpha_0^\sigma \alpha_{\sigma\ 1}\right)(E_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu) \left|0; k\right\rangle \end{aligned} \qquad [4.25]$$

Now

$$\begin{aligned}
\alpha^{\sigma}_{-1}\alpha_{\sigma\ 2}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle &= 0\\
\alpha^{\sigma}_{-1}\alpha_{\sigma\ 2}\alpha^{\mu}_{-2}\left|0;k\right\rangle &= 2\delta^{\mu}_{\sigma}\alpha^{\sigma}_{-1}\left|0;k\right\rangle = 2\alpha^{\mu}_{-1}\left|0;k\right\rangle\\
\alpha^{\sigma}_{0}\alpha_{\sigma\ 1}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle &= \alpha^{\sigma}_{0}(\alpha^{\mu}_{-1}\alpha_{\sigma\ 1}+\delta^{\mu}_{\sigma})\alpha^{\nu}_{-1}\left|0;k\right\rangle\\
&= (\alpha^{\sigma}_{0}\alpha^{\mu}_{-1}\delta^{\nu}_{\sigma}+\alpha^{\mu}_{0}\alpha^{\nu}_{-1})\left|0;k\right\rangle\\
&= \sqrt{2\alpha'}(k^{\nu}\alpha^{\mu}_{-1}+k^{\mu}\alpha^{\nu}_{-1})\left|0;k\right\rangle\\
\alpha^{\sigma}_{0}\alpha_{\sigma\ 1}\alpha^{\mu}_{-2}\left|0;k\right\rangle &= 0
\end{aligned} \tag{4.26}$$

Therefore

$$\begin{aligned}
L_1\left|E,e;k\right\rangle &= \left[\sqrt{2\alpha'}E_{\mu\nu}(k^{\nu}\alpha^{\mu}_{-1}+k^{\mu}\alpha^{\nu}_{-1})+2e_{\mu}\alpha^{\mu}_{-1}\right]\left|0;k\right\rangle\\
&= 2(\sqrt{2\alpha'}k^{\mu}E_{\mu\nu}+e_{\nu})\alpha^{\nu}_{-1}\left|0;k\right\rangle
\end{aligned} \tag{4.27}$$

Setting this to zero implies that

$$\sqrt{2\alpha'}k^{\mu}E_{\mu\nu}+e_{\nu}=0 \tag{4.28}$$

Finally, we consider L_0,

$$\begin{aligned}
L_0\left|E,e;k\right\rangle &= \left(\alpha'p^2+\sum_{n=1}^{\infty}\alpha^{\sigma}_{-n}\alpha_{\sigma\ n}\right)(E_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}+e_{\mu}\alpha^{\mu}_{-2})\left|0;k\right\rangle\\
&= (\alpha'p^2+\alpha^{\sigma}_{-1}\alpha_{\sigma\ 1}+\alpha^{\sigma}_{-2}\alpha_{\sigma\ 2})(E_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}+e_{\mu}\alpha^{\mu}_{-2})\left|0;k\right\rangle
\end{aligned} \tag{4.29}$$

Now

$$\begin{aligned}
\alpha^{\sigma}_{-1}\alpha_{\sigma\ 1}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle &= \alpha^{\sigma}_{-1}(\alpha^{\mu}_{-1}\alpha_{\sigma\ 1}+\delta^{\mu}_{\sigma})\alpha^{\nu}_{-1}\left|0;k\right\rangle\\
&= (\alpha^{\sigma}_{-1}\alpha^{\mu}_{-1}\delta^{\nu}_{\sigma}+\alpha^{\sigma}_{-1}\alpha^{\nu}_{-1}\delta^{\mu}_{\sigma})\left|0;k\right\rangle\\
\alpha^{\sigma}_{-1}\alpha_{\sigma\ 1}\alpha^{\mu}_{-2}\left|0;k\right\rangle &= 0 = 2\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle\\
\alpha^{\sigma}_{-2}\alpha_{\sigma\ 2}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle &= 0\\
\alpha^{\sigma}_{-2}\alpha_{\sigma\ 2}\alpha^{\mu}_{-2}\left|0;k\right\rangle &= \ 2\delta^{\mu}_{\sigma}\alpha^{\sigma}_{-2}\left|0;k\right\rangle = 2\alpha^{\mu}_{-2}\left|0;k\right\rangle
\end{aligned} \tag{4.30}$$

Therefore

$$\begin{aligned}
L_0\left|E,e;k\right\rangle &= \alpha'k^2(E_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}+e_{\mu}\alpha^{\mu}_{-2})\left|0;k\right\rangle+2E_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\left|0;k\right\rangle+2e_{\mu}\alpha^{\mu}_{-2}\left|0;k\right\rangle\\
&= (\alpha'k^2+2)(E_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}+e_{\mu}\alpha^{\mu}_{-2})\left|0;k\right\rangle
\end{aligned} \tag{4.31}$$

Setting $(L_0+A)\left|E,e;k\right\rangle=0$ gives

$$\alpha'k^2+2+A=0 \tag{4.32}$$

or

$$m^2 = \frac{2+A}{\alpha'} = \frac{1}{\alpha'} \qquad [4.33]$$

where we set $A = -1$.

Summarising, $E_{\mu\nu}$ and e_μ need to satisfy the equations

$$0 = 2\sqrt{2\alpha'}k \cdot e + E^\mu_\mu \qquad [4.34]$$

$$0 = \sqrt{2\alpha'}k^\mu E_{\mu\nu} + e_\nu \qquad [4.35]$$

The second equation determines e in terms of E. The first equation then sets an additional condition on E. To see this specifically let us go to the rest frame of this state, which is possible as this state has non-zero mass. In the rest frame we have

$$k = (\frac{1}{\sqrt{\alpha}}, 0, \cdots, 0) \qquad [4.36]$$

and this satisfies $m^2 = -k^2 = 1/\alpha'$ as it should. From [4.35] we then find

$$-\sqrt{2\alpha'}\frac{1}{\sqrt{\alpha'}}E_{0\nu} + e_\nu = 0 \quad \Rightarrow \quad e_\mu = \sqrt{2}E_{0\mu} \qquad [4.37]$$

Plugging this in [4.34] gives

$$0 = -2\sqrt{2\alpha'}\frac{1}{\sqrt{\alpha'}}e_0 + E^\mu_\mu = -2\sqrt{2}\sqrt{2}E_{00} - E_{00} + E_{ii} \qquad [4.38]$$

where i runs over the spacelike indices only. Thus the condition on E becomes

$$E_{ii} = 5E_{00} \qquad [4.39]$$

Let us count the degrees of freedom. E is a symmetric $D \times D$ matrix so has $D(D+1)/2$ entries. The vector e has D entries, but is fully determined by E. There is one more condition [4.34], so we have a total of $\frac{1}{2}D(D+1)-1$ degrees of freedom. Recall that in the lightcone quantisation the physical states were the symmetric tensor $e_{ij}\alpha^i_{-1}\alpha^j_{-1}|0;k\rangle$ and the vector $E_i\alpha^i_{-2}|0;k\rangle$ with $i = 2, \cdots, D-1$, i.e. these are in the $SO(D-2)$ representation. This gives a total number of degrees of freedom

$$\frac{1}{2}(D-2)(D-1) + (D-2) = \frac{1}{2}(D-2)(D+1) = \frac{1}{2}D(D-1) - 1 \qquad [4.40]$$

which corresponds to the traceless symmetric representation of $SO(D-1)$. This is fewer degrees of freedom than we have found so far by

$$\frac{1}{2}D(D+1) - 1 - \left(\frac{1}{2}D(D-1) - 1\right) = D \qquad [4.41]$$

So if everything fits together, we expect that there are D null states at level two in the OCQ approach.

To find the null states, let us first identify the spurious states at level two. Which states at level two are automatically orthogonal to all physical states? A moment's thought reveals that the level two spurious states are formed from linear combinations of $L_{-1}^2\,|0;k\rangle$, $L_{-2}\,|0;k\rangle$ and $L_{-1}\alpha^\mu_{-1}\,|0;k\rangle$, as these three combinations are orthogonal to all physical states. But, writing only the non vanishing terms explicitly

$$L_{-1}^2\,|0;k\rangle = L_{-1}\frac{1}{2}(\cdots + :\alpha_0^\sigma\alpha_{\sigma-1}: + :\alpha^\sigma_{-1}\alpha_{\sigma 0}:)\,|0;k\rangle = \sqrt{2\alpha'}k^\sigma L_{-1}\alpha^\sigma_{-1}\,|0;k\rangle \qquad [4.42]$$

and so $L_{-1}^2\,|0;k\rangle$ is not an independent state. The most general spurious level two state is thus a combination

$$|G_\mu, H; k\rangle = (G_\mu L_{-1}\alpha^\mu_{-1} + HL_{-2})\,|0;k\rangle \qquad [4.43]$$

We have $D+1$ such spurious states. Let us first write this state in terms of the modes only. Here and in the sequel we will only write those terms of the L_k that contribute. We have

$$\begin{aligned} L_{-1}\alpha^\mu_{-1}\,|0;k\rangle &= (\alpha^\sigma_{-2}\alpha_{\sigma\,1} + \alpha^\sigma_{-1}\alpha_{\sigma\,0}\alpha^\mu_{-1}\,|0;k\rangle = (\alpha^\mu_{-2} + \sqrt{2\alpha'}k_\sigma\alpha^\sigma_{-1}\alpha\mu_{-1})\,|0;k\rangle \\ L_{-2}\,|0;k\rangle &= \left(\alpha^\sigma_{-2}\alpha_{\sigma\,0} + \frac{1}{2}\alpha^\sigma_{-1}\alpha_{\sigma\,-1}\right)|0;k\rangle \\ &= \left(\sqrt{2\alpha'}k_\sigma\alpha^\sigma_{-2} + \frac{1}{2}\alpha^\sigma_{-1}\alpha_{\sigma\,-1}\right)|0;k\rangle \end{aligned} \qquad [4.44]$$

So

$$|G_\mu, H; k\rangle = \left[\left(\sqrt{2\alpha'}G_{\{\mu}k_{\nu\}} + \frac{1}{2}H\eta_{\mu\nu}\right)\alpha^\mu_{-1}\alpha^\nu_{-1} + \left(G_\mu + \sqrt{2\alpha'}Hk_\mu\right)\alpha^\mu_{-2}\right]|0;k\rangle \qquad [4.45]$$

Let us check the conditions under which such spurious states are physical. Any L_k with $k \geq 3$ automatically annihilates $|G_\mu, H; k\rangle$. For L_2 we have

$$\begin{aligned} L_2\alpha^\mu_{-2}\,|0;k\rangle &= \left(\alpha_0^\sigma\alpha_{\sigma\,2} + \frac{1}{2}\alpha_1^\sigma\alpha_{\sigma\,1}\right)\alpha^\mu_{-2}\,|0;k\rangle = 2\sqrt{2\alpha'}k^\mu\,|0;k\rangle \\ L_2\alpha^\mu_{-1}\alpha^\nu_{-1}\,|0;k\rangle &= \frac{1}{2}\alpha_1^\sigma\alpha_{\sigma\,1}\alpha^\mu_{-1}\alpha^\nu_{-1}\,|0;k\rangle = \frac{1}{2}\alpha_1^\sigma(\delta^\mu_\sigma + \alpha^\mu_{-1}\alpha_{\sigma\,1})\alpha^\nu_{-1}\,|0;k\rangle \\ &= \frac{1}{2}(\alpha_1^\mu\alpha^\nu_{-1} + \alpha_1^\sigma\alpha^\mu_{-1}\delta^\nu_\sigma)\,|0;k\rangle = \eta^{\mu\nu}\,|0;k\rangle \end{aligned} \qquad [4.46]$$

and thus

$$\begin{aligned} L_2\,|G_\mu, H; k\rangle &= \left[\eta^{\mu\nu}\left(\sqrt{2\alpha'}G_{\{\mu}k_{\nu\}} + \frac{1}{2}H\eta_{\mu\nu}\right) + 2\sqrt{2\alpha'}k^\mu\left(G_\mu + \sqrt{2\alpha'}Hk_\mu\right)\right]|0;k\rangle \\ &= \left(3\sqrt{2\alpha'}G\cdot k + \frac{1}{2}DH + 4\alpha' Hk^2\right)|0;k\rangle \end{aligned} \qquad [4.47]$$

Using the mass shell condition $k^2 = -1/\alpha'$ for level two states this gives the condition

$$3\sqrt{2\alpha'}G \cdot k + \left(\frac{D}{2} - 4\right) H = 0 \tag{4.48}$$

Next we consider the action of L_1 on $|G_\mu, H; k\rangle$. First we compute

$$\begin{aligned} L_1\alpha^\mu_{-2}\,|0;k\rangle &= \left(\alpha^\sigma_{-1}\alpha_{\sigma\,2} + \alpha^\sigma_0\alpha_{\sigma\,1}\right)\alpha^\mu_{-2}\,|0;k\rangle = 2\alpha^\mu_{-1}\,|0;k\rangle \\ L_1\alpha^\mu_{-1}\alpha^\nu_{-1}\,|0;k\rangle &= \left(\alpha^\sigma_{-1}\alpha_{\sigma\,2} + \alpha^\sigma_0\alpha_{\sigma\,1}\right)\alpha^\mu_{-1}\alpha^\nu_{-1}\,|0;k\rangle \\ &= \alpha^\sigma_0(\alpha^\mu_{-1}\alpha_{\sigma\,1} + \delta^\mu_\sigma)\alpha^\nu_{-1}\,|0;k\rangle = \sqrt{2\alpha'}(k^\nu\alpha^\mu_{-1} + k^\mu\alpha^\nu_{-1})\,|0;k\rangle \end{aligned} \tag{4.49}$$

and thus

$$\begin{aligned} L_1\,|G_\mu, H; k\rangle &= \Bigg[\sqrt{2\alpha'}(k^\nu\alpha^\mu_{-1} + k^\mu\alpha^\nu_{-1})\left(\sqrt{2\alpha'}G_{\{\mu}k_{\nu\}} + \frac{1}{2}H\eta_{\mu\nu}\right) \\ &\qquad + 2\left(G_\mu + \sqrt{2\alpha'}Hk_\mu\right)\alpha^\mu_{-1}\Bigg]\,|0;k\rangle \\ &= \left(2\alpha' G_\mu k^2 + 2\alpha' G\cdot k k_\mu + \sqrt{2\alpha'}Hk_\mu + 2G_\mu + 2\sqrt{2\alpha'}Hk_\mu\right)\alpha^\mu_{-1}\,|0;k\rangle \\ &= \left(2(\alpha' k^2 + 1)G_\mu + 2\alpha' G\cdot k k_\mu + 3\sqrt{2\alpha'}Hk_\mu\right)\alpha^\mu_{-1}\,|0;k\rangle \end{aligned} \tag{4.50}$$

Using the mass shell condition $k^2 = -1/\alpha'$ for level two states this gives the condition

$$(\sqrt{2\alpha'}G \cdot k + 3H)k^\mu = 0 \tag{4.51}$$

We leave it as an exercise to the reader to show in a similar way that $(L_0 + 1)\,|G_\mu, H; k\rangle = 0$ is automatically satisfied.

Let us summarise this result. The spurious state

$$|G_\mu, H; k\rangle = \left[\left(\sqrt{2\alpha'}G_{\{\mu}k_{\nu\}} + \frac{1}{2}H\eta_{\mu\nu}\right)\alpha^\mu_{-1}\alpha^\nu_{-1} + \left(G_\mu + \sqrt{2\alpha'}Hk_\mu\right)\alpha^\mu_{-2}\right]|0;k\rangle \tag{4.52}$$

is a null state if G_μ and H satisfy the conditions

$$0 = 3\sqrt{2\alpha'}G \cdot k + \left(\frac{D}{2} - 4\right) H \tag{4.53}$$

$$0 = (\sqrt{2\alpha'}G \cdot k + 3H)k^\mu \tag{4.54}$$

In order to count the number of null states, let us go back to the rest frame. In that frame $k^0 = 1/\alpha'$, $k^i = 0$ and thus $G \cdot k = G_0/\sqrt{\alpha'}$. Eq [4.54] is satisfied for all i, giving $D - 1$ null states. Eqs [4.53] and [4.54] then become

$$\begin{aligned} 0 &= 3\sqrt{2}G_0 + \left(\frac{D}{2} - 4\right) H \\ 0 &= \sqrt{2}G_0 \cdot k + 3H \end{aligned} \tag{4.55}$$

which would seem to give a value for G_0 and H given two more null states, bringing the total to $D+1$, whereas we only need D null states! But these equations are degenerate when

$$0 = \det \begin{pmatrix} 3\sqrt{2} & \frac{D}{2}-4 \\ \sqrt{2} & 3 \end{pmatrix} = 9\sqrt{2} - \sqrt{2}\left(\frac{D}{2}-4\right) = \sqrt{2}\left(13 - \frac{D}{2}\right) \qquad [4.56]$$

Thus when $D = 26$ these equations are not independent and only give one extra null state. We conclude that in $D = 26$ we indeed have D null states and thus indeed $\frac{1}{2}D(D+1) - 1 - D = \frac{1}{2}D(D-1)$ physical states at level two, which is the same result as in the lightcone gauge.

4.9 p 126: Eq (4.2.6) The BRST Invariance of the Quantum Action

As mentioned in the text, S_1 is automatically invariant. For S_2 we have

$$\begin{aligned} \delta_{\rm B} S_2 &= \delta_{\rm B}\big[-iB_A F^A(\phi)\big] = -i\big[(\delta_{\rm B} B_A)F^A(\phi) + B_A(\delta_{\rm B} F^A(\phi))\big] \\ &= -i\big[0 + B_A(-i\epsilon c^\alpha \delta_\alpha F^A(\phi))\big] = -\epsilon B_A c^\alpha \delta_\alpha F^A(\phi) \end{aligned} \qquad [4.57]$$

For S_3 we have

$$\begin{aligned} \delta_{\rm B} S_2 &= \delta_{\rm B}\big[b_A c^\alpha \delta_\alpha F^A(\phi)\big] = (\delta_{\rm B} b_A) c^\alpha \delta_\alpha F^A(\phi) + b_A(\delta_{\rm B} c^\alpha)\delta_\alpha F^A(\phi) + b_A c^\alpha(\delta_{\rm B}\delta_\alpha F^A(\phi)) \\ &= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + b_A\left(\frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\delta_\alpha F^A(\phi) + b_A c^\alpha\left(-i\epsilon c^\beta \delta_\beta \delta_\alpha F^A(\phi)\right) \end{aligned} \qquad [4.58]$$

We can rewrite the second term, using $[\delta_\beta, \delta_\gamma] = f^\alpha_{\beta\gamma}\delta_\alpha$, as

$$\begin{aligned} -\frac{i}{2}\epsilon b_A c^\beta c^\gamma f^\alpha_{\beta\gamma}\delta_\alpha F^A(\phi) &= -\frac{i}{2}\epsilon b_A c^\beta c^\gamma [\delta_\beta, \delta_\gamma] F^A(\phi) = -i\epsilon b_A c^\beta c^\gamma \delta_\beta \delta_\gamma F^A(\phi) \\ &= i\epsilon b_A c^\gamma c^\beta \delta_\beta \delta_\gamma F^A(\phi) \end{aligned} \qquad [4.59]$$

which exactly cancels the third term. We thus have

$$\delta_{\rm B}\left(S_1 + S_2 + S_3\right) = 0 - \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) = 0 \qquad [4.60]$$

4.10 p 127: Ghost Number Conservation

This should be obvious, but let's nevertheless make sure it is correct. Clearly $g_\#(S_1) = g_\#(S_2) = 0$ as $g_\#(\phi) = g_\#(B) = 0$. Finally $g_\#(S_3) = g_\#(b) + g_\#(c) = -1 + 1 = 0$. Moreover we easily see that the BRST transformation preserves the ghost number:

$$\begin{aligned} g_\#\left(\delta_{\rm B}\phi\right) &= g_\#(\epsilon) + g_\#(c) + g_\#(\phi) = -1 + 1 = 0 = g_\#(\phi) \\ g_\#\left(\delta_{\rm B} B_A\right) &= 0 = g_\#(B) \\ g_\#\left(\delta_{\rm B} b_a\right) &= g_\#(\epsilon) + g_\#(B) = -1 + 0 = -1 = g_\#(b) \\ g_\#\left(\delta_{\rm B} c^\alpha\right) &= g_\#(\epsilon) + 2g_\#(c) = -1 + 2 = 1 = g_\#(c) \end{aligned} \qquad [4.61]$$

4.11 p 127: Eq (4.2.7) $\delta_{\mathrm{B}}(b_A F^A) = i\epsilon(S_2 + S_3)$

We have

$$\begin{aligned}\delta_{\mathrm{B}}(b_A F^A) &= (\delta_{\mathrm{B}} b_A)F^A + b_A(\delta_{\mathrm{B}} F^A) = \epsilon B_A F^A + b_A = \epsilon B_A F^A + b_A(-i\epsilon c^\alpha \delta_\alpha F^A) \\ &= i\epsilon(-iB_A F_A + b_A c^\alpha \delta_\alpha F^A) = i\epsilon(S_2 + S_3) \qquad [4.62]\end{aligned}$$

which is what we needed to show.

4.12 p 127: Eq (4.2.8) A Change in the Gauge-Fixing Condition

We have

$$\begin{aligned}\epsilon\,\delta\langle f|i\rangle &= \epsilon\left(\langle f|i\rangle_{F+\delta F} - \langle f|i\rangle_F\right) \\ &= \epsilon\Bigg(\int [d\phi\, dB\, db\, dc]\, e^{-S_1[\phi]+iB_A(F^A+\delta F^A)-b_A c^\alpha \delta_\alpha (F^A+\delta F^A)} \\ &\qquad - \int [d\phi\, dB\, db\, dc]\, e^{-S_1[\phi]+iB_A F^A - b_A c^\alpha \delta_\alpha F^A}\Bigg) \\ &= \epsilon\Bigg(\int [d\phi\, dB\, db\, dc]\, e^{-S_1[\phi]+iB_A F^A - b_A c^\alpha \delta_\alpha F^A}\left(1 + iB_A \delta F^A - b_A c^\alpha \delta_\alpha \delta F^A\right) \\ &\qquad - \int [d\phi\, dB\, db\, dc]\, e^{-S_1[\phi]+iB_A F^A - b_A c^\alpha \delta_\alpha F^A}\Bigg) \\ &= \epsilon \int [d\phi\, dB\, db\, dc]\, e^{-S_1-S_2-S_3}\left(iB_A \delta F^A - b_A c^\alpha \delta_\alpha \delta F^A\right) \qquad [4.63]\end{aligned}$$

Now $\delta_{\mathrm{B}} b_A = \epsilon B_A$ so that we can write the first term between brackets as $iB_A \delta F^A = i(\delta_{\mathrm{B}} b_a)\delta F^A$. We also have $\delta_{\mathrm{B}}\phi = -i\epsilon c^\alpha \partial_\alpha \phi$ which implies, using Leibniz, $\delta_{\mathrm{B}}\delta F(\phi) = -i\epsilon c^\alpha \partial_\alpha \delta F(\phi)$ and so we can write the second term between brackets as $ib_A\big(\delta_{\mathrm{B}}\delta F(\phi)\big)$. Therefore

$$\begin{aligned}\epsilon\delta\,\langle f|i\rangle &= \int [d\phi\, dB\, db\, dc]\, e^{-S_1-S_2-S_3}\left[i(\delta_{\mathrm{B}} b_A)\delta F^A + ib_A\big(\delta_{\mathrm{B}}\delta F(\phi)\big)\right] \\ &= i\int [d\phi\, dB\, db\, dc]\, e^{-S_1-S_2-S_3}\delta_{\mathrm{B}}(b_A \delta F^A) \\ &= i\,\langle f|\,\delta_{\mathrm{B}}(b_A\delta F^A)\,|i\rangle = -\epsilon\,\langle f|\,\{Q_B, b_A \delta F^A\}\,|i\rangle \qquad [4.64]\end{aligned}$$

In the last line we have used $\delta_{\mathrm{B}}\mathcal{A} = i\epsilon\{Q_B, \mathcal{A}\}$ where Q_B is the conserved charge corresponding to the BRST symmetry.

4.13 p 128: Eq (4.2.13) The BRST Charge is Nilpotent

Let us check this for the different fields

$$\begin{aligned}\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)}\phi =&\delta_{\rm B}^{(2)}(-i\epsilon_1 c^\alpha\delta_\alpha\phi) = -i\epsilon_1(\delta_{\rm B}^{(2)}c^\alpha)\delta_\alpha\phi - i\epsilon_1 c^\alpha(\delta_{\rm B}^{(2)}\delta_\alpha\phi)\\ =& - i\epsilon_1\frac{i}{2}\epsilon_2 f^\alpha_{\beta\gamma}c^\beta c^\gamma\delta_\alpha\phi - i\epsilon_1 c^\alpha(-i\epsilon_2 c^\beta\delta_\beta\delta_\alpha\phi)\\ =&\frac{1}{2}\epsilon_1\epsilon_2 c^\beta c^\gamma f^\alpha_{\beta\gamma}\delta_\alpha\phi + \epsilon_1\epsilon_2 c^\alpha c^\beta\delta_\beta\delta_\alpha\phi\\ =&\frac{1}{2}\epsilon_1\epsilon_2 c^\beta c^\gamma[\delta_\beta,\delta_\gamma]\phi + \epsilon_1\epsilon_2 c^\alpha c^\beta\delta_\beta\delta_\alpha\phi\\ =&\epsilon_1\epsilon_2 c^\beta c^\gamma\delta_\beta\delta_\gamma\phi + \epsilon_1\epsilon_2 c^\alpha c^\beta\delta_\beta\delta_\alpha\phi\\ =& - \epsilon_1\epsilon_2 c^\alpha c^\beta\delta_\beta\delta_\alpha\phi + \epsilon_1\epsilon_2 c^\alpha c^\beta\delta_\beta\delta_\alpha\phi = 0\end{aligned} \tag{4.65}$$

It is obvious that $\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)}B_A = 0$ and that $\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)}b_A = 0$. Finally

$$\begin{aligned}\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)}c^\alpha =&\delta_{\rm B}^{(2)}\frac{i}{2}\epsilon_1 f^\alpha_{\beta\gamma}c^\beta c^\gamma = \frac{i}{2}\epsilon_1 f^\alpha_{\beta\gamma}\left[(\delta_{\rm B}c^\beta)c^\gamma + c^\beta(\delta_{\rm B}c^\gamma)\right]\\ =&\frac{i}{2}\epsilon_1 f^\alpha_{\beta\gamma}\left(\frac{i}{2}\epsilon_2 f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma + c^\beta\frac{i}{2}\epsilon_2 f^\gamma_{\delta\epsilon}c^\delta c^\epsilon\right)\\ =& -\frac{1}{4}\epsilon_1\epsilon_2\left(f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma - f^\alpha_{\beta\gamma}f^\gamma_{\delta\epsilon}c^\beta c^\delta c^\epsilon\right)\\ =& -\frac{1}{4}\epsilon_1\epsilon_2\left(f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma - f^\alpha_{\gamma\beta}f^\beta_{\delta\epsilon}c^\gamma c^\delta c^\epsilon\right)\\ =& -\frac{1}{2}\epsilon_1\epsilon_2 f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma\end{aligned} \tag{4.66}$$

We can now rewrite the part with the structure constants and ghosts as

$$\begin{aligned}f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma =&\frac{1}{3}(f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon}c^\delta c^\epsilon c^\gamma + f^\alpha_{\beta\delta}f^\beta_{\epsilon\gamma}c^\epsilon c^\gamma c^\delta + f^\alpha_{\beta\epsilon}f^\beta_{\gamma\delta}c^\gamma c^\delta c^\epsilon)\\ =&\frac{1}{3}(f^\alpha_{\beta\gamma}f^\beta_{\delta\epsilon} + f^\alpha_{\beta\delta}f^\beta_{\epsilon\gamma} + f^\alpha_{\beta\epsilon}f^\beta_{\gamma\delta})c^\delta c^\epsilon c^\gamma = 0\end{aligned} \tag{4.67}$$

by the Jacobi identity satisfied by the structure constants.

4.14 p 129: Eq (4.2.20) The Structure Constants for the BRST Transformation of the Point Particle

$$\begin{aligned}
[\delta_{\tau_1}, \delta_{\tau_2}]X^\mu(\tau) =& \delta_{\tau_1}\delta_{\tau_2}X^\mu(\tau) - \delta_{\tau_2}\delta_{\tau_1}X^\mu(\tau) \\
=& \delta_{\tau_1}\left[-\delta(\tau-\tau_2)\partial_\tau X^\mu(\tau)\right] - \delta_{\tau_2}\left[-\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau)\right] \\
=& -\delta(\tau-\tau_2)\partial_\tau\delta_{\tau_1}X^\mu(\tau) + \delta(\tau-\tau_1)\partial_\tau\delta_{\tau_2}X^\mu(\tau) \\
=& -\delta(\tau-\tau_2)\partial_\tau\left[-\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau)\right] + \delta(\tau-\tau 1)\partial_\tau\left[-\delta(\tau-\tau_2)\partial_\tau X^\mu(\tau)\right] \\
=& \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau) + \delta(\tau-\tau_2)\delta(\tau-\tau_1)\partial_\tau^2 X^\mu(\tau) \\
&- \delta(\tau-\tau_1)\partial_\tau\delta(\tau-\tau_2)\partial_\tau X^\mu(\tau) + \delta(\tau-\tau_1)\delta(\tau-\tau_2)\partial_\tau^2 X^\mu(\tau) \\
=& -\left[\delta(\tau-\tau_1)\partial_\tau\delta(\tau-\tau_2) - \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\right]\partial_\tau X^\mu(\tau)
\end{aligned} \qquad [4.68]$$

Now using (4.2.21) and (4.2.19), i.e. $\delta_{\tau_3}X^\mu(\tau) = -\delta(\tau-\tau_3)\partial_\tau X^\mu(\tau)$, we have

$$\begin{aligned}
\int d\tau_3\, f^{\tau_3}_{\tau_1\tau_2}\delta_{\tau_3}X^\mu(\tau) =& \int d\tau_3\, \left[\delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2) - \delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)\right]\delta_{\tau_3}X^\mu(\tau) \\
=& \int d\tau_3\, \big[-\delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2)\delta(\tau-\tau_3)\partial_\tau X^\mu(\tau) \\
&+ \delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)\delta(\tau-\tau_3)\partial_\tau X^\mu(\tau)\big] \\
=& -\left[\delta(\tau-\tau_1)\partial_\tau\delta(\tau-\tau_2) - \delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1)\right]\partial_\tau X^\mu(\tau)
\end{aligned} \qquad [4.69]$$

which proves (4.2.20).

4.15 p 129: Eq (4.2.22) The BRST Transformation for the Point Particle

Using (4.2.6) and (4.2.19) we find

$$\begin{aligned}
\delta_{\mathrm{B}}X^\mu(\tau) =& -i\epsilon\int d\tau_1 c(\tau_1)\delta_{\tau_1}X^\mu(\tau) = i\epsilon\int d\tau_1 c(\tau_1)\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau) \\
=& \; i\epsilon c(\tau)\partial_\tau X^\mu(\tau) = i\epsilon c\dot{X}^\mu
\end{aligned} \qquad [4.70]$$

Similarly,

$$\begin{aligned}
\delta_{\mathrm{B}}e(\tau) =& -i\epsilon\int d\tau_1 c(\tau_1)\delta_{\tau_1}e(\tau) = i\epsilon\int d\tau_1 c(\tau_1)\partial_\tau[\delta(\tau-\tau_1)e(\tau)] \\
=& i\epsilon\int d\tau_1 c(\tau_1)\left[\partial_\tau\delta(\tau-\tau_1)e(\tau) + \delta(\tau-\tau_1)\partial_\tau e(\tau)\right] \\
=& i\epsilon\int d\tau_1 c(\tau_1)\left[-\partial_{\tau_1}\delta(\tau-\tau_1)e(\tau) + \delta(\tau-\tau_1)\partial_\tau e(\tau)\right] \\
=& i\epsilon\int d\tau_1\left[\partial_{\tau_1}c(\tau_1)\delta(\tau-\tau_1)e(\tau) + c(\tau_1)\delta(\tau-\tau_1)\partial_\tau e(\tau)\right] \\
=& i\epsilon\left[\partial_\tau c(\tau)e(\tau) + c(\tau)\partial_\tau e(\tau)\right] = i\epsilon\partial_\tau(ce)
\end{aligned} \qquad [4.71]$$

Obviously $\delta_{\mathrm{B}}B = 0$ and $\delta_{\mathrm{B}}b = \epsilon B$. So we are left with $\delta_{\mathrm{B}}c$:

$$\begin{aligned}
\delta_{\mathrm{B}}c(\tau) &= \frac{i}{2}\epsilon \int d\tau_2\, d\tau_3\, f^{\tau}_{\tau_2\tau_3} c(\tau_2)c(\tau_3) \\
&= +\frac{i}{2}\epsilon \int d\tau_2\, d\tau_3\, \left[\delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_3) - \delta(\tau-\tau_3)\partial_\tau\delta(\tau-\tau_2)\right] c(\tau_2)c(\tau_3) \\
&= +\frac{i}{2}\epsilon \left[\int d\tau_3\, \partial_\tau\delta(\tau-\tau_3)c(\tau)c(\tau_3) - \int d\tau_2\, \partial_\tau\delta(\tau-\tau_2)c(\tau_2)c(\tau)\right] \\
&= -\frac{i}{2}\epsilon \left[\int d\tau_3\, \partial_{\tau_3}\delta(\tau-\tau_3)c(\tau)c(\tau_3) - \int d\tau_2\, \partial_{\tau_2}\delta(\tau-\tau_2)c(\tau_2)c(\tau)\right] \\
&= +\frac{i}{2}\epsilon \left[\int d\tau_3\, \delta(\tau-\tau_3)c(\tau)\partial_{\tau_3}c(\tau_3) - \int d\tau_2\, \delta(\tau-\tau_2)\partial_{\tau_2}c(\tau_2)c(\tau)\right] \\
&= +\frac{i}{2}\epsilon \left[c(\tau)\partial_\tau c(\tau) - \partial_\tau c(\tau)c(\tau)\right] = i\epsilon c\dot{c}
\end{aligned} \tag{4.72}$$

4.16 p 129: Eq (4.2.23) The BRST Action for the Point Particle

The classical action for the point particle is given by (1.2.5). In Euclidean space this becomes

$$S_1 = \int d\tau\, \left(\frac{1}{2}e^{-1}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}em^2\right) \tag{4.73}$$

For the gauge choice $e(\tau) = 1$, i.e. $F(\tau) = 1 - e(\tau)$, the gauge fixing term is simply (4.2.4)

$$S_2 = -i\int d\tau\, B\left[1 - e(\tau)\right] = i\int d\tau\, B(e-1) \tag{4.74}$$

The Faddeev-Popov determinant gives

$$\begin{aligned}
S_3 &= \int d\tau\, b(\tau) \int d\tau_1\, c(\tau_1)\delta_{\tau_1}(1 - e(\tau)) = -\int d\tau\, d\tau_1\, b(\tau)c(\tau_1)\delta_{\tau_1}e(\tau) \\
&= \int d\tau\, d\tau_1\, b(\tau)c(\tau_1)\partial_\tau[\delta(\tau-\tau_1)e(\tau)] = -\int d\tau\, d\tau_1\, \partial_\tau b(\tau)c(\tau_1)\delta(\tau-\tau_1)e(\tau) \\
&= -\int d\tau\, e(\tau)\partial_\tau b(\tau)c(\tau) = -\int d\tau\, e\dot{b}c
\end{aligned} \tag{4.75}$$

which is (4.2.23), when taking into account the correction on Joe's errata page.

4.17 p 130: Eq (4.2.25) The BRST Transformation of the Gauge Fixed Action for the Point Particle

The equation of motion for the e field in the non-gauge fixed action is

$$-\frac{1}{2}e^{-2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 + iB - e\dot{b}c \qquad [4.76]$$

Solving for B and fixing the gauge $e = 1$ gives

$$B = i\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right) \qquad [4.77]$$

and thus (4.2.22d) becomes

$$\delta_{\rm B}b = i\epsilon\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right) \qquad [4.78]$$

Let us now check that (4.2.25) is indeed a symmetry of the gauge fixed action. The classical part of the action will be invariant as the BRST symmetry is just a gauge symmetry with a gauge parameter $i\epsilon c$. We thus need to calculate

$$\begin{aligned}\delta_{\rm B}\int d\tau\,(-\dot{b}c) &= \int d\tau\,(\delta_{\rm B}c\partial_\tau b + c\partial_\tau\delta_{\rm B}b)\\ &= \int d\tau\,\left[i\epsilon c\partial_\tau c\partial_\tau b + c\partial_\tau i\epsilon\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right)\right]\\ &= \int d\tau\, i\epsilon\left(c\dot{c}\dot{b} + c\ddot{X}^\mu\dot{X}_\mu + c\ddot{b}c + c\dot{b}\dot{c}\right) = 0 \qquad [4.79]\end{aligned}$$

as the X^μ equations of motion is $\ddot{X}^\mu = 0$ and as also $c^2 = 0$.

Let us also check the nilpotency of the BRST transformation. We only need to check the nilpotency on the b ghost as all other transformations are unchanged

$$\begin{aligned}\delta^{(2)}_{\rm B}\delta^{(1)}_{\rm B}b &= \delta^{(2)}_{\rm B}i\epsilon_1\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right)\\ &= i\epsilon_1\left[-\dot{X}^\mu\partial_\tau(\delta^{(2)}_{\rm B}X_\mu) - (\partial_\tau\delta^{(2)}_{\rm B}b)c - \dot{b}\delta^{(2)}_{\rm B}c\right]\\ &= i\epsilon_1\left[-\dot{X}^\mu\partial_\tau(i\epsilon_2\dot{X}_\mu) - i\epsilon_2\partial_\tau\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right)c - \dot{b}i\epsilon_2 c\dot{c}\right]\\ &= \epsilon_1\epsilon_2\left(\dot{X}^\mu\ddot{X}_\mu - \dot{X}^\mu\ddot{X}_\mu - \ddot{b}cc - \dot{b}\dot{c}c - \dot{b}c\dot{c}\right) = 0 \qquad [4.80]\end{aligned}$$

4.18 p 130: Eq (4.2.26) The Canonical Commutation Relations for the Point Particle

Let us be extra careful regarding Euclidean vs Minkowski. The gauge-fixed action in Euclidean time τ is (4.2.24)

$$S_E = \int_{\tau_i}^{\tau_f} d\tau \; \left[\frac{1}{2}\dot{X}^\mu \dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right] \qquad [4.81]$$

We first rotate to Minkowski time, $\tau = it$, and hence $\partial_\tau = -i\partial_t$. The Euclidean action then becomes

$$S_E = \int_{it_i}^{it_f} idt \; \left[-\frac{1}{2}\partial_t X^\mu \partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right] \qquad [4.82]$$

The exponential of the path integral becomes

$$e^{-S_E} = e^{-\int_{it_i}^{it_f} idt\left[-\frac{1}{2}\partial_t X^\mu \partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right]} = e^{i\int_{it_i}^{it_f} dt\left[\frac{1}{2}\partial_t X^\mu \partial_t X_\mu - \frac{1}{2}m^2 - i\partial_t bc\right]} \qquad [4.83]$$

With Minkowski signature this needs to be e^{iS_M} so the Minkowski action is

$$S_M = \int_{it_i}^{it_f} dt \; \left[\frac{1}{2}\partial_t X^\mu \partial_t X_\mu - \frac{1}{2}m^2 - i\partial_t bc\right] \qquad [4.84]$$

We have used partial integration in the last term. The conjugate momenta are now

$$p = \Pi_X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t X_\mu)} = \partial_t X^\mu = i\dot{X}^\mu \quad ; \qquad \Pi_b = \frac{\partial \mathcal{L}}{\partial(\partial_t b)} = -ic \qquad [4.85]$$

c is not a dynamic variable so its conjugate momentum is zero. We can impose the general CCRs $[\hat{x}, \hat{p}]_\pm = i$

$$\begin{aligned} [X^\mu, p^\nu] = i\eta^{\mu\nu} \qquad &\Rightarrow \qquad [p^\mu, X^\nu] = -i\eta^{\mu\nu} \\ \{b, -ic\} = -i \qquad &\Rightarrow \qquad \{b, c\} = 1 \end{aligned} \qquad [4.86]$$

In order to find the BRST charge Q_B we follow the Noether procedure, see [2.36]. Recall how it works. We first work out that the Lagrangian transforms into a total derivative $\Delta\mathcal{L} = \partial_\mu \mathcal{J}^\mu$. The Noether current is then given by $j^\mu = (\partial\mathcal{L}/\partial(\partial_\mu\phi))\Delta\phi - \mathcal{J}^\mu$. We work with the Minkowski Lagrangian

$$\mathcal{L}_M = \frac{1}{2}\partial_t X^\mu \partial_t X_\mu - \frac{1}{2}m^2 - i\partial_t bc \qquad [4.87]$$

In Minkowski time the BRST transformations are

$$\begin{aligned}\delta_{\rm B}X^\mu &= \epsilon c\partial_t X^\mu\\ \delta_{\rm B}b &= i\epsilon\left(\frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right)\\ \delta_{\rm B}c &= \epsilon c\partial_t c\end{aligned} \tag{4.88}$$

We thus find

$$\begin{aligned}\delta_{\rm B}\mathcal{L}_M &= \partial_t X^\mu\partial_t\left(\epsilon c\partial_t X_\mu\right) - i\partial_t\left[i\epsilon\left(\frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right)\right]c - i\partial_t b\epsilon c\partial_t c\\ &= \epsilon\left(\partial_t c\partial_t X^\mu\partial_t X_\mu + c\partial_t X^\mu\partial^2 X_\mu + \partial_t X^\mu\partial^2 X_\mu c + i\partial_t^2 bcc + i\partial_t b\partial_t cc + i\partial_t bc\partial_t c\right)\\ &= \epsilon\partial_t(c\partial_t X^\mu\partial_t X_\mu)\end{aligned} \tag{4.89}$$

We thus have $\mathcal{J} = c\partial_t X^\mu\partial_t X_\mu$ and the BRST Noether current is

$$\begin{aligned}\epsilon j &= \frac{\partial\mathcal{L}_M}{\partial(\partial_t X^\mu)}\delta_{\rm B}X^\mu + \frac{\partial\mathcal{L}_M}{\partial(\partial_t b)}\delta_{\rm B}b - \mathcal{J}\\ &= \partial_t X^\mu\epsilon c\partial_t X^\mu - ici\epsilon\left(\frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right) - \epsilon c\partial_t X^\mu\partial_t X_\mu\\ &= \epsilon\left(-c\partial_t X^\mu\partial_t X_\mu - \frac{1}{2}c\partial_t X^\mu\partial_t X_\mu - \frac{1}{2}cm^2 - ic\partial_t bc + c\partial_t X^\mu\partial_t X_\mu\right)\\ &= -\epsilon c\left(\frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}cm^2\right) = -\epsilon c\frac{1}{2}(p^2+m^2)\end{aligned} \tag{4.90}$$

We can define the BRST Noether current as, keeping in mind that the overall sign is irrelevant,

$$j_{\rm B} = \frac{c}{2}(p^2+m^2) \tag{4.91}$$

We can rewrite this in therms of the Hamiltonian $H = \sum_i \dot{q}_i p_i - \mathcal{L}$, which is

$$\begin{aligned}H &= \partial_t X^\mu\partial_t X_\mu + \partial_t b(-ic) - \frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\\ &= \frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{1}{2}m^2 = \frac{1}{2}(p^2+m^2)\end{aligned} \tag{4.92}$$

We thus conclude that the BRST operator is given by

$$Q_{\rm B} = cH \tag{4.93}$$

4.19 p 131: Eq (4.3.1) The BRST Transformation for the Bosonic String

The BRST transformations for the bosonic string can be easily obtained from those of the point particle as the actions are so similar. The transformation for X^μ follows directly from the worldsheet diffeomorphism setting the gauge parameter equal to $i\epsilon c$. The transformation for c follows a similar pattern. The transformation for b comes from $\delta_{\mathrm{B}} b = \epsilon B$ and replacing B by its value from the equation of motion of the metric. To be specific we have $\delta_{\mathrm{B}} b(z) = \delta_{\mathrm{B}} b_{zz} = \epsilon \delta_{\mathrm{B}} B_{zz}$. We obtain the value of B_{zz} from the equation of motion for g_{zz}. The action is now $S = S_{\mathrm{m}} + S_{\mathrm{g}} + S_{\mathrm{g.f.}}$ with S_{m} and S_{g} the matter and ghost action and

$$S_{\mathrm{g.f.}} = \int d^2\sigma \sqrt{g}\,(-iB^{ab}F_{ab}) = -\frac{i}{4\pi}\int d^2\sigma \sqrt{g}\, B^{ab}(\delta_{ab} - g_{ab}) \qquad [4.94]$$

the gauge fixing action. The normalisation of F_{ab} is just a convention. Note that there is a sign difference in the gauge fixing term compared to Joe's book. Now the equation of motion from varying the worldsheet metric is

$$0 = \frac{\delta}{\delta g_{ab}}(S_{\mathrm{m}} + S_{\mathrm{g}} + S_{\mathrm{g.f.}}) = \frac{\sqrt{g}}{4\pi}(T_X^{ab} + T_g^{ab}) + \frac{\delta}{\delta g_{ab}} S_{\mathrm{g.f.}} \qquad [4.95]$$

We have

$$\begin{aligned}
\delta_{g_{ab}} S_{\mathrm{g.f.}} &= \delta_{g_{ab}}\left[-\frac{i}{4\pi}\int d^2\sigma \sqrt{g}\, B^{ab}(\delta_{ab} - g_{ab})\right] \\
&= -\frac{i}{4\pi}\int d^2\sigma \sqrt{g}\,\left(\frac{1}{2}g^{cd}\delta g_{cd} B^{ab}(\delta_{ab} - g_{ab}) - B^{ab}\delta g_{ab}\right) \\
&= -\frac{i}{4\pi}\int d^2\sigma \sqrt{g}\,\left(\frac{1}{2}g^{ab} B^{cd}(\delta_{cd} - g_{cd}) - B^{ab}\right)\delta g_{ab} \qquad [4.96]
\end{aligned}$$

We now fix the gauge so that the first term between brackets is zero and have

$$\begin{aligned}
\frac{\delta}{\delta g_{ab}(\sigma)} &= +\frac{i}{4\pi}\int d^2\sigma'\, B^{cd}(\sigma')\frac{\delta g_{cd}(\sigma')}{\delta g_{ab}(\sigma)} = +\frac{i}{4\pi}\int d^2\sigma'\, B^{cd}(\sigma')\delta_c^a \delta_d^b \delta^2(\sigma' - \sigma) \\
&= +\frac{i}{4\pi}B^{ab}(\sigma) \qquad [4.97]
\end{aligned}$$

We thus get for the g_{ab} equation of motion

$$0 = \frac{1}{4\pi}(T_X^{ab} + T_g^{ab}) + \frac{i}{4\pi}B^{ab} \qquad [4.98]$$

or

$$B^{ab} = i(T_X^{ab} + T_g^{ab}) \qquad [4.99]$$

and

$$\delta_{\rm B} b = i\epsilon(T_X + T_g) \qquad [4.100]$$

Note that the variation of c is

$$\delta_{\rm B} c = i\epsilon c\partial c \qquad [4.101]$$

and similar for the anti-holomorphic part. See Joe's errata page for an explanation on this.

4.20 p 131: Nilpotency of the BRST Transformation for the Bosonic String

First we consider the action of $\delta_{\rm B}^2$ on X^μ:

$$\begin{aligned}
\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)} X^\mu &= i\epsilon_1 \delta_{\rm B}^{(2)}(c\partial X^\mu + \tilde{c}\bar{\partial} X^\mu) \\
&= i\epsilon_1 \left[i\epsilon_2 c\partial c \partial X^\mu + c i\epsilon_2 \partial(c\partial X^\mu + \tilde{c}\bar{\partial}X^\mu) + i\epsilon_2 \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + \tilde{c} i\epsilon_2 \bar{\partial}(c\partial X^\mu + \tilde{c}\bar{\partial}X^\mu)\right] \\
&= \epsilon_1\epsilon_2 \left[-c\partial c\partial X^\mu + c\partial(c\partial X^\mu + \tilde{c}\bar{\partial}X^\mu) - \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + \tilde{c}\bar{\partial}(c\partial X^\mu + \tilde{c}\bar{\partial}X^\mu)\right] \\
&= \epsilon_1\epsilon_2 \big(- c\partial c\partial X^\mu + c\partial c\partial X^\mu + cc\partial^2 X^\mu + c\partial\tilde{c}\bar{\partial}X^\mu + c\tilde{c}\partial\bar{\partial}X^\mu \\
&\qquad - \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + \tilde{c}\bar{\partial}c\partial X^\mu + \tilde{c}c\bar{\partial}\partial X^\mu + \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + \tilde{c}\tilde{c}\bar{\partial}^2 X^\mu \big) \\
&= \epsilon_1\epsilon_2 \big(- c\partial c\partial X^\mu + c\partial c\partial X^\mu + 0 + 0 + 0 - \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + 0 + 0 + \tilde{c}\bar{\partial}\tilde{c}\bar{\partial}X^\mu + 0 \big) \\
&= 0
\end{aligned} \qquad [4.102]$$

Next, consider

$$\begin{aligned}
\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)} c &= i\epsilon_1 \delta_{\rm B}^{(2)} c\partial c = i\epsilon_1 \left(i\epsilon_2 c\partial c \partial c + c i\epsilon_2 c\partial c\right) \\
&= \epsilon_1\epsilon_2 (c\partial c\partial c + cc\partial c) = 0
\end{aligned} \qquad [4.103]$$

Finally we consider

$$\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)} b = i\epsilon_1 \delta_{\rm B}^{(2)}(T_X + T_g) \qquad [4.104]$$

We work out the BRST transformations of the two energy-momentum tensors separately

$$\begin{aligned}
\delta_{\rm B} T_X &= \delta_{\rm B}\left(-\frac{1}{\alpha'}\partial X^\mu \partial X_\mu \right) = -\frac{2i\epsilon}{\alpha'}\partial X^\mu \partial(c\partial X_\mu + \tilde{c}\bar{\partial}X_\mu) \\
&= -\frac{2i\epsilon}{\alpha'}(\partial c\partial X^\mu \partial X_\mu + c\partial X \partial^2 X)
\end{aligned} \qquad [4.105]$$

where we have used the equations of motion $\partial\tilde{c} = \partial\bar{\partial}X_\mu = 0$. For the ghost energy momentum tensor we find, using (2.5.11) with $\lambda = 2$

$$\begin{aligned}
\delta_{\rm B} T_g =& \delta_{\rm B}(-\partial bc - 2b\partial c) \\
=& -i\epsilon\partial(T_X + T_g)c - \partial bi\epsilon c\partial c - 2i\epsilon(T_X + T_g)\partial c - 2bi\epsilon\partial(c\partial c) \\
=& \frac{2i\epsilon}{\alpha'}\partial X^\mu \partial^2 X_\mu c - i\epsilon(-\partial^2 bc - \partial b\partial c - 2\partial b\partial c - 2b\partial^2 c)c - \partial bi\epsilon c\partial c \\
&+ \frac{2i\epsilon}{\alpha'}\partial X\partial X\partial c - 2i\epsilon(-\partial bc - 2b\partial c)\partial c - 2bi\epsilon\partial c\partial c - 2bi\epsilon c\partial^2 c \\
=& i\epsilon\left(\frac{2}{\alpha'}\partial X^\mu\partial^2 X_\mu c + 3\partial b\partial cc + 2b\partial^2 cc - \partial b\partial cc + \frac{2}{\alpha'}\partial X\partial X\partial c - 2\partial b\partial cc - 2b\partial^2 cc\right) \\
=& \frac{2i\epsilon}{\alpha'}\left(\partial X^\mu\partial^2 X_\mu c + \partial X\partial X\partial c\right)
\end{aligned} \tag{4.106}$$

We thus see that

$$\begin{aligned}
\delta_{\rm B}^{(2)}\delta_{\rm B}^{(1)} b =& i\epsilon_1\delta_{\rm B}^{(2)}(T_X + T_g) \\
=& i\epsilon_1\left[-\frac{2i\epsilon_2}{\alpha'}(\partial c\partial X^\mu\partial X_\mu + c\partial X\partial^2 X) + \frac{2i\epsilon_2}{\alpha'}\left(\partial X^\mu\partial^2 X_\mu c + \partial X\partial X\partial c\right)\right] \\
=& 0
\end{aligned} \tag{4.107}$$

We have thus indeed shown that $\delta_{\rm B}^2 = 0$ up to the equations of motion.

4.21 p 131: Eq (4.3.3) The BRST Current for the Bosonic String

We will not derive the expression of the BRST current for the bosonic string from Noether's theorem. Those with an inclination for tedious calculations are invited to do so. Rather, we will show that the current (4.3.3) generates the correct BRST transformations of the field. We will also show that (1) the BRST current is a spin three primary field if the central charge of the matter sector is 26 and (2) how nilpotency follows from the OPE of the BRST current with itself. To do this, we first need the relevant OPES of the BRST current with the fields.

4.22 p 132: Eq (4.3.4) OPEs with the BRST Current

We start with the OPE of the BRST current with the X^μ field

$$\begin{aligned}
j_{\rm B}(z)X^\mu(w) =& \left[cT^m(z) + bc\partial c(z) + \frac{3}{2}\partial^2 c(z)\right]X^\mu(w) \\
=& c\overline{T^m(z)X^\mu}(w) = \frac{c\partial X^\mu(w)}{z-w}
\end{aligned} \tag{4.108}$$

In order to find the BRST transformation of X^μ we can use the usual contour integration

$$\delta_{\rm B} X^\mu(w) = i\epsilon \oint_{C_w} \frac{dz}{2\pi} j_{\rm B}(z) X^\mu(w) = i\epsilon \oint_{C_w} \frac{dz}{2\pi} \frac{c\partial X^\mu(w)}{z-w} = i\epsilon c\partial X^\mu(w) \qquad [4.109]$$

which is (4.3.1a).

We then consider

$$\begin{aligned} j_{\rm B}(z)b(w) =& \left[cT^m(z) + bc\partial c(z) + \frac{3}{2}\partial^2 c(z)\right] b(w) \\ =& T^m(z)\overline{c(z)b}(w) - b\partial c(z)\overline{c(z)b}(w) + bc(z)\partial\overline{c(z)b}(w) + \frac{3}{2}\partial^2 \overline{c(z)b}(w) \\ =& \frac{T^m(z)}{z-w} - \frac{b\partial c(z)}{z-w} - \frac{bc(z)}{(z-w)^2} + \frac{3}{(z-w)^3} \\ =& \frac{T^m(w)}{z-w} - \frac{b\partial c(w)}{z-w} - \frac{bc(w) + (z-w)\partial(bc)(w)}{(z-w)^2} + \frac{3}{(z-w)^3} \\ =& \frac{3}{(z-w)^3} + \frac{-bc(w)}{(z-w)^2} + \frac{(T^m - \partial bc - 2b\partial c)(w)}{z-w} \\ =& \frac{3}{(z-w)^3} + \frac{j^g(w)}{(z-w)^2} + \frac{(T^m + T^g)(w)}{z-w} \end{aligned} \qquad [4.110]$$

where we have used the definition of the ghost current (2.5.14), $j^g = -bc$ and of the ghost energy momentum tensor $T^g = -\partial bc - 2b\partial c$. The BRST transformation of b follows from the contour integration

$$\begin{aligned} \delta_{\rm B} b(w) =& i\epsilon \oint_{C_w} \frac{dz}{2\pi} j_{\rm B}(z) b(w) = i\epsilon \oint_{C_w} \frac{dz}{2\pi} \left[\frac{3}{(z-w)^3} + \frac{j^g(w)}{(z-w)^2} + \frac{(T^m + T^g)(w)}{z-w}\right] \\ =& i\epsilon\big(T^m(w) + T^g(w)\big) \end{aligned} \qquad [4.111]$$

which is (4.3.1b).

Next we consider

$$\begin{aligned} j_{\rm B}(z)c(w) =& \left[cT^m(z) + bc\partial c(z) + \frac{3}{2}\partial^2 c(z)\right] c(w) \\ =& \overline{b(z)c}(w) c\partial c(z) = \frac{c\partial c(w)}{z-w} \end{aligned} \qquad [4.112]$$

the BRST transformation of c is then given by

$$\delta_{\rm B} c(w) = i\epsilon \oint_{C_w} \frac{dz}{2\pi} j_{\rm B}(z) c(w) = i\epsilon \oint_{C_w} \frac{dz}{2\pi} \frac{c\partial c(w)}{z-w} = i\epsilon c\partial c(w) \qquad [4.113]$$

which is (4.3.1c).

There are, of course, similar relations for $\tilde{j}_\mathrm{B}$ and we thus conclude that these currents indeed generate the BRST transformations.

Finally, consider a general primary matter field $\mathcal{O}^m$

$$\begin{aligned} j_\mathrm{B}(z)\mathcal{O}^m(w) &= \left[cT^m(z) + bc\partial c(z) + \frac{3}{2}\partial^2 c(z)\right]\mathcal{O}^m(w) \\ &= c\overbracket{T^m(z)\mathcal{O}}^m(w) = c(z)\left[\frac{h\mathcal{O}^m(w)}{(z-w)^2} + \frac{\partial\mathcal{O}^m(w)}{z-w}\right] \\ &= \frac{hc\mathcal{O}^m(w)}{(z-w)^2} + \frac{h\partial c\mathcal{O}^m(w) + c\partial\mathcal{O}^m(w)}{z-w} \end{aligned} \qquad [4.114]$$

For completeness, we work out the BRST transformation of such a field

$$\begin{aligned} \delta_\mathrm{B}\mathcal{O}^m(w) =& i\epsilon\oint_{C_w}\frac{dz}{2\pi}j_\mathrm{B}(z)\mathcal{O}^m(w) = i\epsilon\oint_{C_w}\frac{dz}{2\pi}\frac{h\partial c\mathcal{O}^m(w) + c\partial\mathcal{O}^m(w)}{z-w} \\ =& i\epsilon h\partial c\mathcal{O}^m(w) + c\partial\mathcal{O}^m(w) \end{aligned} \qquad [4.115]$$

4.23 p 132: Eq (4.3.6) The Anticommutator $\{Q_\mathrm{B}, b_m\}$

We use (2.6.15) for Grassmann fields

$$\{Q, b(z)\} = \mathrm{Res}_{z_1\to z}\, j(z_1)b(z) \qquad [4.116]$$

Extracting the mode $b_m = \oint \frac{dz}{2\pi i} z^{m+1} b(z)$ we find

$$\begin{aligned} \{Q, b_m\} &= \oint \frac{dz}{2\pi i} z^{m+1}\mathrm{Res}_{z_1\to z}\, j(z_1)b(z) \\ &= \oint \frac{dz}{2\pi i} z^{m+1}\mathrm{Res}_{z_1\to z}\left[\frac{3}{(z_1-z)^3} + \frac{j^g(z)}{(z_1-z)^2} + \frac{(T^m+T^g)(z)}{z_1-z}\right] \\ &= \oint \frac{dz}{2\pi i} z^{m+1}(T^m+T^g)(z) = L^m_m + L^g_m \end{aligned} \qquad [4.117]$$

Let me make two simple remarks. First the reader shall not be confused by the superscript ^{m} denoting the matter field and the subscript $_m$ being a Laurent index. Second, and last, note that Q_B is the conserved charge of the BRST current. Because the BRST current is a dimension one field its Laurent expansion is $j_{\mathrm{B}\,m} = \oint \frac{dz}{2\pi i} z^m j_\mathrm{B}(z)$ and so $Q_\mathrm{B} \equiv j_{\mathrm{B}\,0}$. The anticommutator of b with any other mode of the the BRST current will include contributions form the higher order poles of the OPE as well. See the derivation of the Virasoro operator [2.110] if that is not clear.

4.24 p 132: Eq (4.3.7) The Mode Expansion of the BRST Operator

We focus on the holomorphic part as the anti-holomorphic part is similar. We need to work out $Q_B = \oint dz\, j_{\rm B}(z)$ with $j_{\rm B} = {}^{\circ}_{\circ} cT {}^{\circ}_{\circ} + {}^{\circ}_{\circ} bc\partial c {}^{\circ}_{\circ} + \frac{3}{2}\partial^2 c$ in terms of the underlying modes. We have here included explicitly the (creation-annihilation) normal ordering symbols as they will be needed in the mode expansions. Let us do the three terms separately. We don't write the ^{m} for the matter sector as it will be understood and use the standard Laurent expansion for w dimension h field: $\mathcal{O}^{(h)} = \sum_n \mathcal{O}^{(h)}_n / z^{n+h}$

$$Q^{(1)}_{\rm B} = \oint dz \sum_{m,n=-\infty}^{\infty} {}^{\circ}_{\circ} \frac{c_m}{z^{m-1}} \frac{L_n}{z^{n+2}} {}^{\circ}_{\circ} \tag{4.118}$$

Because the c_m and L_n commute there is no normal ordering ambiguity and we can drop the normal ordering signs and find

$$Q^{(1)}_{\rm B} = \sum_{m,n=-\infty}^{\infty} \oint dz \frac{c_m L_n}{z^{m+n+1}} = \sum_{m,n=-\infty}^{\infty} c_m L_n \delta_{m+n,0} = \sum_{n=-\infty}^{\infty} c_n L_{-n} \tag{4.119}$$

For the second term we find

$$\begin{aligned} Q^{(2)}_{\rm B} &= \oint dz \sum_{\ell,m,n=-\infty}^{\infty} {}^{\circ}_{\circ} \frac{b_\ell}{z^{\ell+2}} \frac{c_m}{z^{m-1}} \frac{-(n-1)c_n}{z^n} {}^{\circ}_{\circ} \\ &= -\sum_{\ell,m,n=-\infty}^{\infty} (n-1) \oint dz \frac{{}^{\circ}_{\circ} b_\ell c_m c_n {}^{\circ}_{\circ}}{z^{\ell+m+n+1}} \end{aligned} \tag{4.120}$$

There is a potential normal ordering ambiguity when b and c don't anti-commute, i.e. when $\ell + m = 0$ or $\ell + n = 0$. Ignoring this for the moment we find

$$Q^{(2)}_{\rm B} = -\sum_{\ell,m,n=-\infty}^{\infty} (n-1) b_\ell c_m c_n \delta_{\ell+m+n} = -\sum_{m,n=-\infty}^{\infty} (n-1) c_m c_n b_{-m-n} \tag{4.121}$$

We have moved the b_{-m-n} to the right. This is fine, because moving it to the right will only add contributions of the form c_0 which we will include in the normal ordering constant that we will determine in a different way. We also note that $\sum_{m,n} c_m c_n b_{-m-n} = 0$ by symmetry considerations. We can also antisymmetrize the expression and get

$$Q^{(2)}_{\rm B} = \sum_{m,n=-\infty}^{\infty} m c_m c_n b_{-m-n} = \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2} {}^{\circ}_{\circ} c_m c_n b_{-m-n} {}^{\circ}_{\circ} \tag{4.122}$$

where we have reintroduced the creation-annihilation normal ordering symbols as the difference will be in the normal ordering constant c_0. Finally, the last term $\frac{3}{2}\partial^2 c$ gives

$$Q_{\mathrm{B}}^{(3)} = \frac{3}{2}\oint dz \sum_{n=-\infty}^{\infty} n(n-1)\frac{c_n}{z^{n+1}} = \frac{3}{2}\sum_{n=-\infty}^{\infty} n(n-1)c_n\delta_{n,0} = 0 \qquad [4.123]$$

As expected, it doesn't contribute to the BRST charge as it is a total derivative. We conclude that the BRST charge is given by

$$Q_{\mathrm{B}} = \sum_{n=-\infty}^{\infty} c_n L_{-n}^m + \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2} {}^{\circ}_{\circ} c_m c_n b_{-m-n} {}^{\circ}_{\circ} + a_{\mathrm{B}} c_0 + \mathrm{c.c.} \qquad [4.124]$$

where a_{B} is the normal ordering constant and c.c. stands for the anti-holomorphic part. We have reintroduced the superscript ^{m} to denote the matter fields and could add creation annihilation normal ordering symbols in the first sum, but it wouldn't make a difference as c_n and L_m^m commute.

4.25 p 132: Eq (4.3.7) The BRST Normal Ordering Constant

We have

$$\{Q_{\mathrm{B}}, b_0\} = \sum_{n=-\infty}^{\infty} \{c_n L_{-n}^m, b_0\} + \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2}\{{}^{\circ}_{\circ} c_m c_n b_{-m-n} {}^{\circ}_{\circ}, b_0\} + a_{\mathrm{B}}\{c_0, b_0\} \qquad [4.125]$$

Split the calculation in three. First,

$$P_1 = \sum_{n=-\infty}^{\infty} L_{-n}^m \delta_{n,0} = L_0^m \qquad [4.126]$$

Next

$$P_2 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\Bigg[\sum_{n=-\infty}^{-1}(m-n)\{-{}^{\circ}_{\circ} c_n c_m b_{-m-n} {}^{\circ}_{\circ}, b_0\} + m\{{}^{\circ}_{\circ} c_m c_0 b_{-m} {}^{\circ}_{\circ}, b_0\}$$
$$+ \sum_{n=1}^{\infty}(m-n)\{-{}^{\circ}_{\circ} c_m b_{-m-n} c_n {}^{\circ}_{\circ}, b_0\}\Bigg] \qquad [4.127]$$

We work this out in three parts as well, $P_2 = \frac{1}{2}(P_{2a} + P_{2b} + P_{2c})$ with

$$\begin{aligned} P_{2a} = &- \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n)\{\circ c_n c_m b_{-m-n} \circ, b_0\} + \sum_{n=-\infty}^{-1} n\{\circ c_n c_0 b_{-n} \circ, b_0\} \\ &- \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n)\{\circ c_n c_m b_{-m-n} \circ, b_0\} \\ = &- \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n)\{c_n c_m b_{-m-n}, b_0\} + \sum_{n=-\infty}^{-1} n\{c_n c_0 b_{-n}, b_0\} \\ &+ \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n)\{c_n b_{-m-n} c_m, b_0\} \end{aligned} \quad [4.128]$$

We now use the fact that for four Grassmann numbers $f_1, \cdots, f_4$ we have

$$\{f_1 f_2 f_3, f_4\} = f_1 f_2 \{f_3, f_4\} - f_1\{f_2, f_4\} f_3 + \{f_1, f_4\} f_2 f_3 \quad [4.129]$$

which can be easily seen by working out both sides. This gives

$$\begin{aligned} P_{2a} = &- \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n)(-c_n \delta_{m,0} b_{-m-n} + \delta_{n,0} c_m b_{-m-n}) + \sum_{n=-\infty}^{-1} n(-c_n b_{-n} + \delta_{n,0} c_0 b_{-n}) \\ &+ \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n)(c_n b_{-m-n} \delta_{m,0} + \delta_{n,0} b_{-m-n} c_m) = - \sum_{n=-\infty}^{-1} n c_n b_{-n} \end{aligned} \quad [4.130]$$

as neither m nor n can be zero in any of the sums. Next, we have

$$\begin{aligned} P_{2b} &= \sum_{m=-\infty}^{-1} m\{\circ c_m c_0 b_{-m} \circ, b_0\} + \sum_{m=1}^{\infty} m\{\circ c_m c_0 b_{-m} \circ, b_0\} \\ &= \sum_{m=-\infty}^{-1} m\{c_m c_0 b_{-m}, b_0\} - \sum_{m=1}^{\infty} m\{b_{-m} c_0 c_m, b_0\} \\ &= \sum_{m=-\infty}^{-1} m(-c_m b_{-m} + \delta_{m,0} c_0 b_{-m}) - \sum_{m=1}^{\infty} (b_{-m} c_0 \delta_{m,0} - b_{-m} c_m) \\ &= - \sum_{m=-\infty}^{-1} m c_m b_{-m} + \sum_{m=1}^{\infty} m b_{-m} c_m \end{aligned} \quad [4.131]$$

The last part of P_2 is

$$\begin{aligned}
P_{2c} = &- \sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}(m-n)\{{}^\circ_\circ c_m b_{-m-n} c_n {}^\circ_\circ, b_0\} + \sum_{n=1}^{\infty} n\{{}^\circ_\circ c_0 b_{-n} c_n {}^\circ_\circ, b_0\} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m-n)\{{}^\circ_\circ c_m b_{-m-n} c_n {}^\circ_\circ, b_0\} \\
= &- \sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}(m-n)\{c_m b_{-m-n} c_n, b_0\} - \sum_{n=1}^{\infty} n\{b_{-n} c_0 c_n, b_0\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m-n)\{b_{-m-n} c_m c_n, b_0\} \\
= &- \sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}(m-n)(c_m b_{-m}\delta_{n,0} + \delta_{m,0} b_{-n} c_n) - \sum_{n=1}^{\infty} n(b_{-n} c_0 \delta_{n,0} - b_{-n} c_n) \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m-n)(b_{-m} c_m \delta_{n,0} - b_{-n}\delta_{m,0} c_n) \\
= &+ \sum_{n=1}^{\infty} n b_{-n} c_n
\end{aligned} \tag{4.132}$$

Collecting the different contributions we have

$$\begin{aligned}
P_2 &= \frac{1}{2}\left(-\sum_{n=-\infty}^{-1} n c_n b_{-n} - \sum_{m=-\infty}^{-1} n c_m b_{-m} + \sum_{m=1}^{\infty} n b_{-m} c_m + \sum_{n=1}^{\infty} n b_{-n} c_n\right) \\
&= -\sum_{n=-\infty}^{-1} n c_n b_{-n} + \sum_{m=1}^{\infty} n b_{-m} c_m = \sum_{n=-\infty}^{\infty} n {}^\circ_\circ b_{-n} c_n {}^\circ_\circ = -\sum_{n=-\infty}^{\infty} n {}^\circ_\circ b_n c_{-n} {}^\circ_\circ
\end{aligned} \tag{4.133}$$

We now use the mode expansion of the ghost energy momentum tensor (2.7.21) with $\lambda = 2$

$$L_0^g = -\sum_{n=-\infty}^{\infty} n {}^\circ_\circ b_n c_{-n} {}^\circ_\circ - 1 \tag{4.134}$$

Therefore

$$P_2 = L_0^g + 1 \tag{4.135}$$

Finally, we have immediately $P_3 = a_{\mathrm{B}}$. Thus

$$\{Q_{\mathrm{B}}, b_0\} = P_1 + P_2 + P_3 = L_0^m + L_0^g + 1 + a_{\mathrm{B}} \tag{4.136}$$

As this needs to be $L_0^m + L_0^g$ we find indeed that

$$a_{\mathrm{B}} = -1 \tag{4.137}$$

4.26 p 132: Eq (4.3.10) The $j_{\mathrm{B}}(z)j_{\mathrm{B}}(w)$ OPE and the Nilpotency of the BRST Charge

We have

$$\begin{aligned} j_{\mathrm{B}}(z)j_{\mathrm{B}}(w) &= \left(cT^m + bc\partial c + \frac{3}{2}\partial^2 c\right)(z)\left(cT^m + bc\partial c + \frac{3}{2}\partial^2 c\right)(w) \\ &= cT^m(z)cT^m(w) + cT^m(z)bc\partial c(w) + bc\partial c(z)cT^m(w) + bc\partial c(z)bc\partial c(w) \\ &\quad + bc\partial c(z)\frac{3}{2}\partial^2 c(w) + \frac{3}{2}\partial^2 c(z)bc\partial c(w) \end{aligned} \qquad [4.138]$$

We split this calculation in six

$$\begin{aligned} \mathrm{jj}_1 &= cT^m(z)cT^m(w) = c(z)c(w)\overbrace{T^m(z)T^m}(w) \\ &= c(z)c(w)\left[\frac{c^m/2}{(z-w)^4} + \frac{2T^m(w)}{(z-3)^2} + \frac{\partial T^m(w)}{z-w}\right] \\ &= \left[c(w) + (z-w)\partial c(w) + \frac{1}{2}(z-w)^2\partial^2 c(w) + \frac{1}{6}(z-w)^3\partial^3 c(w)\right]c(w) \\ &\quad \times \left[\frac{c^m/2}{(z-w)^4} + \frac{2T^m(w)}{(z-w)^2} + \frac{\partial T^m(w)}{z-w}\right] \\ &= \frac{(c^m/2)\partial cc(w)}{(z-w)^3} + \frac{(c^m/4)\partial^2 cc(w)}{(z-w)^2} + \frac{(c^m/12)\partial^3 cc(w) + 2\partial cc T^m(w)}{z-w} \end{aligned} \qquad [4.139]$$

$$\begin{aligned} \mathrm{jj}_2 &= cT^m(z)bc\partial c(w) = \overbrace{c(z)b}(w)T^m(z)c\partial c(w) = \frac{T^m(z)c\partial c(w)}{z-w} \\ &= -\frac{\partial cc T^m(w)}{z-w} \end{aligned} \qquad [4.140]$$

$$\begin{aligned} \mathrm{jj}_3 &= bc\partial c(z)cT^m(w) = \overbrace{b(z)c}(w)c\partial c(z)T^m(w) = \frac{c\partial c(z)T^m(w)}{z-w} \\ &= -\frac{\partial cc T^m(w)}{z-w} \end{aligned} \qquad [4.141]$$

$$\begin{aligned} \mathrm{jj}_4 &= bc\partial c(z)bc\partial c(w) \\ &= \overbrace{b(z)c}(w)\left[\overbrace{c(z)b}(w)\partial c(z)\partial c(w) - \partial\overbrace{c(z)b}(w)c(z)\partial c(w)\right] \\ &\quad + \overbrace{b(z)\partial c}(w)\left[-\overbrace{c(z)b}(w)\partial c(z)c(w) + \partial\overbrace{c(z)b}(w)c(z)c(w)\right] \\ &= \frac{1}{z-w}\left[\frac{\partial c(z)\partial c(w)}{z-w} + \frac{c(z)\partial c(w)}{(z-w)^2}\right] + \frac{1}{(z-w)^2}\left[-\frac{\partial c(z)c(w)}{z-w} - \frac{c(z)c(w)}{(z-w)^2}\right] \\ &= -\frac{c(z)c(w)}{(z-w)^4} + \frac{c(z)\partial c(w) - \partial c(z)c(w)}{(z-w)^3} + \frac{\partial c(z)\partial c(w)}{(z-w)^2} \end{aligned} \qquad [4.142]$$

Combining the denominators

$$\begin{aligned}
\text{jj}_4 &= \frac{-\partial cc(w) + c\partial c(w) - \partial cc(w)}{(z-w)^3} + \frac{-\frac{1}{2}\partial^2 cc(w) - \partial^2 cc(w)}{(z-w)^2} \\
&\quad + \frac{-\frac{1}{6}\partial^3 cc(w) + \frac{1}{2}\partial^2 c\partial c(w) - \frac{1}{2}\partial^3 cc(w) + \partial^2 cc(w)}{z-w} \\
&= \frac{-3\partial cc(w)}{(z-w)^3} + \frac{-\frac{3}{2}\partial^2 cc(w)}{(z-w)^2} + \frac{-\frac{2}{3}\partial^3 cc(w) + \frac{3}{2}\partial^2 c\partial c(w)}{z-w} \qquad [4.143]
\end{aligned}$$

Next

$$\begin{aligned}
\frac{2}{3}\text{jj}_5 &= bc\partial c(z)\frac{3}{2}\partial^2 c(w) = \overbrace{b(z)\partial^2 c}(w)c\partial c(z) = \frac{2c\partial c(z)}{(z-w)^3} \\
&= \frac{-2\partial cc(w)}{(z-w)^3} + \frac{2c\partial^2 c(w)}{(z-w)^2} + \frac{\partial c\partial^2 c(w) + c\partial^3 c(w)}{z-w} \\
&= \frac{-2\partial cc(w)}{(z-w)^3} + \frac{-2\partial^2 cc(w)}{(z-w)^2} + \frac{-\partial^2 c\partial c(w) - \partial^3 cc(w)}{z-w} \qquad [4.144]
\end{aligned}$$

Finally

$$\frac{2}{3}\text{jj}_6 = \partial^2 c(z)bc\partial c(w) = \partial^2 \overbrace{c(z)b}(w)c\partial c(w) = -\frac{2\partial c(w)c}{(z-w)^3} \qquad [4.145]$$

Let us now bring everything together. We start with the numerator of $(z-w)^{-3}$

$$\begin{aligned}
o\big((z-w)^{-3}\big) &= (c^m/2)\partial cc(w) - 3\partial cc(w) - \frac{3}{2}2\partial cc(w) - \frac{3}{2}2\partial cc(w) \\
&\rightsquigarrow \frac{(c^m - 18)\partial cc(w)}{2(z-w)^3} \qquad [4.146]
\end{aligned}$$

Next

$$\begin{aligned}
o\big((z-w)^{-2}\big) &= (c^m/4)\partial^2 cc(w) - \frac{3}{2}\partial^2 cc(w) - \frac{3}{2}2\partial^2 cc(w) \\
&\rightsquigarrow \frac{(c^m - 18)\partial^2 cc(w)}{4(z-w)^2} \qquad [4.147]
\end{aligned}$$

Finally

$$\begin{aligned}
o\big((z-w)^{-1}\big) &= (c^m/12)\partial^3 cc(w) + 2\partial ccT^m(w) - \partial ccT^m(w) - \partial ccT^m(w) \\
&\quad - \frac{2}{3}\partial^3 cc(w) + \frac{3}{2}\partial^2 c\partial c(w) + \frac{3}{2}(-\partial^2 c\partial c(w) - \partial^3 cc(w)) \\
&\rightsquigarrow \frac{(c^m - 26)\partial^3 cc(w)}{12(z-w)} \qquad [4.148]
\end{aligned}$$

We conclude that

$$j_{\rm B}(z)j_{\rm B}(w) = \frac{(c^m-18)\partial cc(w)}{2(z-w)^3} + \frac{(c^m-18)\partial^2 cc(w)}{4(z-w)^2} + \frac{(c^m-26)\partial^3 cc(w)}{12(z-w)} \qquad [4.149]$$

Let us conclude by showing explicitly that this leads a nilpotent charge. We have $Q_{\rm B}^2 = \frac{1}{2}\{Q_{\rm B}, Q_{\rm B}\}$ and we can compute the anti-commutator from a contour integral

$$\begin{aligned}\{Q_{\rm B}, Q_{\rm B}\} &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1\to z_2} j_{\rm B}(z_1) j_{\rm B}(z_2) \\ &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1\to z_2}\left[\frac{(c^m-18)\partial cc(z_2)}{2(z_1-z_2)^3} + \frac{(c^m-18)\partial^2 cc(z_2)}{4(z_1-z_2)^2} + \frac{(c^m-26)\partial^3 cc(z_2)}{12(z_1-z_2)}\right] \\ &= \frac{(c^m-26)}{12}\oint \frac{dz_2}{2\pi i}\partial^3 cc(z_2) \qquad [4.150]\end{aligned}$$

To work this out, first note that $\partial^3 c(z) = -\sum_m (m-1)m(m+1)c_m z^{-m-2}$ and thus

$$c\partial^3 c(z) = -\sum_{m,n=-\infty}^{\infty} \frac{(m^3-m){}^\circ_\circ c_n c_m{}^\circ_\circ}{z^{n+m+1}} \qquad [4.151]$$

Therefore

$$\begin{aligned}\{Q_{\rm B}, Q_{\rm B}\} &= -\frac{(c^m-26)}{12}\oint \frac{dz_2}{2\pi i}\sum_{m,n=-\infty}^{\infty} \frac{(m^3-m){}^\circ_\circ c_n c_m{}^\circ_\circ}{z^{n+m+1}} \\ &= -\frac{(c^m-26)}{12}\sum_{m=-\infty}^{\infty}(m^3-m){}^\circ_\circ c_{-m} c_m{}^\circ_\circ \qquad [4.152]\end{aligned}$$

The normal ordering causes this to be zero on physical states, as it should because physical states are annihilated by the BRST charge. But the operator is only nilpotent if $c^m = 26$.

4.27 p 133: Eq (4.3.11) The BRST Current as a Primary Field

Recalling that $T = T^m + T^g$ and $j_{\rm B} = cT^m(w) + \frac{1}{2}T^g(w) + \gamma\partial^2 c$ we have

$$\begin{aligned}T(z)j_{\rm B}(w) &= (T^m(z) + T^g(z))\left(cT^m(w) + \frac{1}{2}cT^g(w) + \gamma\partial^2 c\right) \\ &= c(w)\wick{\c T^m(z) \c T^m(w)} + T^m(w)\wick{\c T^g(z) \c c(w)} + \frac{1}{2}c(w)\wick{\c T^g(z) \c T^g(w)} \\ &\quad + \frac{1}{2}T^g(w)\wick{\c T^g(z) \c c(w)} + \gamma\wick{\c T^g(z)\partial^2 \c c(w)} \qquad [4.153]\end{aligned}$$

We have replaced momentarily the factor $3/2$ by a parameter γ. As per our Standard Operating Procedure we split this calculation in five

$$\mathsf{Tj}_1 = c(w)T^m(z)T^m(w) = \frac{c^m c(w)}{2(z-w)^4} + \frac{2cT^m(w)}{(z-w)^2} + \frac{c\partial T^m(w)}{z-w}$$
$$\mathsf{Tj}_2 = T^m(w)T^g(z)c(w) = \frac{-cT^m(w)}{(z-w)^2} + \frac{T^m\partial c(w)}{z-w} \qquad [4.154]$$

We take the third and fourth term together. Indeed we need to be careful here because $cT^g(w)$ is a composite operator and so the normal inside this operator is important. So it is easiest to compute these OPEs from the basic underlying fields, i.e. using $T^g(z) = (c\partial b + 2\partial cb)(z)$ and $\frac{1}{2}cT^g(w) = bc\partial c(w)$

$$\mathsf{Tj}_3 + \mathsf{Tj}_4 = T^g(z)\frac{1}{2}cT^g(w) = (c\partial b + 2\partial cb)(z)bc\partial c(w) \qquad [4.155]$$

Let us first work out the terms with double contractions

$$\begin{aligned}\mathsf{Tj}_{34;2} = &-c(z)b(w)\partial b(z)c(w)\partial c(w) + c(z)b(w)\partial b(z)\partial c(w)c(w)\\ &-2\partial c(z)b(w)b(z)c(w)\partial c(w) + 2\partial c(z)b(w)b(z)\partial c(w)c(w)\\ =&\frac{\partial c(w)}{(z-w)^3} + \frac{-2c(w)}{(z-w)^4} + \frac{2\partial c(w)}{(z-w)^3} + \frac{-2c(w)}{(z-w)^4} = \frac{-4c(w)}{(z-w)^4} + \frac{3\partial c(w)}{(z-w)^3} \qquad [4.156]\end{aligned}$$

The terms with only one contraction are

$$\begin{aligned}\mathsf{Tj}_{34;1} = &-c(z)b(w)\partial b(z)c\partial c(w) - \partial b(z)c(w)c(z)b\partial c(w) + \partial b(z)\partial c(w)c(z)bc(w)\\ &-2\partial c(z)b(w)b(z)c\partial c(w) - 2b(z)c(w)\partial c(z)b\partial c(w) + 2b(z)\partial c(w)\partial c(z)bc(w)\\ =&\frac{-\partial b(z)c\partial c(w)}{z-w} + \frac{c(z)b\partial c(w)}{(z-w)^2} + \frac{-2c(z)bc(w)}{(z-w)^3}\\ &+\frac{2b(z)c\partial c(w)}{(z-w)^2} + \frac{-2\partial c(z)b\partial c(w)}{z-w} + \frac{2\partial c(z)bc(w)}{(z-w)^2}\\ =&\frac{-2cbc(w)}{(z-w)^3} + \frac{(cb\partial c - 2\partial cbc + 2bc\partial c + 2\partial cbc)(w)}{(z-w)^2}\\ &+\frac{(-\partial bc\partial c + \partial cb\partial c - \partial^2 bc + 2\partial bc\partial c - \partial cb\partial c + 2\partial^2 cbc)(w)}{z-w}\\ =&\frac{-b\partial cc(w)}{(z-w)^2} + \frac{-(\partial b\partial cc + b\partial^2 cc)(w)}{z-w} \qquad [4.157]\end{aligned}$$

Therefore

$$\mathsf{Tj}_{34} = \frac{-4c(w)}{(z-w)^4} + \frac{3\partial c(w)}{(z-w)^3} + \frac{-b\partial cc(w)}{(z-w)^2} + \frac{-(\partial b\partial cc + b\partial^2 cc)(w)}{z-w} \qquad [4.158]$$

Finally

$$\begin{aligned}
\mathsf{Tj}_5 &= \gamma \overbrace{T^g(z)\partial^2 c}(w) = \gamma\partial_w^2\left[\frac{-c(w)}{(z-w)^2}+\frac{\partial c(w)}{z-w}\right] \\
&= \gamma\partial_w\left[\frac{-2c(w)}{(z-w)^3}+\frac{-\partial c(w)}{(z-w)^2}+\frac{\partial c(w)}{(z-w)^2}+\frac{\partial^2 c(w)}{z-w}\right] \\
&= \gamma\partial_w\left[\frac{-2c(w)}{(z-w)^3}+\frac{\partial^2 c(w)}{z-w}\right] \\
&= \frac{-6\gamma c(w)}{(z-w)^4}+\frac{-2\gamma\partial c(w)}{(z-w)^3}+\frac{+\gamma\partial^2 c(w)}{(z-w)^2}+\frac{\gamma\partial^3 c(w)}{z-w}
\end{aligned} \tag{4.159}$$

Let us look at the orders of $(z-w)$ one by one

$$\begin{aligned}
o\big((z-w)^{-4}\big) &= \left[\frac{1}{2}c^m - 4 - 6\gamma\right]c = \frac{c^m - 8 - 12\gamma}{2}c \\
o\big((z-w)^{-3}\big) &= (3-2\gamma)\partial c
\end{aligned} \tag{4.160}$$

Also

$$\begin{aligned}
o\big((z-w)^{-2}\big) &= 2cT^m - cT^m - b\partial cc + \gamma\partial^2 c = T^m + \frac{1}{2}cT^g + \gamma\partial^2 c = j_{\mathrm{B}} \\
o\big((z-w)^{-1}\big) &= c\partial T^m + T^m\partial c + \partial b + c\partial c + bc\partial^2 c + \gamma\partial^3 c \\
&= \partial\left(cT^m + \frac{1}{2}cT^g + \gamma\partial^3 c\right) = \partial j_{\mathrm{B}}
\end{aligned} \tag{4.161}$$

We come to our final result

$$T(z)j_{\mathrm{B}}(w) = \frac{(c^m - 8 - 12\gamma)c(w)}{2(z-w)^4} + \frac{(3-2\gamma)\partial c(w)}{(z-w)^3} + \frac{j_{\mathrm{B}}}{(z-w)^2} + + \frac{\partial j_{\mathrm{B}}}{z-w} \tag{4.162}$$

We know that nilpotency requires $c^m = 26$. It the follows that choosing $\gamma = 3/2$ ensures not only that the $(z-w)^{-4}$ cancels but also the $(z-w)^{-3}$ term, so that j_{B} is a dimension one primary field, i.e. a conformal tensor:

$$T(z)j_{\mathrm{B}}(w) = \frac{j_{\mathrm{B}}}{(z-w)^2} + + \frac{\partial j_{\mathrm{B}}}{z-w} \tag{4.163}$$

4.28 p 133: Eq (4.3.15) The Algebra Satisfied by the Constraints

The fact that the G_I^m satisfy the constraint algebra is by their definition as it is the algebra of the residual symmetry. Let us therefore check that the G_I^g satisfy this algebra

$$[G_I^g, G_J^g] = -\,g_{IL}^K g_{JN}^M [c^L b_K, c^N b_M] \tag{4.164}$$

We now use the identity for four Grassmann numbers

$$[f_1f_2, f_3f_4] = f_1\{f_2, f_3\}f_4 - f_1f_3\{f_2, f_4\} + \{f_1, f_3\}f_4f_2 - f_3\{f_1, f_4\}f_2 \qquad [4.165]$$

which is easily checked. Keeping only the non-vanishing terms

$$\begin{aligned}[G^g_I, G^g_J] =& - g^K_{IL}g^M_{JN}(\delta^N_K c^L b_M - \delta^L_M c^N b_K) = -g^K_{IL}g^M_{JK}c^L b_M + g^K_{IL}g^L_{JN}c^N b_K \\ =& (-g^K_{IL}g^M_{JK} + g^M_{IK}g^K_{JL})c^L b_M = (g^M_{JK}g^K_{LI} + g^M_{IK}g^K_{JL})c^L b_M \\ =& - g^M_{LK}g^K_{IJ}c^L b_M = ig^K_{IJ}(-ig^M_{KL}c^L b_M) = ig^K_{IJ}G^g_K \end{aligned} \qquad [4.166]$$

We have used the fact that g^I_{JK} is antisymmetric in its lower indices and also that it satisfies the Jacobi identity.

4.29 p 133: Eq (4.3.16) The Nilpotency of the General BRST Charge

We have

$$\begin{aligned}\{Q_\mathrm{B}, Q_\mathrm{B}\} =& \{c^I G^m_I + \frac{1}{2}c^I G^g_I, c^J G^m_J + \frac{1}{2}c^J G^g_J\} \\ =& \{c^I G^m_I, c^J G^m_J\} + \frac{1}{2}\{c^I G^m_I, c^J G^g_J\} + \frac{1}{2}\{c^I G^g_I, c^J G^m_J\} + \frac{1}{4}\{c^I G^g_I, c^J G^g_J\}\end{aligned} \qquad [4.167]$$

As per our standard operating procedure we split this calculation in four parts. To do this we need the identity

$$\{f_1b_1, f_2b_2\} = f_1[b_1, f_2]b_2 + f_1f_2[b_1, b_2] + \{f_1, f_2\}b_2b_1 + f_2[b_2, f_1]b_1 \qquad [4.168]$$

We will also need

$$\begin{aligned}[G^g_I, c^M] =& - ig^K_{IJ}[c^J b_K, c^M] = -ig^K_{IJ}c^J\{b_K, c^M\} = -ig^K_{IJ}c^J\delta^M_K \\ =& - ig^M_{IJ}c^J\end{aligned} \qquad [4.169]$$

Thus we find, keeping only the non-zero (anti)-commutators

$$\mathsf{qq}_1 = \{c^I G^m_I, c^J G^m_J\} = c^I c^J[G^m_I, G^m_J] = ig^K_{IJ}c^I c^J G^m_K \qquad [4.170]$$

$$\begin{aligned}2\mathsf{qq}_2 =& \{c^I G^m_I, c^J G^g_J\} = c^J[G^g_J, c^I]G^m_I = c^J(-ig^I_{JK}c^K)G^m_I \\ =& - ig^I_{JK}c^J c^K G^m_I = -ig^K_{IJ}c^I c^J G^m_K\end{aligned} \qquad [4.171]$$

$$\begin{aligned}2\mathsf{qq}_3 =& \{c^I G^g_I, c^J G^m_J\} = c^I[G^g_I, c^J]G^m_J = c^I(-ig^J_{IK}c^K)G^m_J \\ =& - ig^J IKc^I c^K G^m_J = -ig^K_{IJ}c^I c^J G^m_K\end{aligned} \qquad [4.172]$$

We see that $\mathsf{qq}_1 + \mathsf{qq}_2 + \mathsf{qq}_3 = 0$. So it remains to show that the fourth contribution is zero

$$\begin{aligned}
4\mathsf{qq}_1 =& \{c^I G^g_I, c^J G^g_J\} = c^I[G^g_I, c^J]G^g_J + c^I c^J [G^g_I, G^g_J] + c^J[G^g_J, c^I]G^g_I \\
=& c^I(-ig^J_{IK}c^K)G^g_J + c^I c^J i g^K_{IJ} G^g_K + c^J(-ig^I_{JK}c^K)G^g_I \\
=& i(g^J_{IK}c^K c^I G^g_J + g^K_{IJ}c^I c^J G^g_K + g^I_{JK}c^K c^J G^g_I \\
=& g^J_{IK}c^K c^I g^M_{JN} c^J b_M + g^K_{IJ} c^I c^J g^M_{KN} c^N b_M + g^I_{JK} c^K c^J g^M_{IN} c^N b_M \\
=& g^K_{IJ} c^J c^I g^M_{KL} c^L b_M + g^K_{IJ} c^I c^J g^M_{KL} c^L b_M + g^K_{IJ} c^J c^I g^M_{KL} c^L b_M \\
=& g^K_{IJ} g^M_{KL}(c^J c^I + c^I c^J + c^J c^I) c^L b_M = -g^K_{IJ} g^M_{KL} c^I c^J c^L b_M = 0
\end{aligned} \qquad [4.173]$$

Where this vanishes by the Jacobi identity.

4.30 p 134: Eq (4.3.17) The Hermitian Conjugate of the Ghost Modes

One could argue that it is trivial. But sometimes it is good to check the trivial things in order to be sure that we don't miss any details. It is certainly easy to work out that the definitions (4.3.17) are sufficient conditions for the BRST operator to be Hermitian. But let us also show that it is a necessary condition. Let us focus on the holomorphic modes, the reasoning for the anti-holomorphic modes is the same.

We already know that $L^\dagger_m = L_{-m}$, so

$$\begin{aligned}
Q^\dagger_{\mathrm{B}} &= \sum_n c^\dagger_n L^\dagger_{-n} + \frac{1}{2}\sum_{m,n}(m-n)\circ\!\!\!\circ\, c_m c_n b_{-m-n} \circ\!\!\!\circ^\dagger - c^\dagger_0 \\
&= \sum_n c^\dagger_{-n} L_n + \frac{1}{2}\sum_{m,n}(m-n)\circ\!\!\!\circ\, c_m c_n b_{-m-n} \circ\!\!\!\circ^\dagger - c^\dagger_0
\end{aligned} \qquad [4.174]$$

From the first term we deduce that $c^\dagger_{-n} = c_n$ and Hermiticity of the BRST charge is implies Hermiticity of

$$\tilde{Q} = \sum_{m,n}(m-n)\circ\!\!\!\circ\, c_m c_n b_{-m-n} \circ\!\!\!\circ \qquad [4.175]$$

Let us work this out in detail

$$\tilde{Q} = \Bigg[\sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n) + \sum_{m=-\infty}^{-1} m + \sum_{m=-\infty}^{-1} \sum_{n=1}^{\infty} (m-n) - \sum_{n=-\infty}^{-1} n + 0 - \sum_{n=1}^{\infty} n$$
$$+ \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n) + \sum_{m=1}^{\infty} m + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m-n) \Bigg] \mathopen{\circ\kern-0.3em\circ} c_m c_n b_{-m-n} \mathclose{\circ\kern-0.3em\circ}$$
$$= \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n) c_m c_n b_{-m-n} + \sum_{m=-\infty}^{-1} m c_m c_0 b_{-m} - \sum_{m=-\infty}^{-1} \sum_{n=1}^{\infty} (m-n) c_m b_{-m-n} c_n$$
$$+ \sum_{n=-\infty}^{-1} n c_n c_0 b_{-m-n} - \sum_{n=1}^{\infty} n b_{-m-n} c_0 c_n + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n) c_n b_{-m-n} c_m$$
$$- \sum_{m=1}^{\infty} m b_{-m} c_0 c_m + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m-n) b_{-m-n} c_m c_n \qquad [4.176]$$

From this we find, using $c_m \dagger = c_{-m}$.

$$\tilde{Q}^\dagger = \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (m-n) b^\dagger_{-m-n} c_{-n} c_{-m} + \sum_{m=-\infty}^{-1} m b^\dagger_{-m} c_0 c_{-m} - \sum_{m=-\infty}^{-1} \sum_{n=1}^{\infty} (m-n) c_{-n} b^\dagger_{-m-n} c_{-m}$$
$$+ \sum_{n=-\infty}^{-1} n b^\dagger_{-m-n} c_0 c_{-n} - \sum_{n=1}^{\infty} n c_{-n} c_0 b^\dagger_{-m-n} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (m-n) c_{-m} b^\dagger_{-m-n} c_{-n}$$
$$- \sum_{m=1}^{\infty} m c_{-m} c_0 b^\dagger_{-m} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m-n) c_{-n} c_{-m} b^\dagger_{-m-n} \qquad [4.177]$$

Changing the indices $m \to -m$ and $n \to -n$ gives

$$\tilde{Q}^\dagger = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-m+n) b^\dagger_{m+n} c_n c_m - \sum_{m=1}^{\infty} m b^\dagger_m c_0 c_m - \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} (-m+n) c_n b^\dagger_{m+n} c_m$$
$$- \sum_{n=1}^{\infty} n b^\dagger_{m+n} c_0 c_n + \sum_{n=-\infty}^{-1} n c_n c_0 b^\dagger_{m+n} + \sum_{m=-\infty}^{-1} \sum_{n=1}^{\infty} (-m+n) c_m b^\dagger_{m+n} c_n$$
$$+ \sum_{m=-\infty}^{-1} m c_m c_0 b^\dagger_m + \sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{-1} (-m+n) c_n c_m b^\dagger_{m+n} \qquad [4.178]$$

Keeping in mind that the ghost modes create different states in the Hilbert space, we can equate the different powers of $c_m c_m$ between Q_{B} and $Q_{\mathrm{B}}\dagger$. One then sees that this indeed implies that necessarily $b^\dagger_m = b_{-m}$ for $n \in \mathbb{Z}$.

4.31 p 134: Eq (4.3.18) The Ghost Insertions for the Inner Product of the Ground States

Recall that because of the anti-commutator $\{b_0, c_0\} = 1$ the ghost ground state is degenerate, i.e. we have two ground states $|\downarrow\rangle$ and $|\uparrow\rangle$. Their relations are shown in the figure below

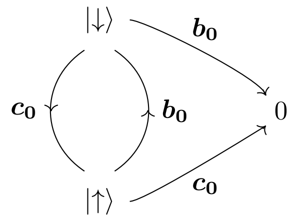


Figure 4.1: The degenerate ghost vacuum

Note that $|0;k\rangle$ actually means $|0;k\rangle \oplus |\downarrow]\rangle$ where the first part is the ground state of the matter part and the latter the ground state of the ghost part. For the closed string

$$(i\,\langle 0;k|\,\tilde{c}_0 c_0\,|0;k\rangle)^* = -i\,\langle 0;k|\,c_0^\dagger \tilde{c}_0^\dagger\,|0;k\rangle = -i\,\langle 0;k|\,c_0\tilde{c}_0\,|0;k\rangle = i\,\langle 0;k|\,\tilde{c}_0 c_0\,|0;k\rangle \quad [4.179]$$

justifying the need for the i in that inner product.

4.32 p 134: Q_B takes $\hat{\mathcal{H}}$ into itself

To see that $[Q_\mathrm{B}, L_0]$ vanishes, recall that the BRST current is a primary dimension one field in the critical dimension, see (4.3.11) and that the BRST charge is the zero mode of the BRST current, see the text below [4.117]. From (2.6.24) it then follows immediately that $[Q_\mathrm{B}, L_0] = 0$.

Now assume $|\psi\rangle \in \hat{\mathcal{H}}$, then

$$\begin{aligned} b_0 Q_\mathrm{B}\,|\psi\rangle &= (\{b_0, Q_\mathrm{B}\} - Q_\mathrm{B} b_0)\,|\psi\rangle = L_0\,|\psi\rangle = 0 \\ L_0 Q_\mathrm{B}\,|\psi\rangle &= ([L_0, Q_\mathrm{B}] - Q_\mathrm{B} L_0)\,|\psi\rangle = 0 \end{aligned} \quad [4.180]$$

and therefore if $|\psi\rangle \in \hat{\mathcal{H}}$ then we also have $Q_\mathrm{B}\,|\psi\rangle \in \hat{\mathcal{H}}$

4.33 p 134: The Need for a New Inner Product on $\hat{\mathcal{H}}$

A level N state with a ghost zero mode $|N;k;c\rangle = c_0\,|N;k\rangle$ has inner product

$$\langle N;k,c|N';k';c\rangle = \langle N;k|\,c_0^\dagger c_0\,|N;k\rangle = \langle N;k|\,c_0 c_0\,|N;k\rangle = 0 \quad [4.181]$$

and similarly for $|N;k;b\rangle = b_0\,|N;k\rangle$. Also the states in $\hat{\mathcal{H}}$ satisfy the mass-shell condition (4.3.20), i.e. $L_0\,|\psi\rangle = \alpha'(p^2+m^2)\,|\psi\rangle = 0$, hence not all k^μ are independent as they have to satisfy the mass shell condition. This means that in $\delta^{26}(k-k')$ there will be one delta function that is automatically satisfies and give a $\delta(0)$. By defining a new inner product without the ghost zero modes and X^0 one takes away the problems so have a good inner product. It seems obvious to me that the BRST charge remains Hermitian with this new inner product.

4.34 p 135: Eq (4.3.23) The Level Zero Mass Shell Condition

The level zero state $|0;\boldsymbol{k}\rangle$ has no ghosts or matter excitations. Therefore $N_{nb} = N_{cn} = N_{\mu n} = 0$ and thus the mass shell condition (4.3.22) implies that $\alpha'(-k^2) = \alpha' m^2 = -1$ or hence $-k^2 = -1/\alpha'$.

4.35 p 135: Eq (4.3.24) The Level Zero Physical State Condition

Using the expansion of the BRST charge in modes (4.3.7) we have

$$Q_\mathrm{B}\,|0;\boldsymbol{k}\rangle = \left[\sum_{n=-\infty}^{\infty} c_n L^m_{-n} + \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n)\,{}^\circ_\circ c_m c_n b_{-m-n}{}^\circ_\circ - c_0\right]|0;\boldsymbol{k}\rangle \qquad [4.182]$$

The first term, for $n \neq 0$, always has either c_k or L_k with $k \geq 1$ which annihilates $|0;k\rangle$. The $n=0$ term gives $c_0 L^m_0\,|0;\boldsymbol{k}\rangle$ with L^m_0 given by (2.7.7), or at least the equivalent relation for the open string,

$$L^m_0 = \alpha' p^2 + \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{\mu\,n} \qquad [4.183]$$

Thus $c_0 L^m_0\,|0;\boldsymbol{k}\rangle = \alpha' k^2 c_0\,|0;\boldsymbol{k}\rangle = c_0\,|0;\boldsymbol{k}\rangle$. This cancels with the third term. In the second term all the terms that have $m \neq 0$ and $m \neq 0$ clearly annihilate $|0;k\rangle$ due to the normal ordering. The $m=n=0$ term has a c_0^2 that is zero by itself. So what remains is terms with either m or n equal to zero, but not both. These are of the form $\sum_{n=-\infty,n\neq 0}^{\infty} n\,{}^\circ_\circ c_0 c_n b_{-n}{}^\circ_\circ$ but by normal ordering there is always an annihilation operator to the right so these terms also annihilate $|0;k\rangle$. We thus find that indeed $Q_\mathrm{B}\,|0;\boldsymbol{k}\rangle = 0$.

4.36 p 135: Eq (4.3.25) The Level One Mass Shell Condition

The state $|\psi_1\rangle$ has one mode α^μ_{-1}, b_{-1} or c_{-1} and thus (4.3.22) gives

$$\alpha'(-k^2) = \alpha' m^2 = 1 - 1 = 0 \qquad \Rightarrow k^2 = 0 \qquad [4.184]$$

4.37 p 135: Eq (4.3.26) The Level One Negative Norm States

The norm of the state is

$$\langle\psi_1||\psi_1\rangle = (e^* \cdot e + \beta^*\gamma + \gamma^*\beta) \langle 0;\boldsymbol{k}||0;\boldsymbol{k}\rangle \tag{4.185}$$

We can take a orthogonal basis for the e_μ where the ν'th element of e_μ is given by δ^ν_μ. We then have $e_0 = (1, 0, \cdots, o)$ and thus $(e_0)^2 = -1$. All tho the e_i have $(e_i)^2 + 1$. That is one negative norm state and 25 positive norm states. For the ghost excitations, let us take a basis for which the corresponding state has $\beta^*\gamma$ real. We then have $\beta^*\gamma = (\beta^*\gamma)* = -\gamma * \beta$, the extra sign because of the Grassmann character. Thus $\beta^*\gamma$ and $\gamma * \beta$ have opposite signs; one of them is necessarily positive norm, and the other one negative norm. We thus have indeed 26 positive norm states and two negative norm states.

4.38 p 135: Eq (4.3.27) The Level One Physical State Condition

The physical state condition is

$$\begin{aligned} Q_\text{B} |\psi_1\rangle = &\left[\sum_{n=-\infty}^{\infty} c_n L_{-n} + \frac{1}{2}\sum_{m,n=-\infty}^{\infty} (m-n) {\circ\atop\circ} c_m c_n b_{-m-n} {\circ\atop\circ} - c_0\right] \\ &\times \left(e_\mu \alpha^\mu_{-1} + \beta b_{-1} + \gamma c_{-1}\right) |0;\boldsymbol{k}\rangle \end{aligned} \tag{4.186}$$

We split this in three

$$\mathsf{q}_1 = \sum_{n=-\infty}^{\infty} c_n L_{-n} \left(e_\mu \alpha^\mu_{-1} + \beta b_{-1} + \gamma c_{-1}\right) |0;\boldsymbol{k}\rangle \tag{4.187}$$

We use

$$[L_k, \alpha^\mu_n] = -n\alpha^\mu_{k+n} \tag{4.188}$$

a relation that is easily checked by working out some examples, and find

$$\begin{aligned} \mathsf{q}_1 &= \sum_{n=-\infty}^{\infty} \left(c_n e_\mu \alpha^\mu_{-n-1} - \beta L_{-n}\delta_{n-1,0}\right) |0;\boldsymbol{k}\rangle \\ &= \sum_{n=-\infty}^{\infty} e_\mu c_n \alpha^\mu_{-n-1} |0;\boldsymbol{k}\rangle - \beta L_{-1} |0;\boldsymbol{k}\rangle \end{aligned} \tag{4.189}$$

We have used the fact that $[c_n, \beta b_{-1}] = -\beta\{c_n, b_{-1}\}$ as β is a Grassmann number. The ghost mode c_n requires $n \leq 0$ and the matter mode requires $-n-1 \leq 0$, or equivalently $n \geq -1$. Combined, only the $n = 0$ and $n = -1$ terms are not zero and we have

$$\mathsf{q}_1 = (e_\mu c_0 \alpha^\mu_{-1} + e_\mu c_{-1}\alpha^\mu_0 - \beta L_{-1}) |0;\boldsymbol{k}\rangle \tag{4.190}$$

Now, $\alpha_0^\mu |0;\boldsymbol{k}\rangle = \sqrt{2\alpha'}k^\mu |0;\boldsymbol{k}\rangle$ and $L_{-1}|0;\boldsymbol{k}\rangle = \alpha_{-1}^\mu \alpha_{\mu 0}|0;\boldsymbol{k}\rangle = \sqrt{2\alpha'}k_\mu \alpha_{-1}^\mu |0;\boldsymbol{k}\rangle$. Thus

$$\mathsf{q}_1 = \left[e_\mu c_0 \alpha_{-1}^\mu + \sqrt{2\alpha'}(k\cdot e c_{-1} - \beta k\cdot\alpha_{-1})\right]|0;\boldsymbol{k}\rangle \quad [4.191]$$

Next

$$\mathsf{q}_2 = \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n)\,{}^\circ_\circ c_m c_n b_{-m-n}{}^\circ_\circ \left(e_\mu \alpha_{-1}^\mu + \beta b_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle \quad [4.192]$$

Acting on the first part $e_\mu \alpha_{-1}^\mu |0;\boldsymbol{k}\rangle$ the ghost mode commute with the α_{-1}^μ and annihilate $|0;\boldsymbol{k}\rangle$, just as they did for the level zero state.

For the second term, it is easiest to work out examples of ${}^\circ_\circ c_m c_n b_{-m-n}{}^\circ_\circ b_{-1}$ for different values of m and n. This is how it goes. We first only consider the the combinations where $n \geq m$:

1. For $m = n$ we have ${}^\circ_\circ c_m c_m b_{-2m}{}^\circ_\circ b_{-1}$. Irrespective whether m is positive, negative or zero, we will the normal ordering will always keep the two c_m's together and hence this is zero.
2. For $m = -3$ we get the series
$${}^\circ_\circ \cdots + c_{-3}c_{-2}b_5 + c_{-3}c_{-1}b_4 + c_{-3}c_0 b_3 + c_{-3}c_1 b_2 + c_{-3}c_2 b_1 + \cdots {}^\circ_\circ b_{-1}$$
Clearly normal ordering ensures each term vanishes.
3. For $m = -2$ we get the series
$${}^\circ_\circ \cdots + c_{-2}c_{-1}b_3 + c_{-2}c_0 b_2 + c_{-2}c_1 b_1 + c_{-2}c_2 b_0 + \cdots {}^\circ_\circ b_{-1}$$
Clearly normal ordering ensures each term vanishes here as well.
4. For $m = -1$ we get the series
$${}^\circ_\circ \cdots + c_{-1}c_0 b_1 + c_{-1}c_1 b_0 + c_{-1}c_2 b_{-1} + \cdots {}^\circ_\circ b_{-1}$$
After normal ordering, each term vanishes. Note that this time we need to use $b_0|0;\boldsymbol{k}\rangle = 0$.
5. For $m = 0$ we get the series
$${}^\circ_\circ \cdots + c_0 c_1 b_{-1} + c_0 c_2 b_{-2} + \cdots {}^\circ_\circ b_{-1}$$
Here we should be concerned about the first term only. It gives $b_{-1}c_0 c_1 b_{-1}|0;\boldsymbol{k}\rangle = b_{-1}c_0|0;\boldsymbol{k}\rangle$.
6. For $m \geq 1$ we have a c_2 or higher and so all terms are zero again.

In conclusion, the only non-vanishing contribution is $(m,n) = (0,1)$ and of course also its opposite $(m,n) = (1,0)$. Thus the second term becomes

$$\frac{1}{2}\left(-{}^\circ_\circ c_0 c_1 b_{-1}{}^\circ_\circ b_{-1} + {}^\circ_\circ c_1 c_0 b_{-1}{}^\circ_\circ b_{-1}\right)|0;\boldsymbol{k}\rangle = -b_{-1}c_0 c_1 b_{-1}|0;\boldsymbol{k}\rangle = -b_{-1}c_0|0;\boldsymbol{k}\rangle \quad [4.193]$$

For the third term we need both m and n to be smaller than one or they end up in the right of the normal ordering, anti-commute through the c_{-1} and annihilate $|0;\boldsymbol{k}\rangle$. If

$(m,n) = (0,0)$ we have a $c_0^2 = 0$. For $(m,n) = (0,-1)$ we have $\,{}^\circ_\circ c_0 c_{-1} b_1 {}^\circ_\circ c_{-1} |0;\boldsymbol{k}\rangle = -c_{-1}c_0 b_1 c_{-1} |0;\boldsymbol{k}\rangle = -c_{-1}c_0 |0;\boldsymbol{k}\rangle$. All the other combinations of m and n annihilate $|0;\boldsymbol{k}\rangle$ because we will have a b_k with $k \geq 2$. Thus the only contribution from the third term is

$$\frac{1}{2}\left(-{}^\circ_\circ c_{-1}c_0 b_1 {}^\circ_\circ c_{-1} + {}^\circ_\circ c_0 c_{-1} b_1 {}^\circ_\circ c_{-1}\right)|0;\boldsymbol{k}\rangle = -c_{-1}c_0 b_1 c_{-1}|0;\boldsymbol{k}\rangle = -c_{-1}c_0|0;\boldsymbol{k}\rangle \qquad [4.194]$$

Bringing the three terms together we find

$$\mathsf{q}_2 = (\beta b_{-1}c_0 + \gamma c_{-1}c_0)|0;\boldsymbol{k}\rangle \qquad [4.195]$$

Finally

$$\begin{aligned}\mathsf{q}_3 =& -c_0\left(e_\mu \alpha^\mu_{-1} + \beta b_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle \\ =& \left(-e_\mu c_0 \alpha^\mu_{-1} - \beta b_{-1}c_0 - \gamma c_{-1}c_0\right)|0;\boldsymbol{k}\rangle\end{aligned} \qquad [4.196]$$

We can now bring $\mathsf{q}_1, \mathsf{q}_2$ and q_3 together and find

$$\begin{aligned}Q_\mathrm{B}|\psi_1\rangle =& \Big[e_\mu c_0 \alpha^\mu_{-1} + \sqrt{2\alpha'}(k\cdot e c_{-1} - \beta k\cdot\alpha_{-1}) + \beta b_{-1}c_0 + \gamma c_{-1}c_0 \\ & - e_\mu c_0 \alpha^\mu_{-1} - \beta b_{-1}c_0 - \gamma c_{-1}c_0\Big]|0;\boldsymbol{k}\rangle \\ =& \sqrt{2\alpha'}(k\cdot e c_{-1} - \beta k\cdot\alpha_{-1})|0;\boldsymbol{k}\rangle\end{aligned} \qquad [4.197]$$

We thus conclude that the state $|\psi_1\rangle = \left(e_\mu \alpha^\mu_{-1} + \beta b_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle$ is annihilated by the BRST charge provided that

$$\sqrt{2\alpha'}(k\cdot e c_{-1} - \beta k\cdot\alpha_{-1})|0;\boldsymbol{k}\rangle = 0 \qquad [4.198]$$

and this implies indeed the physical state conditions $k\cdot e = \beta = 0$. In other words, level one physical states are of the form

$$\left(e_\mu \alpha^\mu_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle \quad \text{with} \quad k\cdot e = 0 \qquad [4.199]$$

Let us now consider the norm of these states. The state $c_{-1}|0;\boldsymbol{k}\rangle$ as follows from (4.3.26) by setting $e_\mu = \beta = 0$. The find the norm of the other states, let us gos to a basis where the momentum is $k^2 = (1,1,0,\cdots,0)$. This satisfies $^2 = 0$ as it should for level one state which are massless. We now need 25 linearly independent vectors e that satisfy $k\cdot e = 0$, the 25 coming from the original 26 minus one condition. These are clearly $(1,1,0,\cdots,0) = k$ and $(0,0,1,0,\cdots,0)$ up to $(0,\cdots,0,1)$. The former has norm zero and the 24 latter have positive norm. The level one physical state thus indeed has 24 positive norm states and two zero norm states. The zero norm states are created by c_{-1} and by $e\cdot\alpha_{-1} = k\cdot\alpha_{-1}$.

4.39 p 139: Eq (4.4.7) The Commutation Relations of the Light-Cone Oscillators

$$\begin{aligned}[\alpha^+_m, \alpha^-_n] &= \frac{1}{2}[\alpha^0_m + \alpha^1_m, \alpha^0_n - \alpha^1_n] = \frac{1}{2}\left([\alpha^0_m, \alpha^0_n] - [\alpha^1_m, \alpha^1_n]\right)\\ &= \frac{1}{2}\left(m\eta^{00}\delta_{m+n,0} - m\eta^{11}\delta_{m+n,0}\right) = -m\delta_{m+n,0}\\ [\alpha^\pm_m, \alpha^\pm_n] &= \frac{1}{2}[\alpha^0_m \pm \alpha^1_m, \alpha^0_n \pm \alpha^1_n] = \frac{1}{2}\left([\alpha^0_m, \alpha^0_n] + [\alpha^1_m, \alpha^1_n]\right)\\ &= \frac{1}{2}\left(m\eta^{00}\delta_{m+n,0} + m\eta^{11}\delta_{m+n,0}\right) = 0 \end{aligned} \quad [4.200]$$

4.40 p 139: Eq (4.4.10) The Splitting of the BRST Operator

The (open string) BRST operator is

$$Q_{\mathrm{B}} = \sum_{n=-\infty}^{\infty}\left(c_n L^X_{-n} + c_n L^K_{-n}\right) + \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n)\circ\!\!\!\circ\, c_m c_n b_{-m-n} \circ\!\!\!\circ - c_0 \quad [4.201]$$

Now

$$\begin{aligned} L^X_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \circ\!\!\!\circ\, \alpha^\mu_{m-n}\alpha_{\mu\, n} \circ\!\!\!\circ\\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(-\circ\!\!\!\circ\, \alpha^0_{m-n}\alpha^0_n \circ\!\!\!\circ + \circ\!\!\!\circ\, \alpha^1_{m-n}\alpha^1_n \circ\!\!\!\circ + \sum_{J=2}^{d-1} \circ\!\!\!\circ\, \alpha^J_{m-n}\alpha^J_n \circ\!\!\!\circ\right) \end{aligned} \quad [4.202]$$

and

$$\begin{aligned} -\alpha^0_{m-n}\alpha^0_n + \alpha^1_{m-n}\alpha^1_n &= -\frac{1}{2}(\alpha^+_{m-n} + \alpha^-_{m-n})(\alpha^+_n + \alpha^-_n) + \frac{1}{2}(\alpha^+_{m-n} - \alpha^-_{m-n})(\alpha^+_n - \alpha^-_n)\\ &= \frac{1}{2}\Big[-\alpha^+_{m-n}\alpha^+_n - \alpha^+_{m-n}\alpha^-_n - \alpha^-_{m-n}\alpha^+_n - \alpha^-_{m-n}\alpha^-_n\\ &\qquad + \alpha^+_{m-n}\alpha^+_n - \alpha^+_{m-n}\alpha^-_n - \alpha^-_{m-n}\alpha^+_n + \alpha^-_{m-n}\alpha^-_n\Big]\\ &= -\alpha^+_{m-n}\alpha^-_n - \alpha^-_{m-n}\alpha^+_n \end{aligned} \quad [4.203]$$

Hence

$$L^X_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(-\circ\!\!\!\circ\, \alpha^+_{m-n}\alpha^-_n \circ\!\!\!\circ - \circ\!\!\!\circ\, \alpha^-_{m-n}\alpha^+_n \circ\!\!\!\circ + \sum_{J=2}^{d-1} \circ\!\!\!\circ\, \alpha^J_{m-n}\alpha^J_n \circ\!\!\!\circ\right) \quad [4.204]$$

and

$$Q_{\rm B} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} \sum_{k=-\infty}^{\infty} \left(-c_n \circ\alpha^+_{-n-k}\alpha^-_k\circ - c_n\circ\alpha^-_{-n-k}\alpha^+_k\circ + \sum_{J=2}^{d-1} c_n\circ\alpha^J_{-n-k}\alpha^J_k\circ \right) + c_n L^K_{-n} \right] + \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m-n)\circ c_m c_n b_{-m-n}\circ - c_0 \qquad [4.205]$$

Let us now look at this. Any terms that have no light-cone oscillators will commute with $N^{\rm lc}$ and be part of Q_0. All terms of the form $\alpha^\pm_\ell \alpha^\mp_m$ for $\ell, m \neq 0$ also commute with $N^{\rm lc}$, as follows from counting the excitations and fall into Q_0. Terms of the form $\alpha^\pm_\ell \alpha^\mp_0$ for $\ell \neg 0$ can be written as $\sqrt{2\alpha'} k^\mp \alpha^\pm_\ell$ and have and belong to Q_1 for α^-_ℓ and to Q_{-1} for α^+_ℓ. Finally, terms that contain $\alpha^\pm_0 \alpha^\mp_0 = 2\alpha' k^\pm k^\mp$ belong again to Q_0

4.41 p 139: Eq (4.4.11) The Ghost Number of the BRST Operator

As c_m has ghost number one and b_m has ghost number minus one – see (2.723), it is clear that $Q_{\rm B}$ has ghost number one, and also each individual Q_j.

4.42 p 139: Eq (4.4.13) The Simplified BRST Operator Q_1

To identify Q_1 we need to find all terms in [4.205] that contain one and only one α^+_0. These can only come from the first two terms and if $k = 0$ or $k = -n$ and $n \neq 0$. Thus

$$Q_1 = -\frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(c_n\circ\alpha^+_0\alpha^-_{-n}\circ + c_n\circ\alpha^-_{-n}\alpha^+_0\circ \right) \qquad [4.206]$$

As we can replace α^+_0 by $\sqrt{2\alpha'}k^+$ we can also drop the normal ordering sign and find

$$Q_1 = -\sqrt{2\alpha'}k^+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \alpha^-_{-n} \qquad [4.207]$$

4.43 p 140: Eq (4.4.13) The Operator S

$$S = \{Q_1, R\} = -\sum_{\substack{m=-\infty \\ m\neq 0}} \sum_{\substack{=-\infty \\ m\neq 0}} \{\alpha^-_{-m}c_m, \alpha^+_{-n}b_n\} \qquad [4.208]$$

We use the identity [4.168]

$$\{f_1 b_1, f_2 b_2\} = f_1[b_1, f_2]b_2 + f_1 f_2[b_1, b_2] + \{f_1, f_2\}b_2 b_1 + f_2[b_2, f_1]b_1 \qquad [4.209]$$

$$\begin{aligned}
S = \{Q_1, R\} &= -\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{=-\infty \\ m\neq 0}}^{\infty} \Big(c_m b_n [\alpha^-_{-m}, \alpha^+_{-n}] + \{c_m, b_n\}\alpha^+_{-n}\alpha^-_{-m} \Big) \\
&= -\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{=-\infty \\ m\neq 0}}^{\infty} \Big(-nc_m b_n \delta_{m+n} + \delta_{m+n}\alpha^+_{-n}\alpha^-_{-m} \Big) \\
&= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \Big(-mc_m b_{-m} - \alpha^+_m \alpha^-_{-m} \Big) \\
&= \sum_{m=-\infty}^{-1} \Big(-mc_m b_{-m} - \alpha^+_m \alpha^-_{-m} \Big) + \sum_{m=1}^{\infty} \Big(-mc_m b_{-m} - \alpha^+_m \alpha^-_{-m} \Big) \\
&= \sum_{m=1}^{\infty} \Big(mc_{-m} b_m - \alpha^+_{-m}\alpha^-_m - mc_m b_{-m} - \alpha^+_m \alpha^-_{-m} \Big) \\
&= \sum_{m=1}^{\infty} \Big(mc_{-m} b_m - \alpha^+_{-m}\alpha^-_m - m\{c_m, b_{-m}\} + mb_{-m}c_m - [\alpha^+_m, \alpha^-_{-m}] - \alpha^-_{-m}\alpha^+_m \Big) \\
&= \sum_{m=1}^{\infty} \Big(mc_{-m} b_m - \alpha^+_{-m}\alpha^-_m - m + mb_{-m}c_m + m - \alpha^-_{-m}\alpha^+_m \Big) \\
&= \sum_{m=1}^{\infty} \Big(mc_{-m} b_m + mb_{-m}c_m - \alpha^+_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^+_m \Big) \qquad [4.210]
\end{aligned}$$

4.44 p 140: The Cohomology of Q_1

We repeat the arguments here more slowly. That always helps me. First we show the Q_1 and S commute.

$$[Q_1, S] \propto \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=1}^{\infty} \Big[\alpha^-_{-m} c_m, nc_{-n} b_n + nb_{-n} c_n - \alpha^+_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^+_n \Big] \qquad [4.211]$$

We use

$$[f_1, f_2 f_3] = \{f_1, f_2\} f_3 - f_2 \{f_1, f_3\} \qquad [4.212]$$

and

$$[b_1, b_2 b_3] = [b_1, b_2]b_3 + b_2[b_1, b_3] \qquad [4.213]$$

to obtain

$$\begin{aligned}
[Q_1, S] \propto & \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=1}^{\infty} \Big(- n\alpha^-_{-m} c_{-n}\{c_m, b_n\} + n\alpha^-_{-m}\{c_m, b_{-n}\}c_n \\
& \qquad - c_m[\alpha^-_{-m}, \alpha^+_{-n}]\alpha^-_n - c_m\alpha^-_n[\alpha^-_{-m}, \alpha^+_n]\Big) \\
= & \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=1}^{\infty} \Big(- n\alpha^-_{-m} c_{-n}\delta_{m+n,0} + n\alpha^-_{-m}\delta_{m-n}c_n + nc_m\delta_{m+n,0}\alpha^-_n - nc_m\alpha^-_n\delta_{m-n,0}\Big) \\
= & \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=1}^{\infty} \Big(m\alpha^-_{-m}c_m + m\alpha^-_{-m}c_m - mc_m\alpha^-_{-m} - mc_m\alpha^-_m\Big) = 0
\end{aligned} \qquad [4.214]$$

We are looking for states satisfying $Q_1 |\psi\rangle = 0$, i.e. for states with zero Eigenvalue under Q_1. As Q_1 and S commute we can diagonalise them simultaneously, so $S|\psi\rangle = s|\psi\rangle$ for some Eigenvalue s. Consider first the case that $s \neq 0$. Then

$$|\psi\rangle = s^{-1}S|\psi\rangle = s^{-1}\{Q_1, R\}|\psi\rangle = s^{-1}(Q_1R + RQ_1)|\psi\rangle = s^{-1}Q_1R|\psi\rangle \qquad [4.215]$$

Thus $|\psi\rangle = Q_1|\chi\rangle$ for some $|\chi\rangle$ and is thus a Q_1 exact state. This means that states with $s \neq 0$ cannot be physical states as physical states are closed but not exact. So we can restrict ourselves to states with zero Eigenvalue under S, i.e. $s = 0$. Now look at the explicit form of S

$$S = \sum_{n=1}^{\infty} \Big(nc_{-n}b_n + nb_{-n}c_n - \alpha^+_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^+_n \Big) \qquad [4.216]$$

We see that S is a number operator for counting the ghosts and the light-cone oscillators. Indeed it annihilates every $b{-}n$ excitation and replaces them by a c_{-n} excitation, it annihilates every $c{-}n$ excitations and replaces them by a b_{-n} excitation Similarly it annihilates every light-cone oscillator excitation $\alpha^{\pm}_{-m}$ and replaces it by its opposite light-cone excitation $\alpha^{\mp}_{-m}$. Requiring that S has Eigenvalue zero thus means that the state ψ cannot have any ghost or light-cone excitations, i.e. no b, c, X^0 or X^1 excitations. The corresponding space is exactly the Hilbert space $\mathcal{H}^{\perp}$ of the transverse excitations. Are there any Q_1 exact states in $\mathcal{H}^{\perp}$? Any such state $|\phi\rangle$ can by definition be written as $|\phi\rangle = Q_1|\chi\rangle$ for some $|\chi\rangle$ in $\mathcal{H}^{\perp}$, i.e.

$$|\phi\rangle = -\sqrt{2\alpha'}k^+ \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \alpha^-_{-m}c_m|\chi\rangle \qquad [4.217]$$

But for $|\phi\rangle$ to be non-zero and have only transverse oscillations, $|\phi\rangle$ needs to have a b and a α^+ excitations and hence $|\chi\rangle \notin \mathcal{H}^\perp$. In other words there are no exact states in $\mathcal{H}^\perp$ and thus the cohomology of Q_1 is $\mathcal{H}^\perp$

Let us also repeat the second argument given in Joe's book that $s = 0$ states, i.e. transverse states are Q_1 invariant. As $s = 0$ states have no ghost excitations they are built from transverse oscillator excitations acting in the ground states $|0;\boldsymbol{k}\rangle$. This ground states includes the ghost ground state $|\downarrow\rangle$ that has ghost number $-1/2$, see (2.7.24). Now as $s = 0$ we have $Q_1 S\,|\psi\rangle = 0$ and because Q_1 and S commute this implies that $SQ_1\,|\psi\rangle = 0$. But Q_1 has ghost number one as it contains just one c and so $Q_1\,|\psi\rangle$ has ghost number $-1/2 + 1 = +1/2$. Furthermore S acting on a ghost number $+1/2$ state has non-zero Eigenvalues. Indeed we just argued that if $s = 0$ then the ghost number is necessarily $-1/2$. If the Eigenvalue under S of $Q_1\,|\psi\rangle$ is not zero, then there exists an S^{-1}. Therefore $S^{-1}SQ_1\,|\psi\rangle = Q_1\,|\psi\rangle = 0$. So an $s = 0$ state is indeed Q_1 closed.

4.45 p 140: The Cohomology of Q_B

Here as well we just repeat the arguments of Joe's book, albeit at a more pedestrian speed. We consider the operator

$$U = \{Q_\mathrm{B}, R\} - S = \{Q_{-1} + Q_0 + Q_1, R\} - S = \{Q_{-1} + Q_0, R\} \qquad [4.218]$$

Now $R \sim \sum \alpha^+_{-m} b_m$ so acting on a state it lowers the light-cone number by one, due to the α^+_{-m}. By definition Q_0 leaves the light-cone number unchanged and Q_{-1} lowers it by one as well. Therefore U lowers the light-cone number by one or two.

If we then write out S as a matrix in a basis, where each basis vector has a given light-cone number, then S maps a state with light-cone number ℓ into a state with light-cone number ℓ and so S is represented by a diagonal matrix. U maps a state with light-cone number ℓ into a state with light-cone number $\ell - 1$ and a state with light-cone number $\ell - 2$. In other words U is represented by a lower triangular matrix, in fact by a strictly triangular matrix. Thus the matrix $M = S + U$ consists of a triangular matrix S and a strictly lower triangular matrix U. In this case, $\mathrm{Ker}(S+U) \subseteq \mathrm{Ker}(S)$. now consider a state $|\psi_0\rangle \in \mathrm{Ker}(S)$, i.e. $S\,|\psi_0\rangle = 0$. Construct now a state

$$|\psi\rangle = \Big(1 - S^{-1}U + S^{-1}US^{-1}U - S^{-1}US^{-1}US^{-1}U + \cdots\Big)\,|\psi_0\rangle \qquad [4.219]$$

Act with $S + U$ on this:

$$\begin{aligned}(S+U)\,|\psi\rangle &= \Big(S - U + US^{-1}U - US^{-1}US^{-1}U + US^{-1}US^{-1}US^{-1}U - \cdots \\ &\qquad + U - US^{-1}U + US^{-1}US^{-1}U - US^{-1}US^{-1}US^{-1}U + \cdots\Big)\,|\psi_0\rangle \\ &= S\,|\psi_0\rangle = 0 \end{aligned} \qquad [4.220]$$

So for every element in $\mathrm{Ker}(S)$ we can write down an element in $\mathrm{Ker}(S+U)$ and thus we in fact that $\mathrm{Ker}(S)$ and $\mathrm{Ker}(S+U)$ are isomorphic.

We can now repeat the same argument we used for deducing the cohomology of Q_1. We first note that $S+U$ and Q_B commute. Indeed

$$\begin{aligned}[S+U, Q_\mathrm{B}] &= [\{Q_\mathrm{B}, R\}, Q_\mathrm{B}] = [Q_\mathrm{B}R + RQ_\mathrm{B}, Q_\mathrm{B}] \\ &= Q_\mathrm{B}RQ_\mathrm{B} + RQ_\mathrm{B}^2 - Q_\mathrm{B}^2 R - Q_\mathrm{B}RQ_\mathrm{B} = 0 \end{aligned} \quad [4.221]$$

by nilpotendcy of Q_B. We can thus diagonalise Q_B and $S+U$ together, $Q_\mathrm{B}\left|\psi\right\rangle = 0$ and $(S+U)\left|\psi\right\rangle = t\left|\psi\right\rangle$. Consider first the case where $t \neq 0$. Then

$$\left|\psi\right\rangle = t^{-1}(S+U)\left|\psi\right\rangle = t^{-1}\{Q_\mathrm{B}, R\}\left|\psi\right\rangle = t^{-1}(Q_\mathrm{B}R + RQ_\mathrm{B})\left|\psi\right\rangle = t^{-1}Q_\mathrm{B}R\left|\psi\right\rangle \quad [4.222]$$

and so $\left|\psi\right\rangle$ is exact and the cohomology of Q_B is non-zero only when the Eigenvalue of $S+U$ is zero. The cohomology of Q_B is thus the same as the kernel of $S+U$. We saw that the Kernel of $S+U$ is isomorphic to the kernel of S and that in turn is the same as the cohomology of Q_1, i,e,

$$\mathrm{coh}(Q_\mathrm{B}) \equiv \mathrm{Ker}(S+U) \equiv \mathrm{Ker}(S+U) \equiv \mathrm{coh}(Q_1) \quad [4.223]$$

The cohomology of Q_B is thus isomorphic to the cohomology of Q_1.

If we can also show the the inner product on the cohomology of Q_B is positive definite, then we know that it is identical to the cohomology of Q_1 and thus consists of only the transverse oscillator excitations. All states in the cohomology of Q_B are necessarily of the form (4.4.19) as we have just argued. To show that this is the case we start by working out the light-cone number of

$$-S^{-1}U + S^{-1}US^{-1}U - S^{-1}US^{-1}US^{-1}U + \cdots \quad [4.224]$$

S had light-cone number zero. R has light-cone number minus one and $U = \{Q_0 + Q_{-1}, R\}$ thus has terms with light-cone number minus one and minus two. I.e. U has only terms with strictly negative light-cone numbers. Thus all the terms of the above sum have strictly negative light-cone numbers. Now, in order to calculate the norm of the state $\left|\psi\right\rangle = (1 - S^{-1}U + S^{-1}US^{-1}U - + \cdots)\left|\psi_0\right\rangle$ we would have to use commutation relations between S^{-1} and U to obtain c-numbers. But in order to have a non-vanishing commutation relations of $[A, B]$ we need A and B to have opposite light-cone numbers.[1] But the only terms in the expansion of $\left\langle\psi\right|\left|\psi'\right\rangle$ that have opposite light-cone numbers are the ones with light-cone number zero, i.e. the first term in the expansion of $\left|\psi\right\rangle$. In other words $\left\langle\psi\right|\left|\psi'\right\rangle = \left\langle\psi_0\right|\left|\psi_0'\right\rangle$, which is positive as the kernel of S has positive definite inner product.

[1] Note that $(\alpha_m^+)^\dagger = \alpha_{-m}^+$, so Hermitian conjugation does not change the light-cone number.

4.46 p 141: Eq (4.4.23) The BRST Operator Acting a a Hilbert Space State

$$Q_\mathrm{B}\left|\psi,\downarrow\right\rangle = \left[\sum_{n=-\infty}^{\infty} c_n L^m_{-n} + \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n){}^\circ_\circ c_m c_n b_{-m-n}{}^\circ_\circ - c_0\right]\left|\psi,\downarrow\right\rangle \qquad [4.225]$$

The contribution from the cubic term vanishes as follows form the breakdown of m and n:

m	n	${}^\circ_\circ c_m c_n b_{-m-n}{}^\circ_\circ$	
$]-\infty,-1]$	$]-\infty,-1]$	$+c_m c_n b_{-m-n}$	$\rightsquigarrow 0$
$''$	0	$-c_n c_0 b_{-m}$	$\rightsquigarrow 0$
$''$	$[1,\infty[$	$-c_m b_{-m-n} c_n$	$\rightsquigarrow 0$
0	$]-\infty,-1]$	$-c_n c_0 b_{-n}$	$\rightsquigarrow 0$
$''$	0	$+c_0 c_0 b_0$	$\rightsquigarrow 0$
$''$	$[1,\infty[$	$+b_{-n} c_0 c_n$	$\rightsquigarrow 0$
$[1,\infty[$	$]-\infty,-1]$	$+c_n b_{-m-n} c_m$	$\rightsquigarrow 0$
$''$	0	$c_0 b_{-m} c_m$	$\rightsquigarrow 0$
$''$	$[1,\infty[$	$+b_{-m-n} c_m c_n$	$\rightsquigarrow 0$

We also have $c_n\left|\psi,\downarrow\right\rangle = 0$ for $n \geq 1$ and are thus left with

$$Q_\mathrm{B}\left|\psi,\downarrow\right\rangle = \left(\sum_{n=1}^{\infty} c_{-n} L^m_n - c_0\right)\left|\psi,\downarrow\right\rangle = \sum_{n=0}^{\infty} c_{-n}(L^m_{-n} - \delta_{n,0})\left|\psi,\downarrow\right\rangle \qquad [4.226]$$

Chapter 5

The String S-Matrix

Open Questions

I have a number of unanswered points for this chapter. They are briefly mentioned here and more detail is given under the respective headings. Any help in resolving them can be sent to hepnotes@hotmail.com and is more than welcome.

- ♣ (5.4.4) At the end of deriving the Weyl invariance of the b ghost insertion in scattering amplitudes, Joe writes that "The b equation of motion comes from $\delta S/\delta c = 0$, so there will be source terms at the c insertions; this is precisely what is needed to account for the effect of the coordinate transformations on the fixed vertex operators". I don't understand this, let alone how you can show this.

5.1 p 147: Eq (5.1.9-11) The Torus as a Parallelogram

The argument from section 3.3. is as follows. Recall that under a Weyl transformation $g_{ab} \to e^{2\omega(\sigma)} g_{ab}$, the Ricci scalar transforms as follows

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega) \tag{5.1}$$

By solving the equation $\nabla^2\omega = \frac{1}{2}R$ we can thus, at least locally, go to a frame that has zero Ricci scalar. As discussed in (3.3.6) and in these notes [3.14], in two dimensions the Riemann curvature is related to the Ricci scalar by $R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R$. A zero Ricci scalar thus implies that locally we have a flat manifold, hence a metric δ_{ab}. In general the coordinate system corresponding to this new metric will not have the same periodicity conditions $(\sigma^1, \sigma^2) \cong (\sigma^1, \sigma^2) + 2\pi(m, n)$. This is similar as for the point particle, where the new tetrad $e'(\tau)$ may not have the same periodicity as the original one. Just as the circle for the point particle may be "stretched" to ℓ, for the torus the periodicity may now be "stretched" in the two directions of the torus

$$\sigma^a \cong \sigma^a + 2\pi(mu^a + nv^a) \tag{5.2}$$

for two vectors $\vec{u} = (u^1, u^2)$ and $\vec{v}(v_1, v_2)$. We can always perform a rotation and scaling of the coordinate system such that $\vec{u}$ lies along the x-axis and has length one. The metric will then remain the δ_{ab} and we have the periodicity

$$\sigma^1 \cong \sigma^1 + 2\pi m + 2\pi n v^1 \quad ; \quad \sigma^2 \cong \sigma^2 + 2\pi n v^2 \qquad [5.3]$$

We now define the complex coordinate $w = \sigma^1 + i\sigma^2$. The periodicity means

$$\begin{aligned} w &= \sigma^1 + i\sigma^2 \cong \sigma^1 + 2\pi m + 2\pi n v^1 + i(\sigma^2 + 2\pi n v^2) \\ &= w + 2\pi m + 2\pi n(v^1 + iv^2) = w + 2\pi m + 2\pi n\tau \end{aligned} \qquad [5.4]$$

with $\tau = v^1 + iv^2$. This is the approach where we keep the metric unchanged $ds^2 = dw d\bar{w}$ but where the periodicity condition changes, $w \cong w + 2\pi m + 2\pi n\tau$. This approach corresponds to setting $e = 1$ and $\tau \in [0, \ell]$ for the point particle.

The alternative approach, corresponding to keeping $\tau \in [0, 1]$ and setting $e = \ell$ for the point particle, is to change the metric. This is achieved by going to another coordinate system,

$$w = \sigma^1 + \tau\sigma^2 \qquad [5.5]$$

The metric is then

$$ds^2 = dw d\bar{w} = (d\sigma^1 + \tau d\sigma^2)(d\sigma^1 + \bar{\tau} d\sigma^2) = |d\sigma^1 + \tau d\sigma^2|^2 \qquad [5.6]$$

The metric is clearly invariant under $\tau \to \bar{\tau}$. Moreover it is of the form

$$g_{ab} = \begin{pmatrix} 1 & \frac{1}{2}(\tau + \bar{\tau}) \\ \frac{1}{2}(\tau + \bar{\tau}) & |\tau|^2 \end{pmatrix} \qquad [5.7]$$

which for τ real gives $\det g = 0$, so this is not an acceptable value of τ. We can thus already restrict our attention to $\mathrm{Im}\,\tau > 0$.

5.2 p 148: Eq (5.1.12) The Transformations S and T

A torus is characterised by a complex parameter τ. This can be represented by a parallelogram in the complex plane with edges $0, 1, \tau$ and $1 + \tau$, see fig. 5.1 and opposite sides identified.

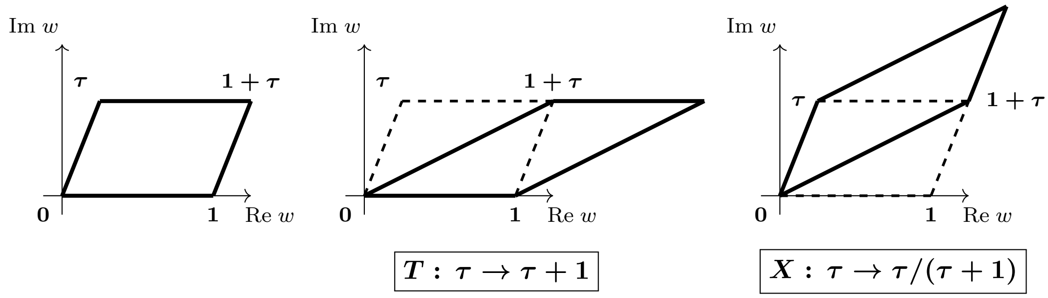


Figure 5.1: Modular transformations of the torus

It should be clear that the two additional figures give an equivalent choice of torus. The first transformed torus corresponds clearly to a modulus $\tau + 1$, hence a transformation $T: \tau \to \tau + 1$. We see that it leaves the periodicity condition unchanged:

$$T : w + 2\pi m + 2\pi n\tau \to w + 2\pi m + 2\pi n(\tau + 1) = w + 2\pi(m + n) + 2\pi n\tau \qquad [5.8]$$

This gives a reciprocity relation $w \cong +2\pi m' + 2\pi n'\tau$ with $(m', n') = (m + n, n)$ and thus (m, n) replaced by $(m - n, n)$.

The second transformed torus corresponds to a transformation $X : \tau \to \tau/(1 + \tau)$. Under the periodicity we have

$$\begin{aligned} X : w + 2\pi m + 2\pi n\tau \to w + 2\pi m + 2\pi n\frac{\tau}{1+\tau} &= \frac{1}{1+\tau}\left[(1+\tau)w + 2\pi m(1+\tau) + 2\pi n\tau\right] \\ &= \frac{1}{1+\tau}\left[(1+\tau)w + 2\pi m + 2\pi(m+n)\tau\right] \end{aligned} \qquad [5.9]$$

We thus have the periodicity

$$w' \cong w + 2\pi m + 2\pi(m + n)\tau \qquad [5.10]$$

where we have rescaled the complex coordinate $w' = (1+\tau)w$. This thus corresponds with replacing (m, n) by $(m, n - m)$.

It is convenient to use another combination than X. We have

$$\begin{aligned} T^{-1}XT^{-1}\tau =& T^{-1}X(\tau - 1) = T^{-1}\left(\frac{\tau - 1}{1 + \tau - 1}\right) = T^{-1}\left(\frac{\tau - 1}{\tau}\right) \\ =& \frac{\tau - 1}{\tau} - 1 = -\frac{1}{\tau} \end{aligned} \qquad [5.11]$$

So, rather than X, we consider the transformation $S: \tau \to -1/\tau$. Note that we have the convenient relations

$$S^2 = 1 \quad ; \quad (ST)^3 = 1 \qquad [5.12]$$

5.3 p 148: Eq (5.1.13) $PSL(2,\mathbb{Z})$ Group

We can represent a general transformation (5.1.13) as a 2×2 matrix

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.13}$$

We then have

$$\begin{aligned} T : \tau \to \tau + 1 \quad &\Rightarrow \qquad T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ S : \tau \to -\frac{1}{\tau} \quad &\Rightarrow \qquad S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{5.14}$$

Composition of transformations is then given by matrix multiplication, e.g.

$$ST(\tau) = S(\tau+1) = -\frac{1}{\tau+1} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \Leftrightarrow ST = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \tag{5.15}$$

The condition that a transformation Y has $ad - bc$ means that $\det(Y) = 1$. Composition of two transformations then automatically preserves this as $\det(YY') = \det(Y)\det(Y')$. These transformations form a group, the projective special linear 2×2 matrices with integer indices. Projective because, if all signs of a, b, c and d are changed, the transformation is the same. Special because is has unit determinant. Mathematically this group is $SL(2,\mathbb{Z})/\mathbb{Z}_2$, also denoted by $PSL(2,\mathbb{Z})$.

5.4 p 148: Eq (5.1.14) $PSL(2,\mathbb{Z})$ Transforming the Metric

Let's apply the transformation (5.1.14) to the metric

$$\begin{aligned} ds^2 = {} & (d\sigma^1)^2 + (\tau + \bar\tau) d\sigma^1 d\sigma^2 + |\tau|^2 (d\sigma^2)^2 \\ = {} & \left(d d\sigma'^1 + b d\sigma'^2\right)^2 + (\tau + \bar\tau)\left(d d\sigma'^1 + b d\sigma'^2\right)\left(c d\sigma'^1 + a d\sigma'^2\right) + |\tau|^2 \left(b d\sigma'^1 + a d\sigma'^2\right)^2 \\ = {} & \left(d^2 + cd(\tau+\bar\tau) + b^2|\tau|^2\right)(d\sigma'^1)^2 + \left(2bd + (ad+bc)(\tau+\bar\tau) + 2|\tau|^2 ab\right) d\sigma'^1 d\sigma'^2 \\ & + \left(b^2 + ab(\tau+\bar\tau) + a^2|\tau|^2\right)(d\sigma'^2)^2 \\ = {} & (c\tau+d)(c\bar\tau+d)(d\sigma'^1)^2 + \left[(c\tau+d)(a\bar\tau+b) + (a\tau+b)(c\bar\tau+d)\right] d\sigma'^1 d\sigma'^2 \\ & + (a\tau+b)(a\bar\tau+b)(d\sigma'^2)^2 \\ = {} & (c\tau+d)(c\bar\tau+d)\left[(d\sigma'^1)^2 + \left(\frac{a\bar\tau+b}{c\bar\tau+d} + \frac{a\tau+b}{c\tau+d}\right) d\sigma'^1 d\sigma'^2 + \left|\frac{a\tau+b}{c\bar\tau+d}\right|^2 (d\sigma'^2)^2\right] \\ = {} & (c\tau+d)(c\bar\tau+d)\left[(d\sigma'^1)^2 + (\tau'+\bar\tau') d\sigma'^1 d\sigma'^2 + |\tau'|^2 (d\sigma'^2)^2\right] \end{aligned} \tag{5.16}$$

with $\tau' = (a\tau+b)/(c\tau+d)$.

5.5 p 148: Eq (5.1.15) The Fundamental Region of $PSL(2,\mathbb{Z})$

We have seen already that the metric is invariant under conjugation of τ so that we can restrict ourselves in the upper half complex plane $\mathbb{H}$.

Let us now consider a point with real part larger than $1/2$. By repeated application of $T^{-1} : \tau \to \tau - 1$ we can bring this modulus into the range $\mathrm{Re}\,\tau \in [-1/2, 1/2]$. Similarly, if $\mathrm{Re}\,\tau < 1/2$ we can bring it in $\mathrm{Re}\,\tau \in [-1/2, 1/2]$ by repeated application of $T : \tau \to \tau + 1$. Every modulus $\tau \in \mathbb{H}$ is thus equivalent to a modulus with real part between $-1/2$ and $1/2$. The boundaries $\mathrm{Re}\,\tau = \pm 1/2$ are identified with one another by the application of T.

Next, consider a modulus with $\mathrm{Re}\,\tau \in [-1/2, 1/2]$. Application of S on this moduli brings it to $S : \tau \to -1/\tau = -\bar{\tau}/|\tau|$ and so it reflects it around the complex axis and moves it out of the unit circle in $\mathbb{H}$. Repeated application of T or T^{-1} then brings it back in the region $[-1/2, 1/2]$ but still outside of the unit circle.

These steps are illustrated in fig. 5.2. So by judicious application of T and S every modulus is thus equivalent using a modular transformation to a modulus τ in the region

$$-\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2} \quad \text{and} \quad |\tau| \geq 1 \tag{5.17}$$

with the borders $\mathrm{Re}\,\tau = \pm 1/2$ identified.

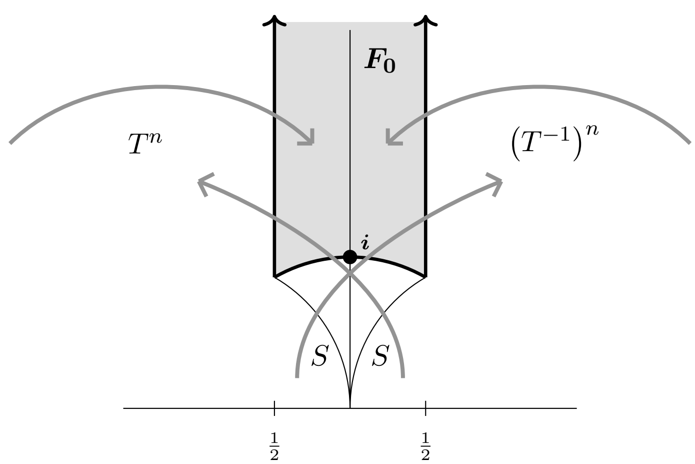


Figure 5.2: The fundamental region of the modular group

The statement that F_0 is the fundamental domain of the modular group consists of two parts. First, it means that every point in $\mathbb{H}$ can be mapped by a modular transformation into F_0, and second that no two points in F_0 can be mapped into one another by a modular transformation. In other words, F_0 contains every torus once and only once. Whilst we have argued that F_0 is the fundamental domain of the modular group, we have been quite cavalier about the boundaries of F_0. Those who are interested in a more detailed approach are referred to Zwiebach's book, section 26.6 of the second edition.

5.6 p 151: Eq (5.2.4) The Diff×Weyl Transformation of the Metric, I

This is just (3.3.16). As a reminder

$$(P_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - g_{ab}\nabla_c\delta\sigma_c) \tag{5.18}$$

and satisfies $(P_1\delta\sigma)_{ab} = (P_1\delta\sigma)_{ba}$ and $g^{ab}(P_1\delta\sigma)_{ab} = 0$.

5.7 p 151: Eq (5.2.5) The Diff×Weyl Transformation of the Metric, II

We want to find all variations of the metric that are not a diff×Weyl variation, i.e. that are not a linear combination of variations of the form (5.2.4). In other words we are looking for all variations $\delta' g_{ab}$ that are orthogonal to (5.2.4), i.e.

$$0 = \int d^2\sigma\,\sqrt{g}\,\delta g_{ab}\delta' g^{ab} = \int d^2\sigma\,\sqrt{g}\,\left[-2(P_1\delta\sigma)^{ab} + (2\delta\omega - \nabla_c\delta\sigma^c)g^{ab}\right]\delta' g_{ab} \tag{5.19}$$

In order to work this out we need some properties of the operators P_1. Note that P_1 takes a vector (a one-index tensor) into a two index tensor. We now define transpose P_1^T that takes a two index tensor u_{ab} into a one index tensor as follows

$$(P_1^T u)_a = -\nabla^b u_{ab} \tag{5.20}$$

We now define an inner product between two symmetric traceless n-index tensors s and t as $(s,t) = \int d^2\sigma\,\sqrt{g}\,s^{a_1\cdots a_n}t_{a_1\cdots a_n}$. We now show that for a symmetric traceless two index tensor u and a one index tensor v we have $(u, P_1 v) = (P_1^T u, v)$. Indeed

$$\begin{aligned}
(u, P_1 v) &= \int d^2\sigma\,\sqrt{g}\,u^{ab}(P_1 v)_{ab} = \int d^2\sigma\,\sqrt{g}\,u^{ab}\frac{1}{2}(\nabla_a v_b + \nabla_b v_a - g_{ab}\nabla_c v^c) \\
&= \int d^2\sigma\,\sqrt{g}\,u^{ab}\left(\nabla_a v_b - \frac{1}{2}g_{ab}\nabla_c v^c\right) \\
&= -\int d^2\sigma\,\sqrt{g}\,v_b\nabla_a u^{ab} = \int d^2\sigma\,\sqrt{g}\,(P_1^T u)^b v_b = (P_1^T u, v)
\end{aligned} \tag{5.21}$$

We have used partial integration, the fact that the metric is covariantly constant and the fact that u^{ab} is traceless. Using this, we can write

$$\int d^2\sigma\,\sqrt{g}\,\delta' g^{ab}(P_1\delta\sigma)^{ab} = (\delta' g, P_1\delta\sigma) = (P_1^T\delta' g, \delta\sigma) = \int d^2\sigma\,\sqrt{g}\,(P_1^T\delta' g)_a\delta\sigma^a \tag{5.22}$$

This relation is only correct if $\delta' g_{ab}$ is traceless. We will imminently see that this is indeed the case. Using this relation we thus find that the variations orthogonal to all diff×Weyl must satisfy

$$= \int d^2\sigma \sqrt{g} \left[-2(P_1^T \delta' g)_a \delta\sigma^a + \delta' g_{ab} g^{ab}(2\delta\omega - \nabla_c \delta\sigma^c) \right] \qquad [5.23]$$

This must be valid for all diff×Weyl, hence for all $\delta\sigma$ and $\delta\omega$. The $\delta\omega$ condition implies $\delta' g_{ab} g^{ab} = 0$ which justifies our assumption that $\delta' g_{ab}$ is traceless.

5.8 p 151: Eq (5.2.5) The Conformal Killing Equation

The conformal Killing vectors are those infinitesimal diff×Weyl transformations that leave the metric unchanged, i.e. satisfy $\delta g_{ab} = 0$. From (5.2.4) this means that conformal Killing vectors are solutions of the equation

$$-2(P_1 \delta\sigma)^{ab} + (2\delta\omega - \nabla \cdot \delta\sigma) g^{ab} = 0 \qquad [5.24]$$

Taking the trace of this, and using the fact that $(P_1 \delta\sigma)^{ab}$ is traceless by construction we find that

$$\delta\omega = \frac{1}{2} \nabla \cdot \delta\sigma \qquad [5.25]$$

which determines the Weyl transformation in terms of the diffeomorphism. The remaining conformal Killing equations are therefore

$$(P_1 \delta\sigma)^{ab} = 0 \qquad [5.26]$$

Note that a conformal Killing vector is thus a specific combination of a diffeomorphism and a Weyl transformation.

5.9 p 151: Eq (5.2.5) The Moduli and Conformal Killing Vectors in the Conformal Gauge

We start with determining the equations for the moduli in the conformal gauge. In that gauge the non-zero metric components are $g_{z\bar{z}} = \frac{1}{2} e^{2\omega}$ and $g^{z\bar{z}} = 2e^{-2\omega}$. It follows that $\nabla^z = g^{z\bar{z}} \nabla_{\bar{z}} = -2e^{-2\omega} \nabla_{\bar{z}}$ and $\nabla^{\bar{z}} = g^{\bar{z}z} \nabla_z = -2e^{-2\omega} \nabla_z$. We also have the connections $\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$. It follows that

$$\begin{aligned}
\Gamma^z_{zz} &= \frac{1}{2} g^{z\bar{z}} (\partial_z g_{z\bar{z}} + \partial_z g_{z\bar{z}} - \partial_{\bar{z}} g_{zz}) = 2\partial\omega \\
\Gamma^{\bar{z}}_{zz} &= \frac{1}{2} g^{\bar{z}z} (\partial_z g_{zz} + \partial_z g_{zz} - \partial_z g_{zz}) = 0 \\
\Gamma^z_{z\bar{z}} &= \frac{1}{2} g^{z\bar{z}} (\partial_z g_{\bar{z}\bar{z}} + \partial_{\bar{z}} g_{z\bar{z}} - \partial_{\bar{z}} g_{z\bar{z}}) = 0
\end{aligned} \qquad [5.27]$$

with all the other connections following from symmetry considerations. I.e. the only non-zero connections are

$$\Gamma^z_{zz} = 2\partial\omega \quad ; \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\bar{\partial}\omega \tag{5.28}$$

The equation for the moduli (5.2.6b) then becomes $\nabla^b \delta' g_{ba} = 0$ or in complex coordinates

$$0 = \nabla^z \delta' g_{zz} + \nabla^{\bar{z}} \delta' g_{\bar{z}z} = \nabla_{\bar{z}} \delta' g_{zz} = \partial_{\bar{z}} \delta' g_{zz} - \Gamma^c_{\bar{z}z} v_c = \partial_{\bar{z}} \delta' g_{zz} \tag{5.29}$$

Where we have used the fact that $g_{z\bar{z}} = 0$ by the tracelessness condition (5.2.6.a) and the fact that $\Gamma^c_{\bar{z}z} = 0$. There is, of course a similar equation for the ant-holomorphic part. Thus the equations for the moduli are

$$\partial_{\bar{z}} \delta' g_{zz} = \partial_z \delta' g_{\bar{z}\bar{z}} = 0 \quad \text{and} \quad \delta' g_{z\bar{z}} = 0 \tag{5.30}$$

The two holomorphic doubly periodical solutions of these equations on the torus are two real constant. They combine together to form the Teichmüller parameter τ.

Let us now turn to the equations for the conformal Killing vectors. They are

$$0 = (P_1 \delta\sigma)_{ab} = \frac{1}{2}(\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab} \nabla \cdot \delta\sigma) \tag{5.31}$$

Let us start with the zz component

$$\begin{aligned} 0 =& \frac{1}{2}(\nabla_z \delta\sigma_z + \nabla_z \delta\sigma_z - g_{zz} \nabla \cdot \delta\sigma) = \nabla_z \delta\sigma_z \\ =& \partial_z \delta\sigma_z - \Gamma^z_{zz} \delta\sigma_z = \partial_z (g_{z\bar{z}} \delta\sigma^{\bar{z}}) - 2\partial_z \omega g_{z\bar{z}} \delta\sigma^{\bar{z}} \\ =& \frac{1}{2} 2\partial_z \omega e^{2\omega} \delta\sigma^{\bar{z}} + g_{z\bar{z}} \partial_z \delta\sigma^{\bar{z}} - 2\partial_z \omega \frac{1}{2} e^{2\omega} \delta\sigma^{\bar{z}} = g_{z\bar{z}} \partial_z \delta\sigma^{\bar{z}} = \partial\bar{z} \end{aligned} \tag{5.32}$$

we have used the fact that the complex coordinates have the indices upstairs; thus $\delta\sigma^{\bar{z}} = \bar{z}$. There is, of course, also the anti-holomorphic equation $\bar{\partial} z = 0$. Finally the mixed indices $(P_1 \delta\sigma)_{z\bar{z}}$ give the trace, which is zero by construction. The equations for the conformal Killing vectors are thus

$$\partial\bar{z} = \bar{\partial} z = 0 \tag{5.33}$$

The two holomorphic doubly periodical solutions of these equations on the torus are two real constants, corresponding to the translations of (5.1.16)

5.10 p 152: Eq (5.2.10) No CKVs for Negative Euler Number and no Moduli for Positive Euler Number

We start by showing that $P_1 T P_1 = -\frac{1}{2}\nabla^2 - \frac{1}{4}R$. Recall that P_1 takes a vector into a two-index tensor and P_1^T takes a two index tensor into a vector, so t $P_1 T P_1$ acts on a vector and changes it into a vector. We have

$$\begin{aligned} P_1^T P_1 v_b &= P_1^T \frac{1}{2}(\nabla_a v_b + \nabla_b v_a - g_{ab}\nabla\cdot v) \\ &= -\frac{1}{2}\nabla^a(\nabla_a v_b + \nabla_b v_a - g_{ab}\nabla\cdot v) \\ &= -\frac{1}{2}\nabla^2 v_b - \frac{1}{2}(\nabla^a\nabla_b v_a - g_{ab}\nabla^a\nabla_c v^c) \\ &= -\frac{1}{2}\nabla^2 v_b - \frac{1}{2}(\nabla^a\nabla_b v_a - g_{cb}\nabla^c\nabla_a v^a) \\ &= -\frac{1}{2}\nabla^2 v_b - \frac{1}{2}(\nabla^a\nabla_b v_a - \nabla_b\nabla^a v_a) \\ &= -\frac{1}{2}\nabla^2 v_b - \frac{1}{2}[\nabla_a, \nabla_b]v^a \end{aligned} \tag{5.34}$$

Now, the commutator of two covariant derivatives acting on a vector gives the Riemann curvature, see e.g. Carroll (3.112), $[\nabla_a, \nabla_b]v^c = R^c_{dab}v^d$. Thus

$$P_1^T P_1 v_b = -\frac{1}{2}\nabla^2 v_b - \frac{1}{2}R^a_{cab}v^c \tag{5.35}$$

But we have already learned in (3.3.76) that in two dimensions $R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R$. Therefore

$$R^a_{bcd} = g^{ae}R_{ebcd} = \frac{1}{2}g^{ae}(g_{ec}g_{bd} - g_{ed}g_{bc})R = \frac{1}{2}(\delta^a_c g_{bd} - \delta^a_d g_{bc}) \tag{5.36}$$

and

$$P_1^T P_1 v_b = -\frac{1}{2}\nabla^2 v_b - \frac{1}{4}(\delta^a_a g_{cb} - \delta^a_b g_{ca})Rv^c = -\frac{1}{2}\nabla^2 v_b - \frac{1}{4}Rv_b \tag{5.37}$$

which is what we set out to show.

Recall that under a Weyl transformation the curvature transforms as (1.2.32)

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\omega) \tag{5.38}$$

This equation can be solved for ω to set R' constant. Now the Euler number is defined as (1.2.31)

$$\chi = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}R \tag{5.39}$$

So if we have a negative Euler number, $\chi < 0$, this necessarily means that the curvature is negative, $R < 0$. And if we have a positive Euler number, $\chi > 0$, this necessarily means that the curvature is positive, $R > 0$.

Let us now work out (5.2.10)

$$\begin{aligned}\int d^2\sigma\,\sqrt{g}\,(P_1\delta\sigma)_{ab}(P_1\delta\sigma)^{ab} &= \int d^2\sigma\,\sqrt{g}\,\delta\sigma_a(P_1^T P_1\delta\sigma)^a \\ &= \int d^2\sigma\,\sqrt{g}\,\delta\sigma_a\left(-\frac{1}{2}\nabla^2-\frac{1}{4}R\right)\delta\sigma^a \\ &= \int d^2\sigma\,\sqrt{g}\,\left(\frac{1}{2}\nabla_a\delta\sigma_b\nabla^a\delta\sigma^b-\frac{1}{4}R\delta\sigma_a\delta\sigma^a\right) \qquad [5.40]\end{aligned}$$

For negative Euler number, R is negative and the integrand is strictly positive, implying that $(P_1\delta\sigma)_{ab}$ is strictly positive, Thus $(P_1\delta\sigma)_{ab}=0$ has no solutions. But this is the equation for conformal Killing vectors. So we conclude that $\chi<0\Rightarrow\kappa=0$.

I have not been able to show in a satisfying way that for positive Euler numbers, the number of moduli is zero. I have tried two ways. First by trying to find an expression for $(P_1P_1^Tu)^{ab}$ that include the Ricci scalar, so that we can link it to the Euler number. Next by trying to work out $\int d^2\sigma\,\sqrt{g}\,\nabla^c\delta' g_{ac}\nabla^d\delta' g_{bd}$ directly and trying to link that to the curvature. But neither way has lead to anything useful. There is, however an indirect way. We know that for closed oriented surfaces, the Euler number is $\chi=2-2g$, with g the number of handles. The only close oriented surface with $\chi>0$ therefore has $g=0$, i.e. is the the sphere, the disk or the projective plane, for which there are no moduli.

5.11 p 155: Eq (5.3.2) The Gauge-Fixed Measure

The measure of the classical action is $[dg\,d\phi]\,d^{2n}\sigma$ where $d^{2n}\sigma=\prod_{i=1}^n d^2\sigma_i$ denotes the product over all vertex operators. Gauge-fixing the action we are left with an integration over the gauge parameters ξ, but still need to integrate over all the μ moduli t^k as these have not been fixed by the Faddeev-Popov procedure. We can, however use the κ conformal Killing vectors, to fix the coordinates of κ of the vertex operators on the worldsheet. Nothing changes w.r.t. the matter fields. We are thus left with a measure $[d\xi\,d\phi]\,d^\mu t\,d^{2n-\kappa}\sigma$.

5.12 p 155: Eq (5.3.5) The Variation of the Metric Including the Moduli

This is just (5.2.4) with in addition a variation in the moduli $\sum_{k=1}^{\mu}\partial_{t^k}\hat{g}_{ab}\,\delta t^k$. Note that, as per Joe's errata page, there is an error in the last term. The correct equation should read

$$\delta g_{ab}=\sum_{k=1}^{\mu}\partial_{t^k}\hat{g}_{ab}\,\delta t^k-2(\hat{P}_1\delta\sigma)_{ab}+(2\delta\omega-\hat{\nabla}\cdot\delta\sigma)\hat{g}_{ab} \qquad [5.41]$$

5.13 p 156: Eq (5.3.6) Inverse Faddeev-Popov Determinant

We basically repeat the analysis of the Faddeev-Popov procedure surrounding (3.3.18) for completeness.

The first line is just the rewriting of (5.3.3) with the $[d\xi] = [d\delta\sigma\, d\delta\omega]$ the integration over the diff×Weyl gauge parameters. We also write

$$\delta\big(\delta\sigma^a(\hat\sigma_i)\big) = \frac{1}{2\pi}\int e^{iy_{ai}\delta\sigma^a(\hat\sigma_i)}dy_{ai} = \int e^{2\pi i x_{ai}\delta\sigma^a(\hat\sigma_i)}dx_{ai} \qquad [5.42]$$

with $x_{ai} = y_{ai}/2\pi$. Thus

$$\prod_{(a,i)\in f}\delta\big(\delta\sigma^a(\hat\sigma_i)\big) = \prod_{(a,i)\in f}\int e^{2\pi i x_{ai}\delta\sigma^a(\hat\sigma_i)}dx_{ai} = \int d^\kappa x\, \exp\left[2\pi i \sum_{(a,i)\in f} x_{ai}\delta\sigma^a(\hat\sigma_i)\right] \qquad [5.43]$$

where $d^\kappa x = \prod_{(a,i)\in f} dx_{ai}$, with κ being the number of conformal Killing vectors that can be fixed. We similarly write the$\delta(\delta g_{ab})$ as an exponential and can thus write for the inverse Faddeev-Popov determinant

$$\begin{aligned}\Delta_{\text{FB}}^{-1} &= n_R\int d^\mu\delta t\, d^\kappa x\,[d\delta\omega\, d\delta\sigma\, d\beta]\, \exp\left[2\pi i\left(\int d^2\sigma\,\sqrt{\hat g}\beta^{ab}\delta_{gab} + \sum_{(a,i)\in f} x_{ai}\delta\sigma^a(\hat\sigma_i\right)\right]\\ &= n_R\int d^\mu\delta t\, d^\kappa x\,[d\delta\omega\, d\delta\sigma\, d\beta]\, \exp\Bigg\{2\pi i\Bigg[\int d^2\sigma\,\sqrt{\hat g}\beta^{ab}\Big(\sum_{k=1}^{\mu}\partial_{t^k}\hat g_{ab}\,\delta t^k - 2(\hat P_1\delta\sigma)_{ab}\\ &\qquad + (2\delta\omega - \hat\nabla\cdot\delta\sigma)\hat g_{ab}\Big) + \sum_{(a,i)\in f} x_{ai}\delta\sigma^a(\hat\sigma_i\Bigg]\Bigg\}\end{aligned} \qquad [5.44]$$

We perform the integration over $\delta\omega$. This gives a factor $\delta(\beta^{ab}\hat g_{ab})$, ensuring that the ghost field β^{ab} is traceless and thus that also the term $\beta^{ab}\hat\nabla\cdot\delta\sigma\hat g_{ab}$ does not contribute. What remains is

$$\begin{aligned}\Delta_{\text{FB}}^{-1} &= n_R\int d^\mu\delta t\, d^\kappa x\,[d\delta\sigma\, d\beta']\, \exp\Bigg\{2\pi i\Bigg[\int d^2\sigma\,\sqrt{\hat g}\beta'^{ab}\Big(\sum_{k=1}^{\mu}\partial_{t^k}\hat g_{ab}\,\delta t^k - 2(\hat P_1\delta\sigma)_{ab}\Big)\\ &\qquad + \sum_{(a,i)\in f} x_{ai}\delta\sigma^a(\hat\sigma_i\Bigg]\Bigg\}\\ &= n_R\int d^\mu\delta t\, d^\kappa x\,[d\delta\sigma\, d\beta']\, \exp\left[2\pi i(\beta', 2\hat P_1\delta\sigma - \delta t^k\partial_{t^k}\hat g_{ab}) + 2\pi i\sum_{(a,i)\in f} x_{ai}\delta\sigma^a(\hat\sigma_i)\right]\end{aligned} \qquad [5.45]$$

Here β' is traceless, $(t,t') = \int d^2\sigma\,\sqrt{g}\,tt'$ and we have changed the sign of what remains from $\delta(\delta g_{ab})$ for convenience.

5.14 p 156: Eq (5.3.8) The Faddeev-Popov Ghosts

We use (A.2.28) from the appendix. For x and y c-numbers and ψ and χ Grassmann numbers we have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{2\pi i\lambda xy} = \frac{1}{\lambda} = \left[\int d\psi \int d\chi\, e^{\lambda\psi\chi}\right]^{-1} \tag{5.46}$$

We make the correspondence, including convenient normalisation

$$\begin{aligned} \delta\sigma^a &\rightsquigarrow -\frac{1}{4\pi}c^a \\ \beta'_{ab} &\rightsquigarrow b_{ab} \\ x_{ai} &\rightsquigarrow \eta_{ai} \\ \delta t^k &\rightsquigarrow -\frac{1}{4\pi}\xi^k \end{aligned} \tag{5.47}$$

This gives

$$\Delta_{\mathrm{FB}} = \frac{1}{n_R}\int [db\, dc] d^\mu\xi\, d^\kappa\eta\, \exp\Big[-\frac{1}{4\pi}\big(b, 2\hat{P}_1 c - \xi^k\partial_{t^k}\hat{g}\big) + \sum_{(a,i)\in f}\eta_{ai}c^a(\hat{\sigma}_i)\Big] \tag{5.48}$$

We can perform the integrations over the Grassmann variables ξ and eta using $\int d\xi e^{\alpha\xi} = \int d\xi(1+\alpha\xi) = \alpha$:

$$\begin{aligned} \Delta_{\mathrm{FB}} &= \frac{1}{n_R}\int [db\, dc]\exp\Big[-\frac{1}{4\pi}\big(b, 2\hat{P}_1 c\big)\Big]\prod_{k=1}^{\mu}\frac{1}{4\pi}(b, \partial_{t^k}\hat{g})\prod_{(a,i)\in f}c^a(\hat{\sigma}_i)\Big] \\ &= \frac{1}{n_R}\int [db\, dc]\exp(-S_g)\prod_{k=1}^{\mu}\frac{1}{4\pi}(b, \partial_{t^k}\hat{g})\prod_{(a,i)\in f}c^a(\hat{\sigma}_i) \end{aligned} \tag{5.49}$$

where we have used the definition of the ghost action (3.3.21)

$$S_g = \frac{1}{2\pi}\int d^2\sigma\,\sqrt{\hat{g}}\, b_{ab}(\hat{P}_1 c)^{ab} = \frac{1}{2\pi}(b, \hat{P}_1 c) \tag{5.50}$$

5.15 p 156: Eq (5.3.9) The *S*-Matrix for the Bosonic String

This formula is obtained by filling in the various parts so is entirely straightforward. But we repeat it here and discuss its different parts because of its sheer importance for string theory

$$S_{j_1\cdots j_n}(k_1,\cdots,k_n)=\sum_{\substack{\text{compact}\\ \text{topologies}}}\frac{1}{n_R}\int_F d^\mu t\int[d\phi\,db\,dc]\,\exp\left(-S_m-S_g-\lambda\chi\right)$$
$$\times\prod_{(a,i)\notin f}\int d\sigma_i^a\prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_{t^k}\hat{g})\prod_{(a,i)\in f}c^a(\hat{\sigma}_i)\prod_{i=1}^{n}\sqrt{\hat{g}(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i) \qquad [5.51]$$

This is the S-matrix element for the scattering of n bosonic strings whose asymptotic states are created by the vertex operators $\mathcal{V}_{j_i}(k_i,\sigma_i)$. Let us explain what this expression means.

- We sum over all compact Riemann surfaces, oriented or unoriented depending on the string type.
- When the gauge symmetry is fully fixed, there may be a residual set of discrete symmetries, e.g. parity on the worldsheet. To avoid over-counting, we divide by the number of such discrete symmetries, n_R.
- We integrate over the μ moduli t^k of the Riemann surface over the fundamental domain F.
- We perform the path integral over the matter fields ϕ and the ghost fields b and c.
- The path integral is weighted by the matter action S_m, the ghost action S_g and the Euler term $\lambda\chi$. Here χ is the Euler number and λ is a parameter, that, as we will see later, is not independent.
- If the Riemann surface has κ conformal Killing vectors we can use this symmetry to fix κ of the coordinates of the vertex operators. We call the set of fixed coordinates f and so we need to integrate over all the coordinates of the vertex operators that are not fixed, i.e. that are not in f.
- We need to take into account the contribution from the variation of the metric under a change of moduli. For each such moduli there is a factor $(4\pi)^{-1}(b,\partial_t\hat{g})$.
- For each coordinate of a vertex operator that was fixed using the symmetry of the conformal Killing vectors, we need to insert a ghost $c^a(\hat{\sigma}^a)$ evaluated at the point of insertion.
- Finally, we add the product of all the vertex operators, weighted with $\sqrt{g}$ at the appropriate point.

5.16 p 157: Eq (5.3.14) $P_1\mathsf{C}_J$ is an Eigenfunction of $P_1P_1^T$

Let us check explicitly that $(P_1P_1^T)P_1=P_1(P_1^TP_1)$. This should, of course, be the case because P_1 and P_1^T are projection operators, but let us check it explicitly to make sure it is indeed the case.

On the one hand, we have

$$(P_1v)_{ab}=\frac{1}{2}(\nabla_a v_b+\nabla_b v_a-g_{ab}\nabla^c v_c) \qquad [5.52]$$

and thus

$$\begin{aligned}\left(P_1^T P_1 v\right)_a &= -\frac{1}{2}\nabla^c(\nabla_a v_c + \nabla_c v_a - g_{ac}\nabla^d v_d)\\ &= -\frac{1}{2}(\nabla^c\nabla_a v_c + \nabla^c\nabla_c v_a - g_{ac}\nabla^c\nabla^d v_d)\end{aligned} \qquad [5.53]$$

and finally

$$\begin{aligned}\Big(P_1\big(P_1^T P_1 v\big)\Big)_{ab} =& -\frac{1}{4}\Big[\nabla_a(\nabla^c\nabla_b v_c + \nabla^c\nabla_c v_b - g_{bc}\nabla^c\nabla^d v_d)\\ &+\nabla_b(\nabla^c\nabla_a v_c + \nabla^c\nabla_c v_a - g_{ac}\nabla^c\nabla^d v_d)\\ &-g_{ab}\nabla^e(\nabla^c\nabla_e v_c + \nabla^c\nabla_c v_e - g_{ec}\nabla^c\nabla^d v_d)\Big]\\ =& -\frac{1}{4}\Big[\nabla_a\nabla^c\nabla_b v_c + \nabla_a\nabla^c\nabla_c v_b - g_{bc}\nabla_a\nabla^c\nabla^d v_d\\ &+\nabla_b\nabla^c\nabla_a v_c + \nabla_b\nabla^c\nabla_c v_a - g_{ac}\nabla_b\nabla^c\nabla^d v_d\\ &-g_{ab}\nabla^e\nabla^c\nabla_e v_c - g_{ab}\nabla^e\nabla^c\nabla_c v_e + g_{ab}\nabla^e g_{ec}\nabla^c\nabla^d v_d\Big]\\ =& -\frac{1}{4}\Big[\nabla_a\nabla^c\nabla_b v_c + \nabla_a\nabla^2 v_b - \nabla_a\nabla_b\nabla\cdot v\\ &+\nabla_b\nabla^c\nabla_a v_c + \nabla_b\nabla^2 v_a - \nabla_b\nabla_a\nabla\cdot v\\ &-g_{ab}\nabla^e\nabla^c\nabla_e v_c - g_{ab}\nabla^e\nabla^2 v_e + g_{ab}\nabla^2\nabla\cdot v\Big]\end{aligned} \qquad [5.54]$$

On the other hand we have

$$(P_1^T u)_a = -\nabla^c u_{ac} \qquad [5.55]$$

and thus

$$\left(P_1(P_1^T u)\right)_{ab} = -\frac{1}{2}\Big(\nabla_a\nabla^c u_{bc} + \nabla_b\nabla^c u_{ac} - g_{ab}\nabla^d\nabla^c u_{dc}\Big) \qquad [5.56]$$

Using $u_{ab} = (P_1 v)_{ab} = \frac{1}{2}(\nabla_a v_b + \nabla_b v_a - g_{ab}\nabla\cdot v$ this gives

$$\begin{aligned}\Big(\big(P_1 P_1^T\big)P_1 v\Big)_{ab} =& -\frac{1}{4}\Big[\nabla_a\nabla^c(\nabla_b v_c + \nabla_c v_b - g_{bc}\nabla\cdot v)\\ &+\nabla_b\nabla^c(\nabla_a v_c + \nabla_c v_a - g_{ac}\nabla\cdot v)\\ &-g_{ab}\nabla^d\nabla^c(\nabla_d v_c + \nabla_c v_d - g_{dc}\nabla\cdot v)\Big]\\ =& -\frac{1}{4}\Big[\nabla_a\nabla^c\nabla_b v_c + \nabla_a\nabla^c\nabla_c v_b - g_{bc}\nabla_a\nabla^c\nabla\cdot v\\ &+\nabla_b\nabla^c\nabla_a v_c + \nabla_b\nabla^c\nabla_c v_a - g_{ac}\nabla_b\nabla^c\nabla\cdot v\\ &-g_{ab}\nabla^d\nabla^c\nabla_d v_c - g_{ab}\nabla^d\nabla^c\nabla_c v_d + g_{ab}\nabla^d\nabla^c g_{dc}\nabla\cdot v\Big]\end{aligned} \qquad [5.57]$$

which gives

$$\Big(\big(P_1P_1^T\big)P_1v\Big)_{ab} = -\frac{1}{4}\Big[\nabla_a\nabla^c\nabla_b v_c + \nabla_a\nabla^2 v_b - \nabla_a\nabla_b\nabla\cdot v + \nabla_b\nabla^c\nabla_a v_c + \nabla_b\nabla^2 v_a - \nabla_b\nabla_a\nabla\cdot v - g_{ab}\nabla^d\nabla^c\nabla_d v_c - g_{ab}\nabla^d\nabla^2 v_d + g_{ab}\nabla^2\nabla\cdot v\Big] \qquad [5.58]$$

We see that indeed $(P_1P_1^T)P_1 = P_1(P_1^TP_1)$.

This thus implies that $P_1\mathsf{C}_J$ is an Eigenfunction of $P_1P_1^T$ with Eigenvalue $\nu_J'^2$. Similarly we have

$$(P_1^TP_1)P_1^T\mathsf{B}_K = P_1^T(P_1P_1^T\mathsf{B}_K) = P_1^T\nu_K^2B_K = \nu_K^2P_1^TB_K \qquad [5.59]$$

and so $P_1^T\mathsf{B}_K$ is an Eigenfunction of $P_1^TP_1$ with Eigenvalue ν_K^2.

5.17 p 158: Eq (5.3.15) The Relation Between The B and C Eigenfunctions

For those Eigenfunctions B_J and C_J that have equal but non zero Eigenvalue $\nu_J = \nu'_J$, we have the normalisation

$$(B_J, B_{J'}) = \frac{1}{\nu_J\nu_{J'}}(P_1\mathsf{C}_J, P_1\mathsf{C}_{J'}) = \frac{1}{\nu_J\nu_{J'}}(\mathsf{C}_J, P_1^TP_1\mathsf{C}_{J'}) = \frac{1}{\nu_J\nu_{J'}}(\mathsf{C}_J, \nu_{J'}^2\mathsf{C}_{J'}) = \frac{\nu_{J'}}{\nu_J}(\mathsf{C}_J, \mathsf{C}_{J'}) = \frac{\nu_{J'}}{\nu_J}\delta_{J,J'} = 1 \qquad [5.60]$$

5.18 p 158: Eq (5.3.16) The Faddeev-Popov Determinant as a Function of the Ghost Eigenfunctions, I

We need to write the ghost action in terms of the ghost Eigenfunctions. From (5.3.11) and (5.3.15) we have

$$S_g = \frac{1}{2\pi}(b, P_1c) = \frac{1}{2\pi}\left(\sum_{k=1}^{\mu} b_{0\,k}\mathsf{B}_{0\,k} + \sum_K b_K\mathsf{B}_K, \sum_{j=1}^{\kappa} c_{0\,j}P_1\mathsf{C}_{0\,j} + \sum_J c_JP_1\mathsf{C}_J\right) \qquad [5.61]$$

But we know that $P_1C_{0\,j} = 0$ and that similarly $P_1^TB_{0\,k} = 0$. There is thus only one combination that survives

$$S_g = \frac{1}{2\pi}\sum_{J,K} b_Kc_J(B_K, P_1C_J) = \frac{1}{2\pi}\sum_{J,K} b_Kc_J(\frac{1}{\nu_K}P_1C_K, P_1C_J) = \frac{1}{2\pi}\sum_{J,K}\frac{b_Kc_J}{\nu_K}(C_K, P_1^TP_1C_J) = \frac{1}{2\pi}\sum_{J,K}\frac{b_Kc_J}{\nu_K}(C_K, \nu_J^2C_J) = \frac{1}{2\pi}\sum_J \nu_Jb_Jc_J \qquad [5.62]$$

(5.3.16) then follows immediately by splitting the integration over the zero Eigenvalue Eigenfunction $b_{0\,k}$ and $c_{0\,j}$ and the non-zero Eigenvalue Eigenfunctions b_J and c_J.

5.19 p 158: Eq (5.3.17) The Faddeev-Popov Determinant as a Function of the Ghost Eigenfunctions, II

Let us first consider the integration over the zero-modes $\int \prod_{j=1}^{\kappa} dc_{0\,j}$. The only place in the integrand where we can find a $c_{0\,j}$ that would make this integral non-vanishing is in the ghost field insertion $\prod_{(a,i\in f)} c^a(\sigma_i)$. Here f is the set of vertex coordinates we can fix, which is equal to the number of conformal Killing vectors, i.e. κ. If we write out his product we find

$$\prod_{(a,i\in f)} c^a(\sigma_i) = \prod_{(a,i\in f)} \left[\sum_{j=1}^{\kappa} c_{0\,j} \mathsf{C}^a_{0\,j}(\sigma_i) + \sum_J c_J \mathsf{C}^a_J(\sigma_i) \right] \tag{5.63}$$

To have a non-zero integral, we need to extract a product of κ zero-modes. In the expansion of the product – that, recall has exactly κ terms – this can only be the case if we take the terms that have only zero modes and no factors $c_J \mathsf{C}_J$. Indeed a term in the expansion with say ℓ factors of $c_J \mathsf{C}_J$ only has $k - \ell$ factors of $c_{0\,j} \mathsf{C}_{0\,j}$ and the integral of that would hence vanish. Thus, that contribution reduced to

$$\prod_{(a,i\in f)} c^a(\sigma_i) \rightsquigarrow \prod_{(a,i\in f)} \sum_{j=1}^{\kappa} c_{0\,j} \mathsf{C}^a_{0\,j}(\sigma_i) \tag{5.64}$$

For entirely similar reasons we find that the contribution of the b-ghost insertions reduces to

$$\prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_{t^k} \hat{g}) \rightsquigarrow \prod_{k=1}^{\mu} \sum_{k'=1}^{\mu} \frac{b_{0\,k'}}{4\pi} (\mathsf{B}_{0\,k'}, \partial_{t^k} \hat{g}) \tag{5.65}$$

What remains is the integration over the non-zero modes and the combination of all this gives directly (5.3.17).

5.20 p 158: Eq (5.3.18) The Faddeev-Popov Determinant as a Function of the Ghost Eigenfunctions, III

We need to show that

$$\int \prod_{j=1}^{\kappa} dc_{0\,j} \prod_{(a,i\in f)} \sum_{j'=1}^{\kappa} c_{0\,j'} \mathsf{C}^a_{0\,j'}(\sigma_i) = \det \mathsf{C}^a_{0\,j'}(\sigma_i) \tag{5.66}$$

Note that the total number of different (a,i) indices is exactly the number of conformal Killing vectors κ. Getting rid of all unnecessary indices we thus need to show that

$$\mathsf{D}^{[\kappa]} = \int \prod_{j=1}^{\kappa} dc_j \prod_{i}^{\kappa} \sum_{\ell=1}^{\kappa} c_\ell \mathsf{C}^i_\ell = \det \mathsf{C}^i_\ell \qquad [5.67]$$

The only non-vanishing terms in the integrand are those that contain $c_1 \ldots c_\kappa$. This the is a general property of determinants, but rather than prove it in general, let us work this out for a few cases so we see the pattern. For $\kappa = 1$ we have

$$\mathsf{D}^{[1]} = \int dc_1\, c_1 \mathsf{C}^1_1 = \mathsf{C}^1_1 = \det \mathsf{C} \qquad [5.68]$$

For $\kappa = 2$ we have

$$\begin{aligned}\mathsf{D}^{[1]} &= \int dc_1\, dc_2\, (c_1\mathsf{C}^1_1 + c_2\mathsf{C}^1_2)(c_1\mathsf{C}^2_1 + c_2\mathsf{C}^2_2) \\ &= \int dc_1\, dc_2\, c_1 c_2 (\mathsf{C}^1_1\mathsf{C}^2_2 - \mathsf{C}^1_2\mathsf{C}^2_1) = \det \mathsf{C} \qquad [5.69]\end{aligned}$$

For $\kappa = 3$ we have

$$\begin{aligned}\mathsf{D}^{[1]} &= \int dc_1\, dc_2\, dc_3\, (c_1\mathsf{C}^1_1 + c_2\mathsf{C}^1_2 + c_3\mathsf{C}^1_3)(c_1\mathsf{C}^2_1 + c_2\mathsf{C}^2_2 + c_3\mathsf{C}^2_3)(c_1\mathsf{C}^3_1 + c_2\mathsf{C}^3_2 + c_3\mathsf{C}^3_3) \\ &= \int dc_1\, dc_2\, dc_3\, c_1c_2c_3(\mathsf{C}^1_1\mathsf{C}^2_2\mathsf{C}^3_3 - \mathsf{C}^1_1\mathsf{C}^3_2\mathsf{C}^2_3 - \mathsf{C}^1_2\mathsf{C}^2_1\mathsf{C}^3_3 + \mathsf{C}^1_2\mathsf{C}^2_3\mathsf{C}^3_1 + \mathsf{C}^1_3\mathsf{C}^2_1\mathsf{C}^3_2 - \mathsf{C}^1_3\mathsf{C}^2_2\mathsf{C}^3_1) \\ &= \frac{1}{3!}\varepsilon^{\ell mn}_{ijk}\mathsf{C}^i_\ell \mathsf{C}^j_m \mathsf{C}^k_n = \det \mathsf{C} \qquad [5.70]\end{aligned}$$

The pattern should now be clear for general κ. The same of course is valid for the b-ghost part.

It remains to show that

$$\tilde{\mathsf{D}}^{[N]} = \int \prod_J db_J\, dc_J \exp\left(-\frac{\nu_J b_J c_J}{2\pi}\right) = \det{}'\left(\frac{\sqrt{P_1^T P_1}}{2\pi}\right) \qquad [5.71]$$

If we expand the exponential, only the linear term remains as b_J and c_J are Grassmann variables. Ignoring the irrelevant signs we thus have

$$\tilde{\mathsf{D}}^{[N]} = \int \prod_J db_J\, dc_J \frac{\nu_J b_J c_J}{2\pi} = \prod_J \frac{\nu_J}{2\pi} \qquad [5.72]$$

But recall from (5.3.12) that ν_J^2 are the (non-zero) Eigenvalues or $P_1^T P_1$. We can thus write symbolically that ν_J are the Eigenvalues of $\sqrt{P_1^T P_1}$. Now $\tilde{\mathsf{D}}^{[N]}$ is the product of the Eigenvalues of $\sqrt{P_1^T P_1}/2\pi$, which is the same as the (functional) determinant. This is what we had to show.

5.21 p 158: Eq (5.3.19) The Weyl Anomaly of the Ghost Current

The OPE of the energy-momentum tensor with the ghost current is given by (2.5.15)

$$T(z)j(0) \sim \frac{1-2\lambda}{z^3} + \frac{j(0)}{z^2} + \frac{\partial j(0)}{z} \qquad [5.73]$$

According to exercise 3.6 the Weyl anomaly of the ghost current is of the form

$$\nabla_a j^a = aR \qquad [5.74]$$

with a read off from the third order pole of the OPE $T(z)j(0) = 4a/z^3 + \cdots$. Thus $4a = 1-2\lambda$, implying $a = (1-2\lambda)/4$. Note there there is a sign error on exercise 3.6, as per Joe's errata page. Therefore

$$\nabla_a j^a = \frac{1-2\lambda}{4}R \qquad [5.75]$$

Let us now, for completeness, derive the Weyl anomaly for the covariant derivative of the ghost current. We will work it out for the holomorphic side. The anti-holomorphic is entirely similar and just gives a doubling of the anomaly.

Recall that the ghost current is $j(z) = bc(z)$ and so its covariant derivative $\nabla_a j^a$ has the same dimension as its energy-momentum tensor. We can therefore use the same reasoning as for the form of the anomaly of T^a_a in (3.4.9) to conclude that the general form of the anomaly is indeed $\nabla j = \alpha R$, with higher order terms in the curvature suppressed by high-momentum cut-off. Here α is some constant that we have to determine. In complex coordinates and expanding around a flat worldsheet and focussing on the holomorphic ghost current $= j_z$ we have

$$\nabla^a j_a = g^{ab}\partial_a j_b = 2\partial_{\bar z} j(z) + 2\partial_z \tilde{j} = 2\bar\partial j + 2\partial\tilde{j} \qquad [5.76]$$

The anomaly equation for the ghost number becomes

$$\bar\partial j + \partial\tilde{j} = \frac{1}{2}\alpha R \qquad [5.77]$$

We will now limit ourselves to the holomorphic side, the anti-holomorphic side being entirely similar. Just as for the calculation of the Weyl anomaly of the energy-momentum tensor, we now take the Weyl transformation of both sides. For the RHS we use [3.93]

$$\delta_{\rm W} R = -2\delta\omega R - 2\nabla^2\delta\omega \qquad [5.78]$$

Near the flat worldsheet, $R = 0$, and using $\nabla^2 = 4\partial\bar\partial$ this gives for the Weyl transformation of the RHS

$$\delta_{\rm W} RHS = -4\alpha\partial\bar\partial\delta\omega \qquad [5.79]$$

For the Weyl transformation of the LHS we use (2.4.12) for a general conformal transformation

$$\delta\mathcal{A}(z) = -\sum_{n=0}^{\infty}\frac{1}{n!}\partial^n v(z)\mathcal{A}^n(z) \tag{5.80}$$

where the $\mathcal{A}^n$ are the coefficients of the different poles in the OPE $T(z)A(0)$ as defined in (2.4.11)

$$T(z)\mathcal{A}(0) \sim \sum_{n=0}^{\infty}\frac{\mathcal{A}(0)}{z^{n+1}} \tag{5.81}$$

We now have for the ghost current

$$T(z)j(0) \sim \frac{4a}{z^3} + \frac{j(0)}{z^2} + \frac{\partial j(0)}{z} \tag{5.82}$$

and thus $\mathcal{A}^0 = \partial j$, $\mathcal{A}^1 = j$ and $\mathcal{A}^3 = 4a$. This gives

$$\delta LHS = \delta\partial_{\bar{z}}j(z) = \partial_{\bar{z}}\delta_{\mathrm{W}}j(z) = \partial_{\bar{z}}\left(-v\partial j - \partial j v - \frac{1}{2}4a\partial^2 v\right) \tag{5.83}$$

The first two terms corresponds to a change in coordinates $\delta z = v$ and can be ignored. A Weyl transformation has $\partial v = 2\delta\omega$, giving

$$\delta_{\mathrm{W}}LHS = -4a\bar{\partial}\partial\delta\omega \tag{5.84}$$

Requiring this to be the same as the Weyl transformation of the RHS in [5.79] gives

$$-4\alpha\partial\bar{\partial}\delta\omega = -4a\bar{\partial}\partial\delta\omega \quad \Rightarrow a = \alpha \tag{5.85}$$

We thus see that the ghost current anomaly is indeed to the form $\delta_a j^a = \alpha R = aR$, where $4a$ is the numerator of the third order pole in the OPE $T(z)j(0)$.

5.22 p 159: Eq (5.3.20) The Riemann-Roch Theorem, I

Just as for any anomaly equation, the ghost number anomaly equation is an operator equation, i.e. it is valid as a path integral equation. It describes the non-conservation of the ghost number symmetry in the path integral. The result is non-conservation of the ghost current

$$\delta_{\mathrm{g\#}}\int[d\phi]e^{-S} \propto \varepsilon\int d^2\sigma\,\sqrt{g}\,\nabla_a j^a \tag{5.86}$$

We can use the expression for the ghost number anomaly (5.3.19) in this

$$\delta_{\mathrm{g\#}}\int[d\phi]e^{-S} \propto \varepsilon\int d^2\sigma\,\sqrt{g}\,\frac{1-2\lambda}{4}R = -3\alpha\pi\varepsilon\times\frac{1}{4\pi}\int = d^2\sigma\,\sqrt{g}\,R = -3\alpha\pi\varepsilon\chi \tag{5.87}$$

We have used $\lambda = 2$, the actual value of the ghosts and have introduced the Euler number.

5.23 p 159: Eq (5.3.21) The Riemann-Roch Theorem, II

The ghost number symmetry transforms the ghosts as $\delta b = -i\varepsilon b$ and $\delta c = i\varepsilon c$, see (2.5.14). The ghost number current just counts the number of c ghosts minus the number of b ghosts. Let us now look at the expression for a general matrix element (5.3.9)

$$S_{j_1\cdots j_n}(k_1,\cdots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \frac{1}{n_R}\int_F d^\mu t \int [d\phi\, db\, dc]\, \exp\left(-S_m - S_g - \lambda\chi\right)$$
$$\times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_{t^k}\hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)}\, \mathcal{V}_{j_i}(k_i,\sigma_i) \qquad [5.88]$$

We number of ghost fields b is the number of moduli μ and the number of ghost fields c is the number of conformal Killing vectors κ. We thus have

$$i\varepsilon(\kappa - \mu) \propto \varepsilon\chi \qquad [5.89]$$

The proportionality constant should be the same for any Riemann surface, so we can find it by considering any example. For a close oriented surface we have $\chi = 2 - 2g$. The torus has one handle and so has $\chi = 0$. This certainly agrees because for the torus $\mu = \kappa = 2$, see (5.2.8), but it doesn't allow us to determine the proportionality constant. For the sphere, we will see in the next chapter in (6.1.5) that there are no moduli, but six conformal Killing vectors. The sphere has zero handles and thus $\chi = 2$. Therefore, the sphere tells us that $\kappa - \mu = \tilde{\alpha}\chi \Rightarrow 6 - 0 = 2\alpha \Rightarrow \alpha = 3$ which leads to the Riemann-Roch theorem (5.2.9)

$$\kappa - \mu = 3\chi \qquad [5.90]$$

5.24 p 160: Eq (5.4.3) Weyl Invariance of the b Insertions

Note the b_{ab} is invariant under Weyl transformations, just as c^a is. Thus

$$(b, \partial_k\hat{g}') = \int d^2\sigma\, \sqrt{\hat{g}'}\, b_{ab}\left(\partial_k\hat{g}'\right)^{ab} = \int d^2\sigma\, \sqrt{\hat{g}'}\, b_{ab}\hat{g}'^{ac}\hat{g}'^{bd}\left(\partial_k\hat{g}'\right)_{cd}$$
$$= \int d^2\sigma\, \sqrt{\hat{g}'}\, b_{ab}\hat{g}'^{ac}\hat{g}'^{bd}\partial_k\hat{g}'_{cd} \qquad [5.91]$$

Now it is just a matter of counting: $\delta\hat{g}_{ab} = +2\omega\hat{g}_{ab}$ and $\delta\hat{g}^{ab} = -2\omega\hat{g}_{ab}$. In addition $\delta\sqrt{\hat{g}} = +2\omega\sqrt{\hat{g}}$.

5.25 p 160: Eq (5.4.4) The Diffeomorphism Invariance of the S-matrix

Let us look at the generic S-matrix element (5.3.9)

$$S_{j_1\cdots j_n}(k_1,\cdots,k_n) = \sum_{\substack{\text{compact}\\\text{topologies}}} \frac{1}{n_R}\int_F d^\mu t \int [d\phi\, db\, dc]\, \exp\left(-S_m - S_g - \lambda\chi\right)$$
$$\times \prod_{(a,i)\notin f}\int d\sigma_i^a \prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_{t^k}\hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i)\prod_{i=1}^{n}\sqrt{\hat{g}(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i) \qquad [5.92]$$

and identify which terms are not manifestly invariant under a worldsheet diffeomorphism. These are the b and c-ghost insertions and the vertex operator insertions for the fixed coordinates, i.e., ignoring an overall constant, the integrand factors

$$\prod_{k=1}^{\mu}(b,\partial_{t^k}\hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{(a,i)\in f}\sqrt{\hat{g}(\sigma_i^a)}\,\mathcal{V}_{j_i}(k_i,\sigma_i^a) \qquad [5.93]$$

From (5.3.5) we have for a general transformation with parameter $\delta\sigma^a = \xi^a$ for diffeomorphisms, $\delta t^k = 0$ for moduli and $\delta\omega = 0$ for Weyl transformations

$$\delta_\xi\,(b,\partial_{t^k}\hat{g}) = (b,\partial_{t^k}\delta_\xi\,\hat{g}) = \big(b,\partial_{t^k}(-2\hat{P}_1\xi - 2\hat{\nabla}\cdot\xi)\hat{g}\big) \qquad [5.94]$$

Consider now

$$\partial_{t^k}\left(\hat{P}_1\xi + \hat{\nabla}\cdot\xi\,\hat{g}\right)_{ab} = \partial_{t^k}\left(\hat{\nabla}_a\xi_b + \hat{\nabla}_b\xi_a - g_{ab}\hat{\nabla}\cdot\xi + \hat{g}_{ab}\hat{\nabla}\cdot\xi\right)$$
$$= \partial_{t^k}\left(\hat{\nabla}_a\xi_b + \hat{\nabla}_b\xi_a\right) = \hat{\nabla}_a\partial_{t^k}\xi_b + \hat{\nabla}_b\partial_{t^k}\xi_a \qquad [5.95]$$

Thus

$$\delta_\xi\,(b,\partial_{t^k}\hat{g}) = -2(b,\hat{\nabla}_a\partial_{t^k}\xi_b + \hat{\nabla}_b\partial_{t^k}\xi_a) \qquad [5.96]$$

By the tracelessness of b_{ab} we can add a term that contains a factor $(b, f\hat{g})$ for some scalar function f. Let us choose $f = -\nabla\cdot(\partial_{t^k}\xi)$:

$$\delta_\xi\,(b,\partial_{t^k}\hat{g}) = -2(b,\hat{\nabla}_a\partial_{t^k}\xi_b + \hat{\nabla}_b\partial_{t^k}\xi_a - g_{ab}\nabla\cdot(\partial_{t^k}\xi))$$
$$= -2(b,\hat{P}_1\partial_{t^k}\xi) = -2(\hat{P}_1^T b,\partial_{t^k}\xi) \qquad [5.97]$$

In the last line we have used $(u, P_1 v) = (P_1^T u, v)$. With the ghost action (3.3.21)

$$S_g = \frac{1}{2\pi}\int d^2\sigma\sqrt{\hat{g}}\,b_{ab}(\hat{P}_1 c)^{ab} = \frac{1}{2\pi}(b,\hat{P}_1 c) = \frac{1}{2\pi}(\hat{P}_1^T b, c) = \frac{1}{2\pi}\int d^2\sigma\sqrt{\hat{g}}\,(\hat{P}_1^T b)_a c^a \qquad [5.98]$$

We see that the equation of motion $\delta S_g/\delta c = 0$ is indeed $\hat{P}_1^T b = 0$. We know that in a general QFT the equations of motions are valid in a correlation functions except at the contact terms. These are the so-called Schwinger-Dyson equations, see e.g. section 6.9 "Symmetries in the Functional Formalism" in Peskin & Schroeder. A review of that section is given in the appendix of this chapter. For a scalar field $\varphi(x)$ the Schwinger-Dyson equation is given in [5.139]

$$\left(\frac{\delta}{\delta\varphi(x)}\int d^4y\,\mathcal{L}[\varphi(y)]\right)\langle\varphi(x_1)\cdots\varphi(x_n)\rangle = \sum_{i=1}^{n}\langle\varphi(x_1)\cdots(-i\delta(x-x_i))\cdots\varphi(x_n)\rangle \tag{5.99}$$

As explained in the appendix, the delta functions are actually the inverse propagators. If we translate this to our string S-matrix then we see that the equations of motions for the b field are satisfied except for the contact terms. To identify these contact terms, we look again at (5.3.9). The contact terms will come from those factor in the second line that have a non-vanishing propagator with b. The only factors in the second line that can give a contact term are thus the c-ghost insertions at the fixed points $\prod_{(a,i)\in f} c^a(\hat{\sigma}_i)$.

I don't understand the rest of the paragraph. What does Joe mean when he says that these contact terms are precisely what is needed for the effect of diffs on the fixed vertex operators to cancel?

5.26 p 161: Eq (5.4.5) The BRST Variation of a Vertex Operator

A vertex operator $\mathcal{V}$ creates a highest weight one state ($L_0^m = 1$), so it is a primary field of dimension one. Therefore

$$\begin{aligned}\delta_B\,\mathcal{V}(w) &= i\varepsilon[Q_B,\mathcal{V}(w)] = i\varepsilon\oint_{C_w} j_B(z)\mathcal{V}(w)\\ &= i\varepsilon\oint_{C_w}\left(c\,T^m + \frac{1}{2}T^g + \frac{3}{2}\partial^2 c\right)(z)\mathcal{V}(w)\end{aligned} \tag{5.100}$$

Because the vertex operator, by definition, only depends on the matter fields, only the first term in j_B contributes

$$\begin{aligned}\delta_B\,\mathcal{V}(w) &= i\varepsilon\oint_{C_w} c(z)\left[\frac{h\mathcal{V}(w)}{(z-w)^2} + \frac{\partial\mathcal{V}(w)}{z-w}\right]\\ &= i\varepsilon\oint_{C_w}\left[\frac{c\,\mathcal{V}(w)}{(z-w)^2} + \frac{\partial c\mathcal{V}(z) + c\,\partial\mathcal{V}(w)}{z-w}\right]\\ &= i\varepsilon\partial(c\,\mathcal{V})(w)\end{aligned} \tag{5.101}$$

(5.4.5) is just this expression in σ coordinates.

5.27 p 161: Eq (5.4.6) The BRST variation of the *b*-Ghost Insertion

From (4.3.1b) we have $\delta_B b = i\varepsilon(T^m + T^g)$. Recalling that $\hat{g}$ is the gauge-fixed metric so it is not affected by the BRST charge, we have

$$\delta_B(b, \partial_{t^k}\hat{g}) = (\delta_B b, \partial_{t^k}\hat{g}) = i\varepsilon(T^m + T^g, \partial_{t^k}\hat{g}) \qquad [5.102]$$

5.28 p 162: Eq (5.4.8) The *b*-Ghost Insertion as a Function of the Beltrami Differential

We have

$$\begin{aligned}\frac{1}{2\pi}(b,\mu_k) &= \frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,b_{ab}\,\mu_k^{ab} = \frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,b_{ab}\,\hat{g}^{ac}\mu_{kc}^{\ \ b} \\ &= \frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,b_{ab}\,\hat{g}^{ac}\frac{1}{2}\hat{g}^{bd}\partial_k\hat{g}_{cd} = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,b^{cd}\,\partial_k\hat{g}_{cd} \\ &= \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,b_{cd}\,\partial_k\hat{g}^{cd} = \frac{1}{4\pi}(b,\partial_k\hat{g}) \end{aligned} \qquad [5.103]$$

which is the b insertion. In complex coordinates in the conformal gauge this becomes

$$\begin{aligned}\frac{1}{2\pi}(b,\mu_k) &= \frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,b_{ab}\,\hat{g}^{ac}\mu_{kc}^{\ \ b} \\ &= \frac{1}{2\pi}\int \frac{1}{2}d^2z\,(b_{zz}\hat{g}^{z\bar{z}}\mu_{k\bar{z}}^{\ \ z} + b_{\bar{z}\bar{z}}\hat{g}^{z\bar{z}z}\mu_{kz}^{\ \ \bar{z}}) \\ &= \frac{1}{2\pi}\int d^2z\,(b_{zz}\mu_{k\bar{z}}^{\ \ z} + b_{\bar{z}\bar{z}}\mu_{kz}^{\ \ \bar{z}}) \end{aligned} \qquad [5.104]$$

5.29 p 162: Eq (5.4.10) The Metric under a Change of Moduli

We keep the coordinate system fixed, but change the moduli by an infinitesimal amount $t^k \to t^k + \delta t^k$. We start with a metric that is, using a Weyl transformation of the form $g \propto dzd\bar{z}$, where we have dropped the superscript ^{m} referring to the patch, as we will be working in the same patch. We need to be careful. Whilst the metric is off-diagonal in the original moduli t, i.e. $g_{zz}(t) = g_{\bar{z}\bar{z}}(t) = 0$, this is no necessarily the case when we deform the moduli, i.e. $g_{z'z'}(t+\delta t)$ and $g_{\bar{z}'\bar{z}'}(t+\delta t)$ do not necessarily vanish. Note that we have added a $'$ to the complex indices. Indeed, we may be talking about the same point P in the patch but they do not necessarily have the same coordinates before and after the moduli transformation. We now have

$$\begin{aligned} g &\propto 2g_{z\bar{z}}(t)dzd\bar{z} + g_{zz}(t)dzdz + g_{\bar{z}\bar{z}}(t)d\bar{z}d\bar{z} \\ &\to 2g_{z\bar{z}}(t+\delta t)dzd\bar{z} + g_{zz}(t+\delta t)dzdz + g_{\bar{z}\bar{z}}(t+\delta t)d\bar{z}d\bar{z} \\ &= 2\big[g_{z\bar{z}}(t) + \partial_k g_{z\bar{z}}(t)\delta t^k\big]dzd\bar{z} + \big[g_{zz}(t) + \partial_k g_{zz}(t)\delta t^k\big]dzdz + \big[g_{\bar{z}\bar{z}}(t) + \partial_k g_{\bar{z}\bar{z}}(t)\delta t^k\big]d\bar{z}d\bar{z} \\ &= \big[1 + 2\partial_k g_{z\bar{z}}(t)\delta t^k\big]dzd\bar{z} + \delta t^k\big[\partial_k g_{zz}(t)dzdz + \partial_k g_{\bar{z}\bar{z}}(t)d\bar{z}d\bar{z}\big] \\ &= dzd\bar{z} + \delta t^k\big[\partial_k g_{zz}(t)dzdz + \partial_k g_{\bar{z}\bar{z}}(t)d\bar{z}d\bar{z}\big] \end{aligned} \qquad [5.105]$$

where we have kept the lowest order in each metric component. Let us now look at the Beltrami differentials. Using the definition (5.4.7) we have

$$\begin{aligned} \mu_{k\bar{z}}^{\;z} &= \frac{1}{2}\hat{g}^{z\bar{z}}\partial_k\hat{g}_{\bar{z}\bar{z}} = \partial_k\hat{g}_{\bar{z}\bar{z}} \\ \mu_{kz}^{\;\bar{z}} &= \frac{1}{2}\hat{g}^{\bar{z}z}\partial_k\hat{g}_{zz} = \partial_k\hat{g}_{zz} \end{aligned} \qquad [5.106]$$

and thus

$$g \propto dzd\bar{z} + \delta t^k\big[\mu_{kz}^{\;\bar{z}}dzdz + \mu_{k\bar{z}}^{\;z}d\bar{z}d\bar{z}\big] \qquad [5.107]$$

which is (5.4.10) taking into account the correction on Joe's errata page.

5.30 p 162: Eq (5.4.11) The Infinitesimal Version of the Beltrami Equations

We abstain from writing the indices ^{m} denoting the patch we are working in. From (5.4.9), $z' = z + \delta t^k v_k^z$, we have

$$dz' = dz + \delta t^k(\partial_z v_k^z dz + \partial\bar{z} v_k^z d\bar{z}) \qquad [5.108]$$

and thus

$$\begin{aligned} dzd\bar{z}' &= \Big[dz + \delta t^k(\partial_z v_k^z dz + \partial_{\bar{z}} v_k^z d\bar{z})\Big]\Big[d\bar{z} + \delta t^k(\partial_{\bar{z}} v_k^{\bar{z}} d\bar{z} + \partial_z v_k^{\bar{z}} dz)\Big] \\ &= dzd\bar{z} + \delta t^k(\partial_{\bar{z}} v_k^{\bar{z}} dzd\bar{z} + \partial_z v_k^{\bar{z}} dzdz + \partial_z v_k^z dzd\bar{z} + \partial_{\bar{z}} v_k^z d\bar{z}d\bar{z}) \\ &= dzd\bar{z} + \delta t^k(\partial_z v_k^{\bar{z}} dzdz + \partial_{\bar{z}} v_k^z d\bar{z}d\bar{z}) \end{aligned} \qquad [5.109]$$

where, just as in [5.105], we have ignored the $dzd\bar{z}$ contribution. Comparing this with (5.4.10) we find

$$\mu_{kz_m}^{\;\bar{z}_m} = \partial_{z_m} v_{k_m}^{\bar{z}_m} \quad \text{and} \quad \mu_{k\bar{z}_m}^{\;z_m} = \partial_{\bar{z}_m} v_{k_m}^{z_m} \qquad [5.110]$$

5.31 p 162: Eq (5.4.12) The b-Insertion in Terms of the Transition Functions, I

Using (5.4.11) in (5.4.8) we have

$$\frac{1}{2\pi}(b,\mu_k) = \frac{1}{2\pi}\int d^2z\,(b_{zz}\partial_{\bar z}v_k^z + b_{\bar z\bar z}\partial_z v_k^{\bar z}) = \frac{1}{2\pi}\int d^2z\,[\partial_{\bar z}(b_{zz}v_k^z) + \partial_z(b_{\bar z\bar z}v_k^{\bar z})] \tag{5.111}$$

where we have used $\bar\partial b = \partial\tilde b = 0$. We now use the divergence theorem (2.1.9)

$$\int_M d^2z\,(\partial_z v^z + \partial_{\bar z}v^{\bar z}) = i\oint_{\partial M}(v^z d\bar z - v^{\bar z}dz) \tag{5.112}$$

to find

Using (5.4.11) in (5.4.8) we have

$$\frac{1}{2\pi}(b,\mu_k) = \frac{i}{2\pi}\sum_m\oint_{C_m}(b_{\bar z\bar z}v_k^{\bar z}d\bar z - b_{zz}v_k^z dz) = \frac{1}{2\pi i}\sum_m\oint_{C_m}(b_{zz}v_k^z dz - b_{\bar z\bar z}v_k^{\bar z}d\bar z) \tag{5.113}$$

Here the sum $\sum_m$ is over all patches m needed to cover the Riemann surface. Indeed de z-integration in [5.111] is over the entire Riemann surface. We can break this down a sum over the integration of the different patches covering the Riemann surface and apply the divergence theorem on each of these patches. For each such a patch m this will give a contour C_m that circles the patch counterclockwise. In the overlap between two patches the contours of the two patches will go in opposite direction and cancel (the integration can be performed in the coordinate system of any of the two patches as they are related by the transition functions). This is illustrated in the figure below.

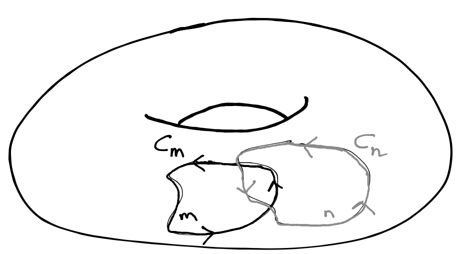


Figure 5.3: Divergence theorem on the torus

5.32 p 162: Eq (5.4.14) The Change in Transition Functions under a Change of Moduli

We evaluate the total derivative in a patch m of dz_m/dt^k in a region where two patches m and n overlap. By Leibniz we have

$$\frac{dz_m}{dt^k} = \frac{\partial z_m}{\partial t^k}\Big|_{z_n}\frac{dt^k}{dt^k} + \frac{\partial z_m}{\partial z_n}\Big|_t\frac{dz_n}{dt^k} = \frac{\partial z_m}{\partial t^k}\Big|_{z_n} + \frac{\partial z_m}{\partial z_n}\Big|_t\frac{dz_n}{dt^k} \qquad [5.114]$$

Using $dz_m/dt^k = v_{km}^{z_m}$ and $dz_n/dt^k = v_{kn}^{z_n}$ and re-arranging we find

$$\frac{\partial z_m}{\partial t^k}\Big|_{z_n} = v_{km}^{z_m} - \frac{\partial z_m}{\partial z_n}\Big|_t v_{kn}^{z_n} \qquad [5.115]$$

But for a general vector v^z going to another coordinate system $z \to w$, whilst holding the moduli fixed, the transformation rule is precisely $v^w = \partial w/\partial z/v^z$ and thus we can write

$$\frac{\partial z_m}{\partial t^k}\Big|_{z_n} = v_{km}^{z_m} - v_{kn}^{z_m} \qquad [5.116]$$

5.33 p 162: Eq (5.4.15) The *b*-Insertion in Terms of the Transition Functions, II

Let us consider (5.4.12) in a region where we have two overlapping patches m and n with contours C_m and C_n, both counterclockwise, as shown in Fig. 5.4. We split the contours C_m and C_n in a part outside of the overlapping region, $C_m^{[0]}$ and $C_n^{[0]}$ and in a part in the overlapping region $C_m^{[\text{over}]}$ and $C_n^{[\text{over}]}$.

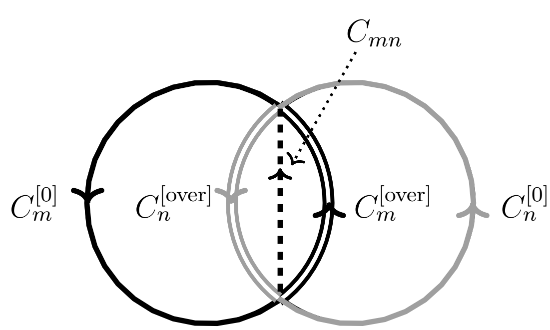


Figure 5.4: Contour integration encircling two patches of a Riemann surface

Restricting ourselves to the contour around these two patches, (5.4.12) becomes

$$\frac{1}{2\pi}(b,\mu_k) = \frac{1}{2\pi}\Bigg(\int_{C_m^{[0]}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_m^{[\text{over}]}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_n^{[0]}} dz_n v_{kn}^{z_n} b_{z_n z_n} + \int_{C_n^{[\text{over}]}} dz_n v_{kn}^{z_n} b_{z_n z_n} + \cdots \Bigg) \quad [5.117]$$

The dots include the anti-holomorphic part, that we will add at the end, and also the contour integrations encircling all other patches of the Riemann surface. What happens with the integration in the overlapping part? First we deform the contours so that to $C_{mn} = C_m^{[\text{over}]} = -C_n^{[\text{over}]}$. This induces a sign change in the integration along $C_n^{[\text{over}]}$ and so we have

$$\frac{1}{2\pi}(b,\mu_k) = \frac{1}{2\pi}\Bigg(\int_{C_m^{[0]}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_{mn}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_n^{[0]}} dz_n v_{kn}^{z_n} b_{z_n z_n} - \int_{C_{mn}} dz_n v_{kn}^{z_n} b_{z_n z_n} + \cdots \Bigg) \quad [5.118]$$

Note that z_n is just an integration variable in the last integral, so we have

$$\frac{1}{2\pi}(b,\mu_k) = \frac{1}{2\pi}\Bigg(\int_{C_m^{[0]}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_n^{[0]}} dz_n v_{kn}^{z_n} b_{z_n z_n} + \int_{C_{mn}} dz_m (v_{km}^{z_m} - v_{kn}^{z_m}) b_{z_m z_m} + \cdots \Bigg)$$
$$\frac{1}{2\pi}\Bigg(\int_{C_m^{[0]}} dz_m v_{km}^{z_m} b_{z_m z_m} + \int_{C_n^{[0]}} dz_n v_{kn}^{z_n} b_{z_n z_n} + \int_{C_{mn}} dz_m \frac{\partial z_m}{\partial t^k}\bigg|_{z_n} b_{z_m z_m} + \cdots \Bigg) \quad [5.119]$$

where we have used (5.4.14). ow that we know how to treat the overlap between two patches we can complete the analysis. The curves $C_m^{[0}]$ and $C_n^{[0}]$ will on a compact Riemann surface also have overlap with other patches and we can apply the same reasoning. We thus have a sum of terms of the form $\int_{C_{mn}} dz_m (\partial z_m)/(\partial t^k)\big|_{z_n} b_{z_m z_m}$ for all possible overlaps. Adding then the anti-holomorphic part we get

$$\frac{1}{2\pi}(b,\mu_k) = \frac{1}{2\pi}\sum_{mn}\int_{C_{mn}} \left(dz_m \frac{\partial z_m}{\partial t^k}\bigg|_{z_n} b_{z_m z_m} - d\bar{z}_m \frac{\partial \bar{z}_m}{\partial t^k}\bigg|_{z_n} b_{\bar{z}_m \bar{z}_m} \right) \quad [5.120]$$

where the sum is over all overlapping boundaries C_{mn} of all the patches m and n.

5.34 p 164: Eq (5.4.18) Simplifying the b-Ghost Insertions

Adding a vertex operator on a closed string amounts to adding a hole on the Riemann surface, see the discussion in section 3.5 for a reminder. The location of the vertex operator is actually a moduli of the Riemann surface. Indeed, a vertex operator $V(z)$ located at the point z_V on a Riemann surface is not equivalent under a diffeomorphism with a vertex operator located at a point z'_V. We thus need to treat the location of the vertex operator as a modulus, or more exactly, since z_V is a complex number, we have two moduli, i.e. $\mu = 2$. The transition function between two patches is as per (5.4.16). But let us take some time to explain this in more detail.

The z' coordinate system corresponds to an m-patch of the previous discussion. There the vertex operator is located at $z'_V = 0$. The z coordinate system is the n-patch of the previous discussion. In that reference frame the vertex operator is located at z_V. The two patches are now related by a simple transition function

$$z = z' + z_V \qquad [5.121]$$

Indeed, the point $z = z_V$ for the location of the vertex operator in the z-patch now corresponds to the point $z'_V = 0$ in the z'-patch.

In order to apply (5.4.15) we need the derivative $\partial z_m/\partial t^k\big|_{z_n}$ and its holomorphic equivalent for both moduli. The moduli are now z_V and $\bar{z}_V$ and the m-patch is the z'-patch. Thus we need

$$\begin{aligned}
\frac{\partial z_m}{\partial t^1}\Big|_{z_n} &\equiv \frac{\partial z'}{\partial z_V}\Big|_z = -1 \quad ; \qquad \frac{\partial \bar{z}_m}{\partial t^1}\Big|_{z_n} \equiv \frac{\partial \bar{z}'}{\partial z_V}\Big|_z = 0 \\
\frac{\partial z_m}{\partial t^2}\Big|_{z_n} &\equiv \frac{\partial z'}{\partial \bar{z}_V}\Big|_z = 0 \quad ; \qquad \frac{\partial \bar{z}_m}{\partial t^2}\Big|_{z_n} \equiv \frac{\partial \bar{z}'}{\partial \bar{z}_V}\Big|_z = -1
\end{aligned} \qquad [5.122]$$

and the b-ghost insertions thus become

$$\begin{aligned}
\prod_{k=1}^{\mu} \frac{1}{2\pi}(b, \mu_k) &= \left(\frac{1}{2\pi i} \int_C \left[dz'(-1) b_{z'z'} - d\bar{z}'(0) b_{\bar{z}'\bar{z}'} \right] \right) \\
&\quad \times \left(\frac{1}{2\pi i} \int_C \left[dz'(0) b_{z'z'} - d\bar{z}'(-1) b_{\bar{z}'\bar{z}'} \right] \right) \\
&= -\frac{1}{2\pi i} \int_C dz'\, b_{z'z'} \frac{1}{2\pi i} \int_C d\bar{z}'\, b_{\bar{z}'\bar{z}'} = b_{-1}\tilde{b}_{-1}
\end{aligned} \qquad [5.123]$$

The contour C encircles the vertex operator and lives in the overlap between the z' and z patch. In the last line we have used the mode expansion for a dimension two field we have $b_\ell = \oint (dz/2\pi i) z^{\ell+1} b(z)$.

5.35 Appendix: The Schwinger Dyson Equations in a QFT

Consider a single real free scalar field $\phi(x)$. The classical field satisfies the Klein-Gordon equation

$$(\partial_x^2 + m^2)\phi(x) = 0 \qquad \text{classical field} \tag{5.124}$$

Is there an equivalent equation of motion for the quantum field $\phi(x)$ when it appears in a correlation function:

$$(\partial_x^2 + m^2)\,\langle\Omega|\,\mathbf{T}\left\{\phi(x)\phi(x_1)\cdots\phi(x_n)\right\}|\Omega\rangle = \,? \qquad \text{quantum field} \tag{5.125}$$

To find the answer to this, let us follow a similar procedure as in classical mechanics. There, the equations of motion are found by requiring that the action is stationary under infinitesimal variations

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \epsilon(x) \tag{5.126}$$

The extension to QFT is to consider this as an infinitesimal change of the field and to consider this in the functional integral. This is essentially a change of integration variables in the path integral. E.g. for a three-point function this leads to the trivial equation

$$\int \mathcal{D}\phi\; e^{i\int d^4x\;\mathcal{L}[\phi]}\phi(x_1)\phi(x_2)\phi(x_3) = \int \mathcal{D}\phi'\; e^{i\int d^4x\;\mathcal{L}[\phi']}\phi'(x_1)\phi'(x_2)\phi'(x_3) \tag{5.127}$$

Now [5.126] is a simple translation of ϕ and so the measure should remain invariant $\mathcal{D}\phi' = \mathcal{D}\phi$ and we find the slightly less trivial equation

$$\int \mathcal{D}\phi\; e^{i\int d^4x\;\mathcal{L}[\phi]}\phi(x_1)\phi(x_2)\phi(x_3) = \int \mathcal{D}\phi\; e^{i\int d^4x\;\mathcal{L}[\phi']}\phi'(x_1)\phi'(x_2)\phi'(x_3) \tag{5.128}$$

Let us now expand the RHS of this using [5.126] to order ϵ. Consider first the product of the three fields

$$\begin{aligned}\phi'(x_1)\phi'(x_2)\phi'(x_3) &= (\phi(x_1)+\epsilon(x_1))(\phi(x_2)+\epsilon(x_2))(\phi(x_3)+\epsilon(x_3))\\ &= \phi(x_1)\phi(x_2)\phi(x_3) + \epsilon(x_1)\phi(x_2)\phi(x_3)\\ &\quad + \phi(x_1)\epsilon(x_2)\phi(x_3) + \phi(x_1)\phi(x_2)\epsilon(x_3)\end{aligned} \tag{5.129}$$

Next, expand the part with the Lagrangian:

$$
\begin{aligned}
e^{i\int d^4x\,\mathcal{L}[\phi']} &= \exp i[\int d^4x\;\frac{1}{2}\partial_\mu\phi'(x)\partial^\mu\phi'(x) - \frac{1}{2}m^2\phi'(x)^2] \\
&= \exp i[\int d^4x\;\frac{1}{2}\partial_\mu(\phi(x)+\epsilon(x))\partial^\mu(\phi(x)+\epsilon(x)) - \frac{1}{2}m^2(\phi(x) \\
&\qquad + \epsilon(x))(\phi(x)+\epsilon(x))] \\
&= \exp i[\int d^4x\;\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \partial_\mu\epsilon(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2 - m^2\epsilon(x)\phi(x) \\
&= e^{i\int d^4x\,\mathcal{L}[\phi]}\exp i\int d^4x\;[\partial_\mu\epsilon(x)\partial^\mu\phi(x) - m^2\epsilon(x)\phi(x) \\
&= e^{i\int d^4x\,\mathcal{L}[\phi]}\exp i\int d^4x\;[\epsilon(x)(-\partial_x^2 - m^2)\phi(x)] \\
&= e^{i\int d^4x\,\mathcal{L}[\phi]}\{1 + i\int d^4x\;[\epsilon(x)(-\partial_x^2 - m^2)\phi(x)]\}
\end{aligned}
\tag{5.130}
$$

In the one but last line we have used partial integration. We can now bring it all together and using shorthand $\phi_x = \phi(x), \phi_1 = \phi(x_1)$ etc we find

$$
\begin{aligned}
\int \mathcal{D}\phi_x\; e^{i\int d^4x\,\mathcal{L}[\phi_x]}\phi_1\phi_2\phi_3 = \int \mathcal{D}\phi_x\; e^{i\int d^4x\,\mathcal{L}[\phi_x]}\{1 + i\int d^4x\;[\epsilon_x(-\partial_x^2 - m^2)\phi_x]\} \\
\times(\phi_1\phi_2\phi_3 + \epsilon_1\phi_2\phi_3 + \phi_1\epsilon_2\phi_3 + \phi_1\phi_2\epsilon_3)
\end{aligned}
\tag{5.131}
$$

or keeping the first order term in ϵ

$$
\begin{aligned}
0 = \int \mathcal{D}\phi_x\; e^{i\int d^4x\,\mathcal{L}[\phi_x]}\Big[i\int d^4x\;\epsilon_x(-\partial_x^2 - m^2)\phi_x\phi_1\phi_2\phi_3 \\
+ \epsilon_1\phi_2\phi_3 + \phi_1\epsilon_2\phi_3 + \phi_1\phi_2\epsilon_3\Big]
\end{aligned}
\tag{5.132}
$$

Now we can write $\epsilon(x_1) = \int d^4x\;\;\epsilon(x)\delta(x - x_1)$ or in shorthand $\epsilon_1 = \int d^4x\;\;\epsilon_x\delta_{x1}$ and similarly for ϵ_2 and ϵ_3 and rewrite this as

$$
\begin{aligned}
0 = \int \mathcal{D}\phi_x\; e^{i\int d^4x\,\mathcal{L}[\phi_x]}\int d^4x\;\epsilon_x\Big[-i(\partial_x^2 + m^2)\phi_x\phi_1\phi_2\phi_3 \\
+ \delta_{x1}\phi_2\phi_3 + \phi_1\delta_{x2}\phi_3 + \phi_1\phi_2\delta_{x3}\Big]
\end{aligned}
\tag{5.133}
$$

This should be independent of ϵ and so we find

$$
0 = \int \mathcal{D}\phi_x\; e^{i\int d^4x\,\mathcal{L}[\phi_x]}\left[(\partial_x^2 + m^2)\phi_x\phi_1\phi_2\phi_3 + i\delta_{x1}\phi_2\phi_3 + i\phi_1\delta_{x2}\phi_3 + i\phi_1\phi_2\delta_{x3}\right] \tag{5.134}
$$

or written down explicitly

$$
\begin{aligned}
(\partial_x^2 + m^2) \langle \Omega | \mathbf{T} \{\phi(x)\phi(x_1)\phi(x_2)\phi(x_3)\} |\Omega\rangle = & \\
& + \langle \Omega | \mathbf{T}\{(-i\delta(x-x_1))\,\phi(x_2)\phi(x_3)\} |\Omega\rangle \\
& + \langle \Omega | \mathbf{T}\{\phi(x_1)\,(-i\delta(x-x_2))\,\phi(x_3)\} |\Omega\rangle \\
& + \langle \Omega | \mathbf{T}\{\phi(x_1)\phi(x_2)\,(-i\delta(x-x_3))\,t\} |\Omega\rangle
\end{aligned} \qquad [5.135]
$$

We can obviously extend this for an n-point function

$$
\begin{aligned}
(\partial_x^2 + m^2) \langle \Omega | \mathbf{T} \{\phi(x)\phi(x_1)\cdots\phi(x_n)\} |\Omega\rangle = & \\
& \sum_{i=1}^{n} \langle \Omega | \mathbf{T}\{\phi(x_1)\cdots(-i\delta(x-x_i))\cdots\phi(x_n)\} |\Omega\rangle
\end{aligned} \qquad [5.136]
$$

So if taken in a correlation function, a field $\phi(x)$ satisfies the Klein-Gordon equation, except at a number of discrete points that coincide with the location of the other fields in the correlation function. These terms on the RHS where the equation of motion is not valid in a correlation function are called contact terms.

Eq.[5.136] shouldn't really come as a surprise. Let us first consider the special case of one field $\phi(x_1)$

$$
(\partial_x^2 + m^2) \langle \Omega | \mathbf{T} \{\phi(x)\phi(x_1)\} |\Omega\rangle = \langle \Omega | \mathbf{T}\,(-i\delta(x-x_1))\} |\Omega\rangle = -i\delta(x-x_1) \qquad [5.137]
$$

So this is nothing else than a rewriting of the definition of the propagator $D = (\partial^2 + m^2)^{-1}$. Eq.[5.136] is a straightforward generalisation where the field $\phi(x)$ is contracted with each of the fields $\phi(x_1), \cdots, \phi(x_2)$ yielding each time a delta function, which is the result of the commutator of the two fields.

This result can be easily generalised to a more general field theory with field $\varphi(x)$ and Lagrangian $\mathcal{L}[\varphi]$. The variation of the action gives the Euler-Lagrange equations

$$
\frac{\delta}{\delta\varphi(x)} \left(\int d^4y \; \mathcal{L}[\varphi(y)] \right) \equiv \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \qquad [5.138]
$$

Eq.[5.136] generalises immediately to

$$
\begin{aligned}
\left(\frac{\delta}{\delta\varphi(x)} \int d^4y \; \mathcal{L}[\varphi(y)] \right) \langle \varphi(x_1)\cdots\varphi(x_n)\rangle = & \\
& \sum_{i=1}^{n} \langle \varphi(x_1)\cdots(-i\delta(x-x_i))\cdots\varphi(x_n)\rangle
\end{aligned} \qquad [5.139]
$$

In this equation, the fields are assumed to be in time ordering, but with derivatives acting on $\varphi(x)$ are taken outside the correlation function. This is the Schwinger-Dyson equation that states that the classical equations of motion are obeyed within correlation functions, up to the contact terms where a delta function appears due to non-trivial commutation relations.

Chapter 6

Tree-Level Amplitudes

Open Questions

I have a number of unanswered points for this chapter. They are briefly mentioned here and more detail is given under the respective headings. Any help in resolving them can be sent to hepnotes@hotmail.com and is more than welcome.

- ♣ (6.2.38) I believe the expression for the Green's function on RP_2 in Joe's book contains an error. This is strange because it is not mentioned on his errata page, which, otherwise, is very complete. I believe the correct expression should be

$$G(\sigma_2, \sigma_2) = -\frac{\alpha'}{2} \ln |z_1 - z_2|^2 - \frac{\alpha'}{2} \ln \left| \frac{1}{z_1 \bar{z}_2} + 1 \right|^2$$

6.1 p 166: The Two-Sphere S_2

Some general comments on the two-sphere S_2 and how it can be covered by two patches. The standard description of S_2 is with spherical coordinates θ and φ:

$$x = \sin\varphi \cos\theta; \qquad y = \sin\varphi \sin\theta; \qquad z = \cos\varphi \tag{6.1}$$

We see that there is a problem because the points $\varphi = 0$ and $\varphi = \pi$ do not have a single value for θ. This means that we cannot have one coordinate system to describe the entire two-sphere. At this point it is convenient to introduce the stereographic projection. Draw a line from the north pole of S_2 through a point $P \equiv (x, y, z)$ on the sphere and consider the point where this line crosses the xy plane, see figure 6.1. Call this new point P' and call its coordinates in the xy plane u and v.

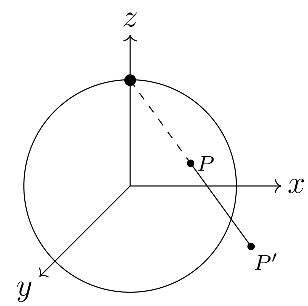


Figure 6.1: Stereographic projection for S_2. The point P on S_2 with coordinates (x, y, z) is mapped into the xy plane to the point P' with coordinates (u, v) in that plane.

Elementary geometry then relates the (u, v) coordinates of P' to the (x, y, z) coordinates of P as follows

$$x = \frac{2u}{u^2 + v^2 + 1}; \qquad y = \frac{2v}{u^2 + v^2 + 1}; \qquad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \tag{6.2}$$

or in terms of spherical coordinates

$$\cos\varphi = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}; \qquad \tan\theta = \frac{v}{u} \tag{6.3}$$

We also have

$$\sin^2\varphi = \frac{4(u^2 + v^2)}{u^2 + v^2 + 1} \tag{6.4}$$

and so we see again the problem at $\varphi = 0$ and $\varphi = \pi$ as at these points $u = v = 0$ and then $\tan\theta$ is not well-defined. The issue is easily seen to be that the point at the north pole doesn't have a stereographic projection in the xy-plane; i.e. it crosses that plane somewhere at infinity.

Introducing $z = u + iv$ we thus see that we can describe the two-sphere by the complex plane plus a point at infinity[1] In order to describe S_2 completely, the complex plane is no enough; we need to introduce a second patch. This can be done by considering the stereographic projection from the south pole. Now all point on S_2 will be mapped to the complex plane expect for the south pole. But these two patches give together a complete coordinate system for S_2.

Note that the great circle at $z = 0$ is mapped to the unit circle in the complex z plane, $|z| = 1$. Points on the upper half two-sphere are mapped to points outside the unit circle

[1]There will be no confusion between the complex coordinate z and the third coordinate z as we will not use the latter anymore.

and have $|z| > 1$. Points on the lower half two-sphere are mapped to points in the unit circle and have $|z| < 1$. We can thus take a $\rho > 1$ and take two patches: one with coordinates $|z| < \rho$ and one with coordinates $|u| < \rho$ where u is the complex coordinate obtained from the stereographic projection from the south pole. Where the patches overlap, i.e. for $1 < |z| < \rho$ and $1 < |u| < \rho$ both coordinate systems are linked by the coordinate transformation

$$u = 1/z \qquad [6.5]$$

6.2 p 167: Eqs. (6.4.5a,b) The CKVs on S_2

We require δz and δu to be well-defined in their respective coordinate patches, in particular to have no singularity at $z = 0$ and at $u = 0$ respectively. The former implies that we can expand δZ in a Taylor series

$$\delta z = \sum_{n=0}^{\infty} a_n z^n \qquad [6.6]$$

As we have (6.1.4a) that $\delta u = -z^{-2}\delta z$ we have

$$\delta u = -u^2 \sum_{n=0}^{\infty} a_n u^{-n} = \sum_{n=0}^{\infty} a_n u^{2-n} \qquad [6.7]$$

For δu to be well-defined at $u = 0$ we thus need $a_n = 0$ for $n > 3$, which means that only a_0, a_1 and a_2 can be non-zero. Therefore we have (6.1.5.a), i.e.

$$\delta z = a_0 + a_1 z + a_2 z^2 \qquad [6.8]$$

6.3 p 168: The Two-Disk D_2

Some general comments on the two-disk D_2. The two-disk can be obtained by the identifying the points z and $z' = 1/\bar{z}$ of the representation of the two-sphere S_2 in the complex plane. It identifies the upper half sphere with the lower half sphere and creates a boundary, the unit circle

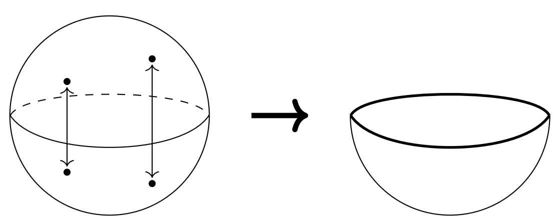


Figure 6.2: The two-disk D_2 is obtained from the two-sphere S_2 by identifying points z and $1/\bar{z}$.

In polar coordinates in the z plane, $z = re^{i\phi}$, this amounts to identifying $re^{i\phi} \equiv \frac{1}{r}e^{i\phi}$. Graphically this amounts to identifying points outside the unit circle with points inside the unit circle. The unit circle is then the boundary of the disk; it consists of the fixed points of the transformation $z \to 1/\bar{z}$, i.e. those points with $r = 1$.

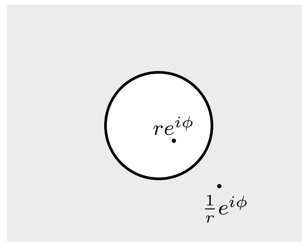


Figure 6.3: The two-disk D_2 is obtained from the two-sphere S_2 by identifying points $z = re^{i\phi}$ and $1/\bar{z} = \frac{1}{r}e^{i\phi}$ in the complex plane. The boundary of the disk is now the unit circle.

An alternative description that is often more convenient is to identify the points z and $z' = \bar{z}$. This is the same as the previous identification up to a conformal transformation. Now a point $z = re^{i\phi}$ is identified with $\bar{z} = re^{-i\phi}$ hence with the point on the opposite side of the real axis. D_2 is thus represented by the upper half complex plane $\mathbb{H} = \mathbb{C}^+$ with the real axis as the boundary of the disk.

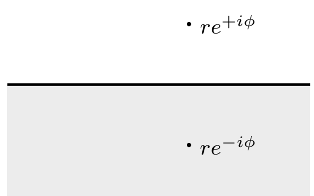


Figure 6.4: The two-disk D_2 as the upper half complex plane $\mathbb{C}^+$ by identifying points $z = re^{+i\phi}$ and $\bar{z} = re^{-i\phi}$ in the complex plane. The boundary of the disk is now the real axis.

6.4 p 168: The Two-Dimensional Projective Plane RP_2

Some general comments on two-dimensional projective plane RP_2. We start from representation of the sphere in the complex plane and identify the points $z = re^{i\phi}$ with $z' = -1/\bar{z} = -\frac{1}{r}e^{i\phi} = \frac{1}{r}e^{i(\phi+\pi)}$. Contrary to the two-disk this transformation has no fixed points[2]. RP_2 thus has no boundary.

[2]The fixed point equation is $re^{i\phi} = -\frac{1}{r}e^{i\phi}$ or hence $r^2 = -1$. But as r is a radius it has to be real and so there is no solution to this equation.

To see where a point on the two-sphere S_2 is projected by this transformation, let us take a point P_0 on the sphere. For convenience assume it is on the upper half part of the sphere (and not at the north pole). We first perform a stereographic projection to a point P_1 in the complex plane. P_1 is then transformed via $z \to -1/\bar{z}$, i.e. $r \to 1/r$ and $\phi \to \phi + \pi$, to the point P_2 in the complex plane. The inverse stereographic projection of P_2 then leads to the point P_0' on the sphere. As P_0 was on the upper half part of the sphere, P_1 lies outside of the unit circles in the complex plane and P_2 lies inside the unit circle. P_0' is then on the lower half of the sphere. The points P_0 and P_0' are identified. Note that there is no fixed points, hence no boundary.

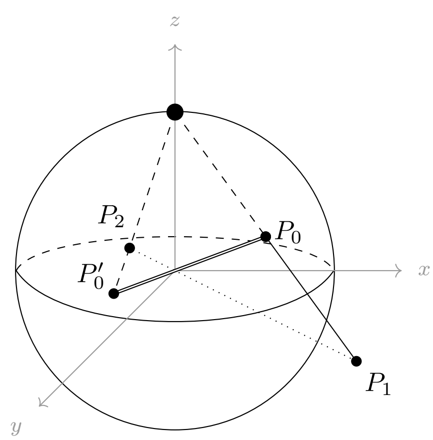


Figure 6.5: The projective plane RP_2 from the two-sphere S_2. The point P_0 on the sphere has stereographic projection to the point P_1 on the complex plane. P_1 is then transformed via $z \to -1/\bar{z}$, i.e. $r \to 1/r$ and $\phi \to \phi + \pi$, to the point P_2 in the complex plane. The inverse stereographic projection of P_2 then leads to the point P_0' on the sphere. The points P_0 and P_0' are identified. Note that there is no fixed points, hence no boundary.

6.5 p 169: Eq. (6.2.3) The Functional Integral in Terms of a Complete Set of Fields

Including the source term the action in the generating functional $Z[J]$ is

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\partial^a X^\mu \partial_a X_\mu + i\int d^2\sigma\, J^\mu X_\mu \\ &= +\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g} X^\mu \nabla^2 X_\mu + i\int d^2\sigma\, J^\mu X_\mu \end{aligned} \qquad [6.9]$$

In the second line we have performed a partial integration. We now fill in (6.2.2.a)

$$\begin{aligned} S &= +\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\sum_{I,J} x_I^\mu \mathsf{X}_I \nabla^2 x_{\mu J}\mathsf{X}_J + i\int d^2\sigma\, J^\mu \sum_I x_{\mu I}\mathsf{X}_I \\ &= -\frac{1}{4\pi\alpha'}\sum_{I,J}\omega_J^2 x_I^\mu x_{\mu J}\int d^2\sigma\,\sqrt{g}\,\mathsf{X}_I\mathsf{X}_J + i\sum_I x_{\mu I}\int d^2\sigma\, J^\mu \mathsf{X}_I \\ &= -\frac{1}{4\pi\alpha'}\sum_I \omega_I^2 x_I^\mu x_{\mu I} + i\sum_I x_{\mu I} J_I^\mu \end{aligned} \qquad [6.10]$$

which is (6.2.3) Note that the I and μ's in the measure should really not be called that, they are not the same indices in the action, but denote integration over all such values.

6.6 p 169: Eq. (6.2.5) The Zero Mode Normalisation

The zero mode X_0 is the solution to $\nabla^2\mathsf{X}_I = 0$, i.e. $\mathsf{X}_0 = c$, a constant. The constant is determined by the normalisation condition

$$1 = \int d^2\sigma\,\sqrt{g}\,\mathsf{X}_0\mathsf{X}_0 = \int d^2\sigma\,\sqrt{g}\,c^2 \qquad [6.11]$$

i.e.

$$c = \left(\int d^2\sigma\,\sqrt{g}\right)^{-1/2} \qquad [6.12]$$

6.7 p 170: Eq. (6.2.6) The Functional Integral as a Determinant

This is standard stuff in the path integral approach to QFTs. First we complete the square. For $\omega_i \neq 0$:

$$\begin{aligned} -\frac{\omega_I^2 x_I^\mu x_{I\mu}}{4\pi\alpha'} + i x_I^\mu J_{I\mu} &= -\frac{\omega_I^2}{4\pi\alpha'}\left(x_I^\mu x_{I\mu} - \frac{4\pi i\alpha'}{\omega_I^2}x_I^\mu J_{I\mu}\right) \\ &= -\frac{\omega_I^2}{4\pi\alpha'}\left[\left(x_I^\mu - \frac{2\pi i\alpha'}{\omega_I^2}J_I^\mu\right)\left(x_{I\mu} - \frac{2\pi i\alpha'}{\omega_I^2}J_{I\mu}\right) + \frac{4\pi^2\alpha'^2}{\omega_I^4}J_I^\mu J_{I\mu}\right] \\ &= -\frac{\omega_I^2}{4\pi\alpha'}y_I^\mu y_{I\mu} - \frac{\pi\alpha'}{\omega_I^2}J_I^\mu J_{I\mu} \end{aligned} \qquad [6.13]$$

Thus

$$Z[J] = \int d^d x_0\, \exp(i x_0^\sigma J_{0\sigma}) \times \left(\prod_{K\neq 0,\nu}\int dy_K^\nu\right)\exp\left(-\frac{\omega_I^2}{4\pi\alpha'}y_I^\mu y_{I\mu} - \frac{\pi\alpha'}{\omega_I^2}J_I^\mu J_{I\mu}\right) \qquad [6.14]$$

We have split out the zero mode from the other modes. We can now perform the integration. The zero mode gives a delta function and the other integrations are just Gauusioan The integration is now Gaussian and simply gives

$$Z[J] = i(2\pi)^d \delta^d(J_0) \times \prod_{I\neq 0} \left(\frac{4\pi^2\alpha'}{\omega_I^2}\right)^{d/2} \exp\left(-\frac{\pi\alpha'}{\omega_I^2} J_I^\mu J_{I\mu}\right) \qquad [6.15]$$

Using the fact that $-\omega_I^2$ are the Eigenvalues of the operator ∇^2, see (6.2.2.b), we can rewrite the product of the Eigenvalues as the determinant of the operator:

$$Z[J] = i(2\pi)^d \delta^d(J_0) \times \left(\det{}' \frac{-\nabla^2}{4\pi^2\alpha'}\right)^{-d/2} \prod_{I\neq 0} \exp\left(-\frac{\pi\alpha'}{\omega_I^2} J_I^\mu J_{I\mu}\right) \qquad [6.16]$$

The $'$ denoting that the determinant excludes the zero mode. Finally, we can write the last factor as

$$\begin{aligned}
\prod_{I\neq 0} \exp\left(-\frac{\pi\alpha'}{\omega_I^2} J_I^\mu J_{I\mu}\right) &= \exp\sum_{I\neq 0}\left(-\frac{\pi\alpha'}{\omega_I^2} \int d^2\sigma_1\, J^\mu(\sigma_1) X_I(\sigma_1) \int d^2\sigma_2\, J_\mu(\sigma_2) X_I(\sigma_2)\right) \\
&= \exp\left(-\frac{1}{2}\int d^2\sigma_1\, d^2\sigma_2\, J(\sigma_1)\cdot J(\sigma_2) \sum_{I\neq 0} \frac{2\pi\alpha'}{\omega_I^2} X_I(\sigma_1) X_I(\sigma_2)\right) \\
&= \exp\left(-\frac{1}{2}\int d^2\sigma_1\, d^2\sigma_2\, J(\sigma_1)\cdot J(\sigma_2) G'(\sigma_1,\sigma_2)\right) \qquad [6.17]
\end{aligned}$$

Bringing it all together gives (6.2.6).

6.8 p 170: Eq. (6.2.8) Green's Function PDE

The PDE for the Green's function (6.2.8) seems to have an extra term $-\mathsf{X}_0$ compared to the standard Green's function. The reason for this is that we are working on a compact surface without boundary. The "standard" Green's function PDE (the Poisson equation) $\nabla^2\phi(\sigma) = \delta^2(\sigma)$ does not have a solution on a compact surface. One way to understand this is to think about this equation defining an electric potential. Let us then look at the flux of the electric field $E_a = -\partial_a\phi$ through a closed loop around the delta function. Performing the contour integration we get the charge, but on a compact surface we can also consider the complement of the curve; that is also a closed surface but the integration result is zero as that region contains no charge. So having a single charge on a compact surface is inconsistent. A more physical explanation is that the electric field lines coming out of the charge have nowhere to end on a compact surface.

This can be resolved by adding a constant term to the PDE: $\nabla^2\phi(\sigma) = \delta^2(\sigma) - \kappa^{-1}$. This constant κ can be viewed as a constant density charge over the compact surface, that cancels the charge from the delta source. Both contour integrals, the original around the delta function and its complement, can then be made equal by a judicious choice of the constant. Working this out implies that κ is the volume of the compact surface $\int d^2\sigma\sqrt{g}$, which gives exactly (6.2.5).

Linking this directly back to Joe's book, let us consider (2.1.18):

$$0 = \eta^{\mu\nu}\left\langle\delta^2(z-z',\bar{z}-\bar{z}')\right\rangle + \frac{1}{\pi\alpha'}\partial\bar{\partial}\left\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\right\rangle \qquad [6.18]$$

This looks indeed like the equation for the Green's function. However, the way to derive this equation is by starting with the fact that the path integral of a total derivative is zero

$$0 = \int[dX]\,\frac{\delta}{\delta X_\mu(z,\bar{z})}\left[e^{-S}X^\nu(z,\bar{z}')\right] \qquad [6.19]$$

This assumes that the integral is convergent. But if the X^μ have a non-vanishing zero mode, i.e. a constant, then this integral does not converge as $X^\mu \to \pm\infty$ and so (2.1.18) is not valid.

Let us expand X in the complete set (6.2.2.a)

$$X^\mu(\sigma) = \sum_I x_I^\mu \mathsf{X}_I \qquad [6.20]$$

The path integral measure then becomes $[dX] = \prod_{I,\mu} dx_I^\mu$. If the action is Gaussian then the zero-mode has $w_0^2 = 0$ and so the integral over x_0^μ diverges. The formal way to resolve this, is by putting X into a box, i.e. giving it an upper and lower bound. Note that this is about X. The worldsheet coordinate σ already live on a compact surface.

This zero mode is actually very important: we will see that it leads to momentum conservation in amplitudes.

Note also (6.2.5) i.e. that $\mathsf{X}_0 = \left(\int d^2\sigma\sqrt{g}\right)^{-1/2}$, i.e. X_0 is inversely proportional to the square root of the total surface of the manifold. For a non-compact manifold the surface is infinite and so X_0 vanishes and we recover the standard Possion equation.

6.9 p 170: Eq. (6.2.9) Green's Function on S_2

Solving PDEs is hard, so it is better to show that the given solution satisfies the PDE. We are working with a general conformal gauge metric (6.1.2)

$$ds^2 = e^{2\omega(z,\bar{z})}dz d\bar{z} \qquad [6.21]$$

This means that the only non-zero metric components are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^{2\omega} \qquad [6.22]$$

and $g^{z\bar{z}} = g^{\bar{z}z} = 2e^{-2\omega}$. This[3] implies that $\sqrt{g} = e^{2\omega}$. Recall that g is indeed the determinant with downstairs indices, the contravariant tensor if you wish to sound intelligent. The Laplacian is then

$$\nabla^2 = g^{ab}\partial_a\partial_b = 2g^{z\bar{z}}\partial\bar{\partial} = 4e^{-2\omega}\partial\bar{\partial} = 4g^{-1/2}\partial\bar{\partial} \qquad [6.23]$$

Next we calculate

$$\nabla^2 \ln|z_{12}|^2 = 4g^{-1/2}\partial\bar{\partial}\ln|z_{12}|^2 = 8g^{-1/2}\pi\delta^2(z_{12},\bar{z}_{12}) = 4\pi g^{-1/2}\delta^2(\sigma^1-\sigma^2) \qquad [6.24]$$

where we have used (2.1.24) and the our convention that $\delta^2(z) = \frac{1}{2}\delta^2(\sigma)$. Thus

$$-\frac{1}{2\pi\alpha'}\nabla^2\left(-\frac{\alpha'}{2}\ln|z_{12}|^2\right) = \frac{g^{-1/2}}{4\pi}\times 4\pi\delta^2(\sigma^1-\sigma^2) = g^{-1/2}\delta^2(\sigma^1-\sigma^2) \qquad [6.25]$$

Next we calculate

$$\begin{aligned}\nabla^2 f(z,\bar{z}) &= \nabla^2\left(\frac{\alpha'\mathsf{X}_0^2}{4}\int d^2w\, e^{2\omega(w,\bar{w})}\ln|z-w|^2 + k\right)\\ &= \frac{\alpha'\mathsf{X}_0^2}{4}\int d^2w\, e^{2\omega(w,\bar{w})}\nabla^2\ln|z-w|^2\\ &= \frac{\alpha'\mathsf{X}_0^2}{4}\int d^2w\, e^{2\omega(w,\bar{w})}8\pi g^{-1/2}\delta^2(z_{12},\bar{z}_{12})\end{aligned} \qquad [6.26]$$

Now $\sqrt{g} = e^{2\omega}$ in the conformal gauge, so we have

$$\nabla^2 f(z,\bar{z}) = 2\pi\alpha'\mathsf{X}_0^2\int d^2w\,\delta^2(z_{12},\bar{z}_{12}) = 2\pi\alpha'\mathsf{X}_0^2\int d^2\sigma\,\delta^2(\sigma) = 2\pi\alpha'\mathsf{X}_0^2 \qquad [6.27]$$

and thus

$$-\frac{1}{2\pi\alpha'}\nabla^2 f(z,\bar{z}) = -\frac{1}{2\pi\alpha'}2\pi\alpha'\mathsf{X}_0^2 = -\mathsf{X}_0^2 \qquad [6.28]$$

We have shown that $-\frac{\alpha'}{2}\ln|z_{12}|^2 + f(z_1,\bar{z}_1)$ satisfies the PDE for the Green's function G', but we know that G' must be symmetric in the two points, so we need to symmetrise the result. This means that

$$G'(\sigma_1,\sigma_2) = -\frac{\alpha'}{2}\ln|z_{12}|^2 + f(z_1,\bar{z}_1) + f(z_2,\bar{z}_2) \qquad [6.29]$$

which is (6.2.9) and satisfies (6.2.8) indeed.

[3] To calculate the determinant we need the metric in the σ coordinates. We have $ds^2 = e^{2\omega}dzd\bar{z} = e^{2\omega}(dx+idy)(dx-idy) = e^{2\omega}(dx^2+dy^2)$ and so $g_{11} = g_{22} = e^{2\omega}$ and $g_{12} = 0$. Therefore $g = \det g = e^{4\omega}$ and $\sqrt{g} = e^{2\omega}$.

6.10 p 171: Eq. (6.2.13) From the Zero Mode to Momentum Conservation and the Renormalised Green's Function

From (6.2.4) and using the fact that the zero mode X_0 is constant, we have

$$J_0^\mu = \int d^2\sigma\, J^\mu(\sigma)\mathsf{X}_0 \int d^2\sigma\, \sum_{i=1}^n k_i^\mu \delta^2(\sigma-\sigma_i)\mathsf{X}_0 = \mathsf{X}_0 \sum_{i=1}^n k_i^\mu \tag{6.30}$$

The $\delta^d(J_0)$ in (6.2.6) then becomes

$$\delta^d(\mathsf{X}_0 \sum_{i=1}^n k_i^\mu) = \mathsf{X}_0^{-d}\delta^d(\sum_{i=1}^n k_i^\mu) \tag{6.31}$$

which gives the momentum conservation in (6.2.13) and the X_0^{-d} in $C_{S_2}^X$ in (6.2.14).

The second important point in (6.2.13) is the appearance of the renormalised Green's function. This comes from the fact that the tachyon vertex operators in (6.2.11) are renormalised. Our definition of renormalisation is given by (3.6.5)

$$[F]_r = \exp\left(\frac{1}{2}\int d^2\sigma d^2\sigma'\, \Delta(\sigma,\sigma')\frac{\delta}{\delta X^\mu(\sigma)}\frac{\delta}{\delta X_\mu(\sigma')}\right) F \tag{6.32}$$

with $\Delta(\sigma,\sigma') = \frac{\alpha'}{2}\ln d^2(\sigma,\sigma')$ and $d^2(\sigma,\sigma')$ the geodesic distance between two points. Applying this on the tachyon vertex operator we find that

$$\begin{aligned}\left[e^{ik_i\cdot X(\sigma_i)}\right]_r &= \exp\left(\frac{1}{2}\int d^2\sigma d^2\sigma'\, \Delta(\sigma,\sigma')i^2 k_i\cdot k_i\,\delta^2(\sigma-\sigma_i)\delta^2(\sigma'-\sigma_i)\right)e^{ik_i\cdot X(\sigma_i)}\\ &= \exp\left(-\frac{1}{2}\frac{\alpha' k_i^2}{2}\ln d^2(\sigma_i,\sigma_i)\right)e^{ik_i\cdot X(\sigma_i)}\end{aligned} \tag{6.33}$$

We thus find for the amplitude (6.2.11)

$$\begin{aligned}A_{S_2}^n(k,\sigma) &= \left\langle \left[e^{ik_1\cdot X(\sigma_1)}\right]_r \left[e^{ik_2\cdot X(\sigma_2)}\right]_r \cdots \left[e^{ik_n\cdot X(\sigma_n)}\right]_r\right\rangle\\ &= e^{-\frac{\alpha'}{4}\sum_{i=1}^n k_i^2 \ln d^2(\sigma_i,\sigma_i)}\left\langle e^{i\sum_{j=1}^n k_j\cdot X(\sigma_j)}\right\rangle\end{aligned} \tag{6.34}$$

We can bring this in the form of a generating functional by using (6.2.12)

$$J^\mu = \sum_{i=1}^n k_i^\mu \delta^2(\sigma-\sigma_i) \tag{6.35}$$

Indeed

$$\begin{aligned}\left\langle e^{i\int d^2\sigma\, J^\mu(\sigma)X_\mu(\sigma)}\right\rangle &= \left\langle e^{i\int d^2\sigma\, \sum_{i=1}^n k_i^\mu \delta^2(\sigma-\sigma_i)X_\mu(\sigma)}\right\rangle\\ &= \left\langle e^{i\sum_{j=1}^n k_j\cdot X(\sigma_j)}\right\rangle\end{aligned} \tag{6.36}$$

From (6.2.6) we thus have, ignoring all the pre-factors,

$$Z[J] \propto e^{-\frac{\alpha'}{4}\sum_{i=1}^{n} k_i^2 \ln d^2(\sigma_i,\sigma_i)} e^{-\frac{1}{2}\int d^2\sigma d^2\sigma' \, J^\mu(\sigma) J_\mu(\sigma') G'(\sigma,\sigma')} \quad [6.37]$$

Let us focus on the last exponential; it is

$$\begin{aligned}
&\exp -\frac{1}{2}\int d^2\sigma d^2\sigma' \sum_{i=1}^{n} k_i^\mu \delta^2(\sigma-\sigma_i) \sum_{j=1}^{n} k_{\mu\, j}\delta^2(\sigma'-\sigma_j) G'(\sigma,\sigma') \\
&= \exp -\frac{1}{2}\sum_{i,j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) \\
&= \exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i=1}^{n} k_i^2 G'(\sigma_i,\sigma_i)\right] \quad [6.38]
\end{aligned}$$

So that

$$\begin{aligned}
Z[J] &\propto \exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i=1}^{n} k_i^2 G'(\sigma_i,\sigma_i) - \frac{\alpha'}{4}\sum_{i=1}^{n} k_i^2 \ln d^2(\sigma_i,\sigma_i)\right] \\
&= \exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i=1}^{n} k_i^2 \left(G'(\sigma_i,\sigma_i) + \frac{\alpha'}{2}\ln d^2(\sigma_i,\sigma_i)\right)\right] \\
&= \exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i=1}^{n} k_i^2 G_r'(\sigma_i,\sigma_i)\right] \quad [6.39]
\end{aligned}$$

where we defined

$$G_r'(\sigma_i,\sigma_j) = G'(\sigma_i,\sigma_j) + \frac{\alpha'}{2}\ln d^2(\sigma_i,\sigma_j) \quad [6.40]$$

This gives us precisely (6.2.13) and (6.2.15) with the pre-factor of (6.2.14). The additional term in G_r' comes from the renormalisation of the tachyon vertex operator.

6.11 p 171: Eq. (6.2.16) The Renormalised Green's Function

We can now work out the renormalised Green's function, using the equation for the geodesic distance at short distance (3.6.9)

$$d^2(\sigma_1,\sigma_2) = (\sigma_1-\sigma_2)^2 e^{2\omega(\sigma)} = |z_{12}|^2 e^{2\omega(\sigma)} \quad [6.41]$$

Thus

$$\begin{aligned} G'_r(\sigma_1,\sigma_1) &= \lim_{z_2\to z_1}\left[-\frac{\alpha'}{2}\ln|z_{12}|^2 + f(z_1,\bar z_1) + f(z_2,\bar z_2) + \frac{\alpha'}{2}\ln|z_{12}|^2 e^{2\omega(\sigma)}\right] \\ &= 2f(z_1,\bar z_1) + \alpha'\omega(z,\bar z') \end{aligned} \qquad [6.42]$$

This is, not surprisingly as we have regularised the tachyon operators, a finite result.

6.12 p 171: Eq. (6.2.16) The Tachyon amplitude on S_2: Final Result

The exponential part of (6.2.13) is, writing f_i for $f(z_i,\bar z_i)$

$$\begin{aligned} &\exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j\left(-\frac{\alpha'}{2}\ln|z_{ij}|^2 + f_i + f_j\right) - \frac{1}{2}\sum_{i=1}^{n} k_i^2\left(2f_i + \alpha'\omega(z_i)\right)\right] \\ =&\exp\left(-\sum_{i<j=1}^{n} k_i\cdot k_j\left(f_i+f_j\right) - \sum_{i=1}^{n} k_i^2 f_i - \frac{\alpha'}{2}\sum_{i=1}^{n} k_i^2\omega(z_i)\right)\prod_{i<j=1}^{n}|z_{ij}|^{\alpha' k_i\cdot k_j} \end{aligned} \qquad [6.43]$$

It remains to show that the f functions drop out. It is easily seen that

$$-\sum_{i<j=1}^{n} k_i\cdot k_j\left(f_i+f_j\right) - \sum_{i=1}^{n} k_i^2 f_i = -\sum_{i=1}^{n} k_i\cdot\sum_{j=1}^{n} k_j f_j \qquad [6.44]$$

If this is not immediately clear, let us work out the case $n=3$. Ignoring the minus sign:

$$\begin{aligned} &k_1\cdot k_2(f_1+f_2) + k_1\cdot k_3(f_1+f_3) + k_2\cdot k_3(f_2+f_3) + k_1^2 f_1 + k_2^2 f_2 + k_3^2 f_3 \\ =&f_1 k_1\cdot(k_1+k_2+k_3) + f_2 k_2\cdot(k_1+k_2+k_3) + f_3 k_3\cdot(k_1+k_2+k_3) \\ =&(k_1+k_2+k_3)\cdot(k_1 f_1 + k_2 f_2 + k_3 f_3) \end{aligned} \qquad [6.45]$$

and so the f contribution vanishes by momentum conservation $\sum_{i=1}^{n} k_i = 0$, as enforced by the delta function, which itself came from the zero mode.

6.13 p 172: Eq. (6.2.19) Amplitudes for Higher Order Vertex Operators

This formula is rather confusing, so let us go slowly. We first consider one derivative.

$$\begin{aligned} A_{S_2}^{(n,1,0)} &= \left\langle \prod_{i=1}^{n}\left[e^{ik_i\cdot X(z_i,\bar z_i)}\right]_r \partial X^\mu(z_1')\right\rangle \\ &= e^{-\frac{\alpha'}{2}\sum_{i=1}^{n} k_i^2\omega(z_i)}\left\langle\prod_{i=1}^{n} e^{ik_i\cdot X(z_i,\bar z_i)}\partial X^\mu(z_1')\right\rangle \\ &= e^{-\frac{\alpha'}{2}\sum_{i=1}^{n} k_i^2\omega(z_i)}\tilde A_{S_2}^{(n,1,0)} \end{aligned} \qquad [6.46]$$

where we have taken out immediately the regularised part of the vertex operators. Let us now look at $\tilde{A}^{(n,1,0)}_{S_2} = \langle \prod_{i=1}^n e^{ik_i\cdot X(z_i,\bar{z}_i)}\partial X^\mu(z')\rangle$. We need to take all possible contractions to compute this. Start with

$$\begin{aligned}
\tilde{A}^{(1,1,0)}_{S_2} &= \left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}\partial X^\mu(z')\right\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}\left\langle \big(k\cdot X(z_1)\big)^n \partial X^\mu(z')\right\rangle \\
&= \sum_{n=1}^{\infty}\frac{i^n}{n!}\left\langle nk^\sigma X_\sigma(z_1)\partial X^\mu(z')\big(k\cdot X(z_1)\big)^{n-1}\right\rangle \\
&= ik^\sigma X_\sigma(z_1)\partial X^\mu(z')\sum_{n=1}^{\infty}\frac{i^{n-1}}{(n-1)!}\big(k\cdot X(z_1)\big)^{n-1} \\
&= +\frac{i\alpha'}{2}\frac{k_1^\mu}{z_1-z'}\left\langle e^{ik_1\cdot X(z_1)}\right\rangle = -\frac{i\alpha'}{2}\frac{k_1^\mu}{z'-z_1}\left\langle e^{ik_1\cdot X(z_1)}\right\rangle
\end{aligned} \quad [6.47]$$

From there we find that

$$\tilde{A}^{(n,1,0)}_{S_2} = -\frac{i\alpha'}{2}\sum_{j=1}^{n}\frac{k_j^\mu}{z'-z_j}\left\langle\prod_{i=1}^{n}e^{ik_i\cdot X(z_i)}\right\rangle = -\frac{i\alpha'}{2}\sum_{i=1}^{n}\frac{k_i^\mu}{z'-z_i}\prod_{i<j=1}^{n}|z_{ij}|^{\alpha' k_i\cdot k_j} \quad [6.48]$$

Let us now consider two derivatives

$$\tilde{A}^{(1,2,0)}_{S_2} = \left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}\partial X^\mu(z_1')\partial X^\mu(z_2')\right\rangle \quad [6.49]$$

We need to contract out the two derivatives. They can either each be contracted with an X in the exponential or they can be contracted with one another. We thus find

$$\begin{aligned}
\tilde{A}^{(1,2,0)}_{S_2} =& \big[ik_1^\sigma X_\sigma(z_1)\partial X^\mu(z_1')\big]\times\big[ik_1^\rho X_\rho(z_1)\partial X^\mu(z_2')\big]\left\langle e^{ik_1\cdot X(z_1)}\right\rangle \\
&+ \partial X^\mu(z_1')\partial X^\nu(z_2')\left\langle e^{ik_1\cdot X(z_1)}\right\rangle \\
=& \left[\left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z_1'-z_1}\right)\times\left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z_2'-z_1}\right) - \frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z_1'-z_2')^2}\right]\left\langle e^{ik_1\cdot X(z_1)}\right\rangle
\end{aligned} \quad [6.50]$$

Let us now look at what (6.2.19) says for this case? It gives, ignoring all the pre-factors,

$$\begin{aligned}
\tilde{A}^{(1,2,0)}_{S_2} &\propto \left\langle \big[v^\mu(z_1')+Q^\mu(z_1')\big]\times\big[v^\mu(z_2')+Q^\mu(z_2')\big]\right\rangle \\
&= v^\mu(z_1')v^\mu(z_2') + \left\langle Q^\mu(z_1')Q^\nu(z_2')\right\rangle
\end{aligned} \quad [6.51]$$

Indeed $v^\mu(z_i') = -(i\alpha'/2)k_1^\mu/(z_i'-z_1)$ is just a function and $Q^\mu(z_i')$ needs to be viewed as an field with two-point function

$$\left\langle Q^\mu(z_1')Q^\nu(z_2')\right\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z_1'-z_2')^2} \quad [6.52]$$

We cannot have mixed terms with an odd number of Q left or such a term will vanish in an expectation value. I.e. all Q's need to be contracted out. And so direct inspection shows that our $\tilde{A}^{(1,2,0)}_{S_2}$ is indeed the same as (6.2.19).

Baby steps: consider three derivatives:

$$\tilde{A}^{(1,3,0)}_{S_2} = \left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}\partial X^\mu(z'_1)\partial X^\mu(z'_2)\partial X^\sigma(z'_3)\right\rangle \qquad [6.53]$$

There are two types of terms. Either we contract each ∂X with an X from the exponential, or we contract only one ∂X with an X from the exponential and then contract the remaining two ∂X's with one another. The result is

$$\begin{aligned}\tilde{A}^{(1,3,0)}_{S_2} = \Bigg[&\left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_1}\right) \times \left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_2}\right) \times \left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_3}\right) \\ &- \left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_1}\right)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z'_2 - z'_3)^2}\right) - \left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_2}\right)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z'_1 - z'_3)^2}\right) \\ &- \left(-\frac{i\alpha'}{2}\frac{k_1^\mu}{z'_1 - z_3}\right)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z'_1 - z'_2)^2}\right)\Bigg]\left\langle e^{ik_1\cdot X(z_1)}\right\rangle\end{aligned} \qquad [6.54]$$

We can write this as

$$\begin{aligned}\tilde{A}^{(1,3,0)}_{S_2} = \Big[&v^\mu(z'_1)v^\mu(z'_2)v^\mu(z'_2) + v^\mu(z'_1)\big\langle Q^\mu(z'_2)Q^\nu(z'_3)\big\rangle \\ &+ v^\mu(z'_2)\big\langle Q^\mu(z'_1)Q^\nu(z'_3)\big\rangle + v^\mu(z'_3)\big\langle Q^\mu(z'_1)Q^\nu(z'_2)\big\rangle\Big]\left\langle e^{ik_1\cdot X(z_1)}\right\rangle\end{aligned} \qquad [6.55]$$

Which is once more (6.2.19). The pattern should now be clear. So if we have four ∂X we will obtain something of the form

$$\tilde{A}^{(1,4,0)}_{S_2} \propto v\,v\,v\,v \text{ (1 term)} + v\,v\langle QQ\rangle \text{ (6 terms)} + \langle QQ\rangle\,\langle QQ\rangle \text{ (3 terms)} \qquad [6.56]$$

again we recover (6.2.19).

Let us now turn to the case of more than one exponential factor. There must certainly be a simple proof of the formula, but we will restrict ourselves to the physicist proof and show that it is correct for two exponentials and three ∂X's after which the general pattern should emerge. So we consider

$$\tilde{A}^{(2,3,0)}_{S_2} = \left\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}e^{ik_2\cdot X(z_2,\bar{z}_2)}\partial X^\mu(z'_1)\partial X^\mu(z'_2)\partial X^\sigma(z'_3)\right\rangle \qquad [6.57]$$

We can have two types of terms: either all the ∂X's contract with the exponential, or only one of them does. Let us first focus on the terms when all partial ∂X' are contracted with an X from an exponential. We thus have three contractions. Let us denote by $[ij']$ the contraction between $X(z_i, \bar{z}_i)$ and $\partial X(z'_j)$. Three contractions thus give

$$\begin{aligned} &[11'][12'][13'] + [11'][12'][23'] + [11'][22'][13'] + [11'][22'][23'] \\ +&[21'][12'][13'] + [21'][12'][23'] + [21'][22'][13'] + [21'][22'][23'] \end{aligned} \quad [6.58]$$

But we can rewrite these 8 terms as

$$\big([11'] + [21']\big)\big([12'] + [22']\big)\big([13'] + [23']\big) \sim v(z'_1)v(z'2)v(z'3) \quad [6.59]$$

where, with some abuse of notation $v(z'_j) = -(i\alpha'/2)\sum_{i=1}^{2} k_i^\mu/(z'_j - z_i)$ which is indeed the term with three v from (6.2.19). Let us now turn to the terms with only one contraction between a ∂X and an X from the exponential. Denoting by $(i'j')$ a contraction between $\partial X(z'_)$ and $\partial X(z'_j)$ we have

$$\begin{aligned} &[11'](2'3') + [21'](2'3') + [12'](1'3') + [22'](1'3') + [13'](1'2') + [23'](1'2') \\ &\sim v(z'_1)\langle Q(z'_2)Q(z'_3)\rangle + v(z'_2)\langle Q(z'_1)Q(z'_3)\rangle + v(z'_3)\langle Q(z'_1)Q(z'_2)\rangle \end{aligned} \quad [6.60]$$

and we find this same term in (6.2.19).

From here it should be clear that the relation is valid for any number of exponentials and for any number of ∂X's. It remains to consider mixed ∂X's and $\bar{\partial} X$'s, but it should be immediately clear that this is just a duplication of the previous result. We can thus consider (6.2.19) proven.

6.14 p 172: Eqs. (6.2.21-6.2.23) How Holomorphicity can Determine Expectation Values

we know that the OPE of ∂X^μ with itself is

$$\partial X^\mu(z)\partial X^\nu(w) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} + \text{regular terms} \quad [6.61]$$

The regular terms are, by definition, holomorphic. Thus the expectation value on the two-sphere s necessarily of the form

$$\langle \partial X^\mu(z)\partial X^\nu(w)\rangle_{S_2} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}\langle \mathbb{1}\rangle_{S_2} + g(z,w) \quad [6.62]$$

where $\langle \mathbb{1}\rangle_{S_2}$ is the expectation value of the unit operator on the two-sphere and $g(z,w)$ is some holomorphic function. The coordinate z lives in the patch that covers the two-sphere without the north pole. Let us now look at the other patch that contains the north pole, but not the south pole, described by the coordinate u. The transition function between the two patches is simply $u = 1/z$. Therefore

$$\partial_u X^\mu = \frac{\partial z}{\partial u}\partial_z X^\mu = -\frac{1}{u^2}\partial_z X^\mu = -z^2\partial_z X^\mu \qquad [6.63]$$

Our field must of course be holomorphic over the entire two-sphere and thus also in the u-patch. Now, let us assume that $\partial_z X^\mu$ is not holomorphic and has a Laurent expansion $\partial_z X^\mu = \sum_{n=-\infty}^{\infty} a_n z^{-n}$. Then

$$\partial_u X^\mu = -z^2\partial_z X^\mu = -z^2\sum_{n=-\infty}^{\infty}\frac{a_n}{z^n} = -\sum_{n=-\infty}^{\infty}\frac{a_n}{z^{n-2}} = -\sum_{n=-\infty}^{\infty} a_n u^{n-2} \qquad [6.64]$$

Now $\partial_u X^\mu$ being holomorphic implies that it can have no poles at $u = 0$. This implies that $a_n = 0$ for $n - 2 < 0$. I.e. $\partial_z X^\mu$ must be of the form

$$\partial_z X^\mu = \cdots + \frac{a_3}{z^3} + \frac{a_2}{z^2} \qquad [6.65]$$

Thus indeed, as $z \to \infty$, we have that $\partial_z X^\mu \to z^{-2}$.

Let us now use this on [6.62]. Requiring that $\partial X^\mu(z)$ falls off as z^-2, whilst we keep w fixed means that $g(z, w)$ must also fall-off as z^{-2} at the least. But $g(z, w)$ must be holomorphic, so it can't have poles at $z = 0$. We thus conclude that $g(z, w)$ must be zero.

6.15 p 173: Eq. (6.2.25) The Expectation with a Level One Vertex Operator

This should be straightforward by now. We start with one exponential

$$\begin{aligned}\partial X^\mu(z) : e^{ik_1\cdot X(z_1)} &= \partial X^\mu(z)\sum_{n=0}^{\infty}\frac{i^n}{n!}\big(k_1\cdot X(z_1)\big)^n \\ &= ik_{1\,\nu}\overbrace{\partial X^\mu(z) X^\nu}(z_1)\sum_{n=1}^{\infty}\frac{i^{n-1}}{(n-1)!}\big(k_1\cdot X(z_1)\big)^{n-1} \\ &= -\frac{i\alpha'}{2}\frac{k_1^\mu}{z-z_1}e^{ik_1\cdot X(z_1)} \end{aligned} \qquad [6.66]$$

(6.2.25) follows from that immediately.

6.16 p 173: Eq. (6.2.26) Momentum Conservation in the Expectation Value

This is exactly the same argument as with the discussion around Eqs (6.2.21)-(6.2.23) but it is worth repeating it. The requirement of holomorphicity on the two-sphere implies that

∂X^μ falls as z^{-2} as z goes to infinity. This means that (6.2.25) cannot have regular terms, but it also means that it cannot have a single pole $\sim 1/z$. Hence the numerator of the single pole in (6.2.25) needs to vanish and this implies momentum conservation $\sum_i k_i^\mu = 0$.

6.17 p 174: Eq. (6.2.28) Expanding around $z \to z_1$

We are just taking (6.2.25), splitting out the $i = 1$ term and expanding the sum of $i = 2$ to n around z_1:

$$-\frac{i\alpha'}{2} A^n_{S_2}(k,\sigma) \left(\frac{k_1^\mu}{z - z_1} + \sum_{i=2}^{n} \frac{k_i^\mu}{z - z_i} \right) \qquad [6.67]$$

Now, as $z \to z_1$ for $i = 2, \cdots, n$:

$$\begin{aligned} \frac{1}{z - z_i} &= \frac{1}{z_1 + (z - z_1) - z_i} = \frac{1}{z_1 - z_i} \frac{1}{1 + (z - z_1)/(z_1 - z_i)} \\ &= \frac{1}{z_1 - z_i} \left[1 - \frac{z - z_1}{z_1 - z_i} + o\big((z - z_1)^2\big) \right] = \frac{1}{z_1 - z_i} + o(z - z_1) \end{aligned} \qquad [6.68]$$

and we find (6.2.28).

6.18 p 174: Eq. (6.2.31) The Expectation Value of Vertex Operators on S_2 from the Holomorphicity Condition

We first observe that the delta function for the momentum comes from (6.2.26). It is then easiest to check that (6.2.31) does satisfy the differential equation (6.2.30). For the holomorphic part, we need top show that

$$\partial_{z_1} \prod_{i<j=1}^{n} z_{ij}^{\alpha' k_i \cdot k_j/2} = \frac{\alpha'}{2} \prod_{i<j=1}^{n} z_{ij}^{\alpha' k_i \cdot k_j} \sum_{i=2}^{n} \frac{k_1 \cdot k_i}{z_1 - z_i} \qquad [6.69]$$

Let us show this for $n = 3$ and the general pattern should be clear:

$$\begin{aligned} &\partial_{z_1} \left(z_{12}^{\alpha' k_1 \cdot k_2/2} z_{13}^{\alpha' k_1 \cdot k_3/2} z_{23}^{\alpha' k_2 \cdot k_3/2} \right) \\ &= \frac{\alpha'}{2} k_1 \cdot k_2 \, z_{12}^{\alpha' k_1 \cdot k_2/2 - 1} z_{13}^{\alpha' k_1 \cdot k_3/2} z_{23}^{\alpha' k_2 \cdot k_3/2} + \frac{\alpha'}{2} k_1 \cdot k_3 \, z_{12}^{\alpha' k_1 \cdot k_2/2} z_{13}^{\alpha' k_1 \cdot k_3/2 - 1} z_{23}^{\alpha' k_2 \cdot k_3/2} \\ &= \frac{\alpha'}{2} \left(\frac{k_1 \cdot k_2}{z_1 - z_2} + \frac{k_1 \cdot k_3}{z_1 - z_3} \right) z_{12}^{\alpha' k_1 \cdot k_2/2} z_{13}^{\alpha' k_1 \cdot k_3/2} z_{23}^{\alpha' k_2 \cdot k_3/2} \end{aligned} \qquad [6.70]$$

which is what we needed to show.

6.19 p 174: Eq. (6.2.32) The Green's Function on the Two-Disk D_2

The two-disk D_2 is obtained from the two-sphere S_2 by identifying the points z and $\bar{z}$ and restricting z to the upper half complex plane, see (6.1.8). The Green's function should thus be the same if we replace one of the points by its conjugate. Let us transform $z_1 \to \bar{z}_1$:

$$\begin{aligned} G'(\sigma_1, \sigma_2) &= -\frac{\alpha'}{2} \ln |z_1 - z_2|^2 - \frac{\alpha}{2} \ln |z_1 - \bar{z}_2|^2 \\ &\to -\frac{\alpha'}{2} \ln |\bar{z}_1 - z_2|^2 - \frac{\alpha'}{2} \ln |\bar{z}_1 - \bar{z}_2|^2 \\ &= -\frac{\alpha'}{2} \ln |z_1 - \bar{z}_2|^2 - \frac{\alpha'}{2} \ln |z_1 - z_2|^2 = G'(\sigma_1, \sigma_2) \end{aligned} \tag{6.71}$$

as $|\bar{z}_1 - z_2| = |z_1 - \bar{z}_2|$ and $|\bar{z}_1 - \bar{z}_2| = |z_1 - z_2|$. The same, of course, also holds for $z_2 \to \bar{z}_2$. Let us also check that this Green's function is consistent with the Neumann boundary conditions $\partial_\sigma X|_{\sigma=0,\pi} = 0$ for an open string. We first translate this boundary condition into complex coordinates. We have $\partial_\sigma = \partial_2 = i(\partial - \bar{\partial})$. Thus

$$\begin{aligned} \partial_{\sigma_1} G'(\sigma_1, \sigma_2) &\propto (\partial_{z_1} - \partial_{\bar{z}_1})\big(\ln |z_1 - z_2|^2 + \ln |z_1 - \bar{z}_2|^2 \big) \\ &= (\partial_{z_1} - \partial_{\bar{z}_1})\big(\ln(z_1 - z_2) + \ln(\bar{z}_1 - \bar{z}_2) + \ln(z_1 - \bar{z}_2) + \ln(\bar{z}_1 - z_2) \big) \\ &= \frac{1}{z_1 - z_2} + 0 + \frac{1}{z_1 - \bar{z}_2} + 0 - 0 - \frac{1}{\bar{z}_1 - \bar{z}_2} - 0 - \frac{1}{\bar{z}_1 - z_2} \\ &= \frac{1}{z_1 - z_2} + \frac{1}{z_1 - \bar{z}_2} - \frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{\bar{z}_1 - z_2} \end{aligned} \tag{6.72}$$

This needs to vanish at the boundaries of the open string. But these boundaries correspond to the boundaries of the disk, which is the real line where $\operatorname{Im} z = 0$ and hence $z = \bar{z} = u$ with $u \in \mathbb{R}$. Thus

$$\partial_{\sigma_1} G'(\sigma_1, \sigma_2) \propto \frac{1}{u_1 - u_2} + \frac{1}{u_1 - u_2} - \frac{1}{u_1 - u_2} - \frac{1}{u_1 - u_2} = 0 \tag{6.73}$$

Similarly we find of course that $\partial_{\sigma_2} G'(\sigma_1, \sigma_2) = 0$.

Note that D_2 is a compact surface with a boundary. It is not clear to me, and frankly neither Joe, nor Kiritsis nor Tong make it clear whether or not there is a zero mode contribution. Joe e.g. says "up to terms that drop out due to momentum conservation". So he suggests that there is a zero mode and appropriate f functions. But the fact is that it is only on compact surfaces without boundaries that the Poisson equation has no solutions and one needs to include a zero mode. So in a way this statement is a bit surprising.

6.20 p 175: Eq. (6.2.33) The Tachyon Vertex Amplitude in the Two-Disk D_2

The Green's function has an extra term $-\frac{\alpha'/2}{2}\ln|z_1-\bar{z}_2|^2$ compared to the S_2. This immediately leads to an extra factor

$$\prod_{i,j=1}^{n}|z_i-\bar{z}_j|^{\alpha' k_i\cdot k_j/2}=\prod_{i<j=1}^{n}|z_i-\bar{z}_j|^{\alpha' k_i\cdot k_j}\prod_{i=1}^{n}|z_i-\bar{z}_i|^{\alpha' k_i^2/2} \qquad [6.74]$$

which is (6.2.33).

6.21 p 175: Eq. (6.2.34) Boundary Normal Ordering

Recall the normal ordering procedure (2.1.21b)

$$:X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2):=X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)+\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2 \qquad [6.75]$$

This ensures that a two-point function of a normal ordered product is not divergent at coinciding points as it subtracts the divergence. This certainly works for the two-sphere, where the Green's function is given by (6.2.9)

$$G'(\sigma_1,\sigma_2)=-\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2+\text{finite terms} \qquad [6.76]$$

This implies that the two-point function of a normal ordered product is finite:

$$\begin{aligned}\langle :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2):\rangle &=\langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle+\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2\\ &=\eta^{\mu\nu}G'(\sigma_1,\sigma_2)+\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2=\text{finite terms}\end{aligned} \qquad [6.77]$$

However we have a problem on the two-disk D_2. Indeed at the boundary we have $z=\bar{z}$ and so

$$\begin{aligned}G'(\sigma_1,\sigma_2)\Big|_{\partial D_2}&=-\frac{\alpha'}{2}\eta^{\mu\nu}\lim_{z_1->\bar{z_1},z_2->\bar{z_2}}\left(\ln|z_1-z_2|^2+\ln|z_1-z_2|^2\right)+\text{finite terms}\\ &=-\alpha'\eta^{\mu\nu}\lim_{z_1->\bar{z_1},z_2->\bar{z_2}}\ln|z_1-z_2|^2=-\alpha'\eta^{\mu\nu}\ln(z_1-z_2)^2\end{aligned} \qquad [6.78]$$

As a result the two-point function of a normal ordered product is not finite, but has a divergence:

$$\begin{aligned}\langle: X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) :\rangle\Big|_{\partial D_2} &= \langle X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2)\rangle\Big|_{\partial D_2} + \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z_1-z_2)^2\\ &= \eta^{\mu\nu}G'(\sigma_1,\sigma_2)\Big|_{\partial D_2} + \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z_1-z_2)^2\\ &= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z_1-z_2)^2 + \text{finite terms}\end{aligned} \qquad [6.79]$$

That is why (6.2.34) introduces a boundary normal ordering when two points are on the boundary

$$\overset{\times}{\times}X^\mu(y_1)X^\nu(y_2)\overset{\times}{\times} = X^\mu(y_1)X^\nu(y_2) + 2\alpha'\eta^{\mu\nu}\ln(y_1-y_2) \qquad [6.80]$$

The two-point function of a normal ordered product on the boundary is now finite, as it should.

6.22 p 176: Eq. (6.2.38) The Green's Function on the Projective Plane RP_2

I believe the expression for the Green's function on RP_2 in Joe's book contains an error as explained here. This is strange because it is not mentioned on his errata page, which, otherwise, is very complete.

The Green's function on RP_2 is given by (6.2.38)

$$G'(\sigma_2,\sigma_2) = -\frac{\alpha'}{2}\ln|z_1-z_2|^2 - \frac{\alpha'}{2}\ln|1+z_1\bar z_2|^2 \qquad [6.81]$$

RP_2 is defined by identifying the points z and $-1/\bar z$, so both points should give the same Green's function. Let thus transform z_1:

$$\begin{aligned}G'(\sigma_2,\sigma_2) &= -\frac{\alpha'}{2}\ln(z_1-z_2)(\bar z_1-\bar z_2)(1+z_1\bar z_2)(1+\bar z_1 z_2)\\ &\to -\frac{\alpha'}{2}\ln\Big(-\frac{1}{\bar z_1}-z_2\Big)\Big(-\frac{1}{z_1}-\bar z_2\Big)\Big(1-\frac{1}{\bar z_1}\bar z_2\Big)\Big(1-\frac{1}{z_1}z_2\Big)\\ &= -\frac{\alpha'}{2}\ln\frac{1+\bar z_1 z_2}{\bar z_1}\,\frac{1+z_1\bar z_2}{z_1}\,\frac{\bar z_1-\bar z_2}{\bar z_1}\,\frac{z_1-z_2}{z_1}\\ &= -\frac{\alpha'}{2}\ln\frac{(z_1-z_2)(\bar z_1-\bar z_2)(1+z_1\bar z_2)(1+\bar z_1 z_2)}{|z_1|^4}\end{aligned} \qquad [6.82]$$

But this is not equal to G'. If there is a sign error in the second term we get

$$\begin{aligned}G'(\sigma_2,\sigma_2) &= -\frac{\alpha'}{2}\ln\frac{(z_1-z_2)(\bar z_1-\bar z_2)}{(1+z_1\bar z_2)(1+\bar z_1 z_2)}\\ &\to -\frac{\alpha'}{2}\ln\big(-\frac{1}{\bar z_1}-z_2\big)\big(-\frac{1}{z_1}-\bar z_2\big)/\big(1-\frac{1}{\bar z_1}\bar z_2\big)\big(1-\frac{1}{z_1}z_2\big)\\ &= -\frac{\alpha'}{2}\ln\frac{1+\bar z_1 z_2}{\bar z_1}\,\frac{1+z_1\bar z_2}{z_1}\Big/\frac{\bar z_1-\bar z_2}{\bar z_1}\,\frac{z_1-z_2}{z_1}\\ &= -\frac{\alpha'}{2}\ln\frac{(1+z_1\bar z_2)(1+\bar z_1 z_2)}{(z_1-z_2)(\bar z_1-\bar z_2)}\\ &= -G'(\sigma_1,\sigma_2)\end{aligned} \qquad [6.83]$$

so this is not correct either.

I am pretty sure that this is not correct. Indeed GSW have for the Green's function on $RP2$ Eq. (8.3.21):

$$G(z;z';q^2) = \eta^{\mu\nu}\left(\ln|z-z'| + \ln\left|\frac{q^2}{z\bar z'}+1\right|\right) \qquad [6.84]$$

where the points with z and $-q^2/\bar z$ are identified. In Joe's convention this would correspond to

$$G_{\mathrm{GSW}}(\sigma_2,\sigma_2) = -\frac{\alpha'}{2}\ln|z_1-z_2|^2 - \frac{\alpha'}{2}\ln\left|\frac{1}{z_1\bar z_2}+1\right|^2 \qquad [6.85]$$

Under $z_1\to -1/\bar z_1$ this transforms as

$$\begin{aligned}G_{\mathrm{GSW}}(\sigma_2,\sigma_2) &= -\frac{\alpha'}{2}\ln(z_1-z_2)(\bar z_1-\bar z_2)\left(\frac{1}{z_1\bar z_2}+1\right)\left(\frac{1}{\bar z_1 z_2}+1\right)\\ &\to -\frac{\alpha'}{2}\ln\left(-\frac{1}{\bar z_1}-z_2\right)\left(-\frac{1}{z_1}-\bar z_2\right)\left(-\frac{\bar z_1}{\bar z_2}+1\right)\left(-\frac{z_1}{z_2}+1\right)\\ &= -\frac{\alpha'}{2}\ln\frac{(1+\bar z_1 z_2)(1+z_1\bar z_2)(\bar z_1-\bar z_2)(z_1-z_2)}{\bar z_1 z_1\bar z_2 z_2}\\ &= -\frac{\alpha'}{2}\ln\left(\frac{1}{\bar z_1 z_2}+1\right)\left(\frac{1}{z_1\bar z_2}+1\right)(\bar z_1-\bar z_2)(z_1-z_2)\\ &= G_{\mathrm{GSW}}(\sigma_2,\sigma_2)\end{aligned} \qquad [6.86]$$

Both Greens functions are related:

$$G'(\sigma_1,\sigma_2) = G_{\mathrm{GSW}}(\sigma_2,\sigma_2) - \frac{\alpha'}{2}\ln|z_1 z_2|^2 \qquad [6.87]$$

6.23 p 176: Eq. (6.3.1) The Simplest Ghost Non-Vanishing Expectation Value

Recall the general expression for the S-matrix (5.3.9)

$$S_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\text{comp topos}} \int_F \frac{d^\mu t}{n_R} \int [d\phi\, db\, dc]\, e^{-S_m - S_g - \lambda\chi}$$
$$\times \prod_{(a,i)\notin f} \int d\sigma_i^\mu \prod_{k=1}^{\mu} \frac{1}{4\pi}(g, \partial_k \hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \hat{g}(\sigma_i)^{1/2} \mathcal{V}_{j_i}(k_i, \sigma_i) \qquad [6.88]$$

we refer to that equation for the explanation of all symbols, expect for what we need here. f is the set of fixed coordinates we can chose, i.e. the number of conformal Killing vectors (CKV). The integral is also over the conformal Killing group (CKG), denoted by its moduli. The sphere, disk and projective space are genus zero, and as a result have no moduli. The sphere has six CKVs, see (6.1.5 a,b) and so what remains from the FP determinant is $\prod_{(a,i)\in f} c^a(\hat{\sigma}_i)$, i.e. six c-ghost insertions, $c(z_1)c(z_2)c(z_3)\tilde{c}(z_1)\tilde{c}(z_2)\tilde{c}(z_3)$. The simplest non-vanishing expectation value we can make on the sphere is hence indeed $\langle c(z_1)c(z_2)c(z_3)\tilde{c}(z_1)\tilde{c}(z_2)\tilde{c}(z_3)\rangle_{S_2}$.

6.24 p 177: Eq. (6.3.5) The Multi-Ghost Field Amplitude

This should be rather obvious, but here we go. We consider only the holomorphic sector, the anti-holomorphic being a copy. We need three more c-ghosts than b-ghost, so that once we have contracted all possible bc pairs we are left with a non-vanishing result. So we contract p ghosts $c(z_i)$ with p ghosts $b(z'_j)$ and this gives us p factors of $1/(z_i - z_j)$ and an expectation value of three remaining c-ghosts that gives us $(z_{p+1}-z_{p+2})(z_{p+1}-z_{p+3})(z_{p+2}-z_{p+3})$. In addition we have, of course, to take all possible permutations of the p ghosts c and b and how these can be contracted.

6.25 p 177: Eq. (6.3.6) The Holomorphic Derivation for the Need for Three c-Ghost Insertions

Eq (2.5.17) shows how the ghost current transforms under a conformal transformation. For $\lambda = 2$ it gives

$$(\partial_z u) j_u(u) = j_z(z) + \frac{3}{2}\frac{\partial_z^2 u}{\partial_z u} \qquad [6.89]$$

With $u = 1/z$ we have $\partial_z u = -1/z^2 = -u^2$ and $\partial_z^2 u = 2/z^3 = 2u^3$. Moreover $dz = -du/u^2$. Thus

$$-u^2 j_u(u) = j_z(z) + \frac{3}{2}\frac{2u^3}{-u^2} = j_z(z) - 3u \tag{6.90}$$

We thus find that

$$\oint_C \frac{dz}{2\pi i} j_z(z) = \oint_C \frac{-du/u^2}{2\pi i}\left(-u^2 j_u(u) + 3u\right) = \oint_C \frac{du}{2\pi i} j_u(u) - \oint_C \frac{du}{2\pi i}\frac{3}{u} \tag{6.91}$$

The contour C is counter-clockwise in the z-patch so, it is clockwise in the u-patch. Making it counter-clockwise flips the sign and we have

$$\oint_C \frac{dz}{2\pi i} j_z(z) = -\oint_C \frac{du}{2\pi i} j_u(u) + 3 \tag{6.92}$$

which is (6.3.6) with the correction from Joe's errata page included.

6.26 p 177: Eq. (6.3.8) The Alternative Expression for the Multi-Ghost Field Amplitude

There must be a neat mathematical way to show that (6.3.8) is equivalent to (6.3.5). But why the permutations "evidently" sum up, is a mystery to me. We can however check it for a couple of cases, to convince ourselves.

Let us start with $p = 1$ and work out (6.3.5). There are four c-ghosts and one b-ghost, so there is one propagator. Let us denote by

$$(abc|d) = \frac{z_{ab} z_{ac} z_{bc}}{z_d - w_1} \tag{6.93}$$

Up to the pre-factor, (6.3.5) is then

$$\langle\,\rangle^{p=1} = (123|4) - (124|3) + (134|2) - (234|1) \tag{6.94}$$

Using Mathematica, see the code in fig.6.6, where to be equal to (6.3.8), i.e.

$$\frac{z_{12} z_{13} z_{23} z_{14} z_{24} z_{34}}{(z_1 - w_1)(z_2 - w_1)(z_3 - w_1)(z_4 - w_1)} \tag{6.95}$$

Next, we consider $p = 2$. There are nor five c-ghosts and two b-ghosts, hence we have two propagators. We define

$$(abc|de) = \frac{z_{ab} z_{ac} z_{bc}}{(z_d - w_1)(z_e - w_2)} \tag{6.96}$$

and (6.3.5) is then given by

$$\langle\ \rangle^{p=2} = (123|45) - (124|35) + (125|34) + (134|25) - (135|24) + (145|23) - (234|15) + (235|14) - (245|13) + (345|12) \qquad [6.97]$$

This isn't something one would like to work out by hand, so Mathematica comes to help once more, see fig.6.6

```
ClearAll [z, w, cc, co];
z[a_, b_] := z[a] - z[b]; w[a_, b_] := w[a] - w[b];
cc[a_, b_, c_] := (z[a] - z[b]) * (z[a] - z[c]) * (z[b] - z[c]);
co[a_, b_] := 1 / (z[a] - w[b]);
cc[a_, b_, c_, d_, e_] :=
  Simplify [Expand [cc[a, b, c] * (co[e, 1] * co[d, 2] - co[d, 1] * co[e, 2])]];

In[71]:= (* p=1 *)
p1635 = Simplify [
    cc[1, 2, 3] * co[4, 1] - cc[1, 2, 4] * co[3, 1] + cc[1, 3, 4] * co[2, 1] - cc[2, 3, 4] * co[1, 1]] /.
   {z[1] → z1, z[2] → z2, z[3] → z3, z[4] → z4, w[1] → w1, w[2] → w2};
p1638 = z[1, 2] * z[1, 3] * z[1, 4] * z[2, 3] * z[2, 4] * z[3, 4] * co[1, 1] * co[2, 1] * co[3, 1] *
    co[4, 1] /. {z[1] → z1, z[2] → z2, z[3] → z3, z[4] → z4, w[1] → w1, w[2] → w2};
Simplify [(p1635 - p1638 )]

Out[73]= 0

In[74]:= (* p=2 *)
p2635 = ( cc[1, 2, 3, 4, 5] - cc[1, 2, 4, 3, 5] +
     cc[1, 2, 5, 3, 4] + cc[1, 3, 4, 2, 5] - cc[1, 3, 5, 2, 4] + cc[1, 4, 5, 2, 3] -
     cc[2, 3, 4, 1, 5] + cc[2, 3, 5, 1, 4] - cc[2, 4, 5, 1, 3] + cc[3, 4, 5, 1, 2]) /.
   {z[1] → z1, z[2] → z2, z[3] → z3, z[4] → z4, z[5] → z5, w[1] → w1, w[2] → w2} ;
p2638 = z[1, 2] * z[1, 3] * z[1, 4] * z[1, 5] * z[2, 3] * z[2, 4] * z[2, 5] *
    z[3, 4] * z[3, 5] * z[4, 5] * w[1, 2] * co[1, 1] * co[1, 2] * co[2, 1] *
    co[2, 2] * co[3, 1] * co[3, 2] * co[4, 1] * co[4, 2] * co[5, 1] * co[5, 2] /.
   {z[1] → z1, z[2] → z2, z[3] → z3, z[4] → z4, z[5] → z5, w[1] → w1, w[2] → w2};
Simplify [(p2635 - p2638 )]

Out[76]= 0
```

Figure 6.6: Mathematica code for multi-ghost expectation value. p1635 and p1638 are the formula (6.3.5) and (6.3.8) for $p = 1$ and similarly for p2635 and p2638. We show that for both cases $p = 1, 2$ these expressions are the same.

6.27 p 179: Eq. (6.4.1) The Three Tachyon Open String Amplitude, I

Let us look at what makes out this expectation value. First, we have three open string tachyon vertex operators, given by (3.6.25), i.e. ${}^{\times}_{\times} g_o e^{ik.X}(y)^{\times}_{\times}$, with boundary normal ordering as the asymptotic states are on the boundary of the disk, i.e. on the real axis. Next we have the three c-ghost insertions corresponding to the three conformal Killing vectors of the two disk. Then we have the Euler term $e^{-\lambda\chi}$, where for the tow-disk we find, using the Riemann-Roch theorem, that $3\chi = \kappa - \mu$, with κ the number of conformal Killing vectors and μ the number of moduli. For the two-disk this becomes $3\chi = 3-0$ or hence $\chi = 1$. The three fixed coordinates can now be fixed on the real axis of the complex plane in two ways that are linked by a $PSL(2,\mathbb{R})$ transformation, depending on the cyclic order, as shown in the figure below. Bringing this all together we find (6.4.1).

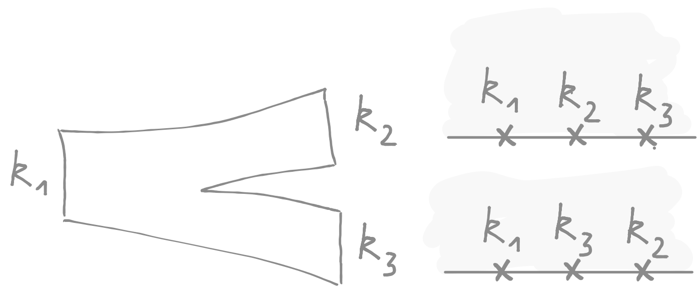


Figure 6.7: Mapping the three open string tachyon amplitude to the upper half complex plane. There are two cyclic ordering that are not related by a $PSL2(\mathbb{R})$ transformation.

6.28 p 179: Eq. (6.4.2) The Three Tachyon Open String Amplitude, II

The three c-ghost insertions are given by (6.3.11), but with $z_i = y_i$ for $i = 1, 2, 3$ real: $C^g_{D_2} y_{12} y_{13} y_{23}$. The matrix element needs to be positive, so without loss of generality we can replace y_{ij} by $|y_{ij}|$. The matter contribution was worked out in (6.2.35) and is given by $iC^X_{D_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{12}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3}$. We thus find

$$S_{D_2}(k_1;k_2;k_3) = ig_0^3 e^{-\lambda} C^g_{D_2} C^X_{D_2}(2\pi)^{26}\delta^{26}(\sum_{i=1}^{3} k_i)|y_{12}|^{1+2\alpha' k_1\cdot k_2}|y_{12}|^{1+2\alpha' k_1\cdot k_3}|y_{23}|^{1+2\alpha' k_2\cdot k_3} + \langle k_2 \leftrightarrow k_3\rangle \qquad [6.98]$$

Defining

$$C_{D_2} = e{-}\lambda C^g_{D_2} C^X_{D_2} \qquad [6.99]$$

this becomes

$$S_{D_2}(k_1; k_2; k_3) = ig_0^3 C_{D_2}(2\pi)^{26}\delta^{26}(\sum_{i=1}^{3} k_i)|y_{12}|^{1+2\alpha' k_1\cdot k_2}|y_{12}|^{1+2\alpha' k_1\cdot k_3}|y_{23}|^{1+2\alpha' k_2\cdot k_3} + \langle k_2 \leftrightarrow k_3\rangle \qquad [6.100]$$

which is (6.4.2) taking into account the correction on Joe's errata page.

6.29 p 179: Eq. (6.4.3-6.4.4) The Three Tachyon Open String Amplitude, III

Momentum conservation is $k_1 + k_2 = k_3$ which implies that $(k_1 + k_2)^2 = k_3^2$, which gives using $k_i^2 = 1/\alpha'$

$$\frac{2}{\alpha'} + 2k_1 \cdot k_2 = \frac{1}{\alpha'} \quad \Rightarrow \quad 2\alpha' k_1 \cdot k_2 = -1 \qquad [6.101]$$

And of course as well $k_1 \cdot k_3 = k_2 \cdot k_3 = -1$. This then immediately gives (6.4.4).

6.30 p 179: Eq. (6.4.5) The Four Tachyon Open String Amplitude, I

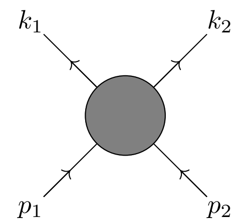
The four tachyon open string amplitude is similar as the three tachyon open string amplitude, with an extra open string vertex operator. The location of the that extra vertex operator needs to be integrated over as it cannot be fixed by the conformal Killing Group. The integration is over the real axis as it lies on the boundary of the two-disk. This immediately gives the result (6.4.5).

6.31 p 180: Eq. (6.4.7) The Mandelstam Variables

It is a standard consequence of the definition of the Mandelstam variables that their sum is equal to the sum of the mass squared of the particles. Indeed

$$s + t + u = -(k_1 + k_2)^2 - (k_1 + k_3)^2 - (k_1 + k_4)^2 \qquad [6.102]$$

We give here a short reminder of Mandelstam arises and how the above property arises. This is taken almost verbatim from my QFT Notes. In order to keep the reader sharp, I am still using the mostly negative signature that is used in these notes. Consider a scattering process of two incoming and two outgoing particles:



We introduce so-called Mandelstam Variables:

$$s = (p_1 + p_2)^2 = (k_1 + k_2)^2 \quad [6.103]$$
$$t = (k_1 - p_1)^2 = (k_2 - p_2)^2 \quad [6.104]$$
$$u = (k_2 - p_1)^2 = (k_1 - p_2)^2 \quad [6.105]$$

There is clearly some arbitrariness in the definition of t and u, but this should not bother us here. There is no ambiguity in the definition of s; it is always the sum squared of the incoming momenta.

To get a better understanding of the Mandelstam variables it is useful to work them out in a centre of mass reference frame, i.e. in a frame where the total three-momentum of the two incoming particles is zero. We also assume all incoming and outgoing particles have the same mass.

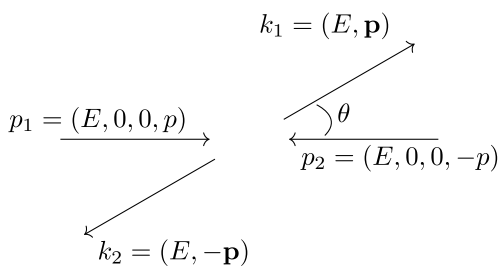


Figure 6.8: Kinematics for the Mandelstam Variables. We are using negative signature for a change.

We chose the z-axis along the line of the incoming particles and we can always select the x and y axis such that the outgoing particles are entirely in the $y - z$ plane. We then have for the incoming momenta: $p_1 = (E, 0, 0, p)$ and $p_2 = (E, 0, 0, -p)$ and for the outgoing momenta $k_1 = (E, 0, p\sin\theta, p\cos\theta)$ and $k_2 = (E, 0, -p\sin\theta, -p\cos\theta)$. The Mandelstam variables become:

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (2E)^2 = E_{\text{c.o.m.}} && [6.106] \\
t &= (k_1 - p_1)^2 = [(E, 0, p\sin\theta, p\cos\theta) - (E, 0, 0, p)]^2 \\
&= (0, 0, p\sin\theta, p(1-\cos\theta))^2 \\
&= -p^2\sin^2\theta - p^2(1-\cos\theta)^2 = -p^2(\sin^2\theta + 1 + \cos^2\theta - 2\cos\theta) \\
&= -2p^2(1-\cos\theta) && [6.107] \\
u &= (k_2 - p_1)^2 = [(E, 0, -p\sin\theta, -p\cos\theta) - (E, 0, 0, p)]^2 \\
&= (0, 0, -p\sin\theta, -p(1+\cos\theta))^2 \\
&= -p^2\sin^2\theta - p^2(1+\cos\theta)^2 = -p^2(\sin^2\theta + 1 + \cos^2\theta + 2\cos\theta) \\
&= -2p^2(1+\cos\theta) && [6.108]
\end{aligned}
$$

where $E_{\text{c.o.m.}}$ is the centre of mass energy.

We can note a few important properties of these Mandelstam variables :

- s is strictly positive, whilst t and u are negative:

$$s > 0 \quad , \quad t \leq 0 \quad \text{and} \quad u \leq 0 \qquad [6.109]$$

- $t = 0$ for $\theta = 0$ and $u = 0$ for $\theta = \pi$. If t or u appear in the denominator of an amplitude, it will blow up. This is e.g. the case in electron-muon scattering. Note that s is strictly positive and cannot blow up an amplitude. So the particles interchanged through the different channels will result in different angular dependencies of the scattering amplitudes.

- The sum of the Mandelstam variables is constant: $s + t + u = 4E^2 - 4p^2 = 4m^2$. This is a special case of a general rule when the particles do not necessarily have the same mass:

$$s + t + u = \sum_{i=1}^{4} m_i^2 \qquad [6.110]$$

We also note that in the center of mass frame $s = 4E^2$ and $t = -2p^2(1-\cos\theta)$. But as $0 = E^2 - p^2$ we have

$$s^2 - t^2 = (2E^2)^2 - (2E^2)^2(1-\cos\theta)^2 = 4E^4(1-(1-\cos\theta)^2 \qquad [6.111]$$

But as $0 \leq 1 - \cos\theta \leq 1$ we have $s^2 - t^2 \geq 0$. As $s > 0$ and $t \leq 0$ this gives us another constraint:

- s is larger or equal than the absolute value of t:

$$s \geq |t| \qquad [6.112]$$

6.32 p 180: Eq. (6.4.8) The Four Tachyon Open String Amplitude, II

Setting $y_1 = 0$, $y_2 = 1$ and $y_3 = \infty$ in (6.4.5) should only raise concerns regarding the y_3 leading to an infinity. Let us keep $y_3 = y$ as it is and consider $\lim_{y\to\infty}$. Ignoring the unimportant pre-factors (6.4.5) gives

$$\begin{aligned} S_{D_2} &\propto \lim_{y\to\infty} |(0-1)\times(0-y)\times(1-y)| \int_{-\infty}^{+\infty} dy_4\, |0-1|^{2\alpha' k_1\cdot k_2} |0-y|^{2\alpha' k_1\cdot k_3} \\ &\quad \times |0-y_4|^{2\alpha' k_1\cdot k_4} |1-y|^{2\alpha' k_2\cdot k_3} |1-y_4|^{2\alpha' k_2\cdot k_4} |y-y_4|^{2\alpha' k_3\cdot k_4} \\ &\propto \lim_{y\to\infty} y^2 y^{2\alpha' k_1\cdot k_3} y^{2\alpha' k_2\cdot k_3} y^{2\alpha' k_3\cdot k_4} \int dy_4\, |y_4|^{2\alpha' k_1\cdot k_4} |1-y_4|^{2\alpha' k_2\cdot k_4} \end{aligned} \quad [6.113]$$

We can justify replacing the $y - y_4$ by y by introducing a regulator $\int_{-\Lambda}^{+\Lambda}$ and then first take the limit of $y \to \infty$ allowing us to take it outside of the integral sign, and only then taking $\Lambda \to \infty$. We now have

$$\begin{aligned} -t = (k_1+k_3)^2 = k_1^2 + k_3^2 + 2k_1\cdot k_3 = \frac{2}{\alpha'} + 2k_1\cdot k_3 \\ \Rightarrow 2\alpha' k_1\cdot k_3 = -2 - \alpha' t \end{aligned} \quad [6.114]$$

where we have used the tachyon on-shell condition $k^2 = 1/\alpha'$. Similarly we have

$$\begin{aligned} -u = (k_1+k_4)^2 = (k_2+k_3)^2 = k_2^2 + k_3^2 + 2k_2\cdot k_3 = \frac{2}{\alpha'} + 2k_2\cdot k_3 \\ \Rightarrow 2\alpha' k_2\cdot k_3 = -2 - \alpha' u \end{aligned} \quad [6.115]$$

and

$$\begin{aligned} -s = (k_1+k_2)^2 = (k_3+k_4)^2 = k_3^2 + k_4^2 + 2k_3\cdot k_4 = \frac{2}{\alpha'} + 2k_3\cdot k_4 \\ \Rightarrow 2\alpha' k_3\cdot k_4 = -2 - \alpha' s \end{aligned} \quad [6.116]$$

Therefore

$$\begin{aligned} S_{D_2} &\propto \lim_{y\to\infty} y^{2-2-\alpha' t-2-\alpha' u-2-\alpha' s} \int dy_4\, |y_4|^{2\alpha' k_1\cdot k_4} |1-y_4|^{2\alpha' k_2\cdot k_4} \\ &\propto \lim_{y\to\infty} y^{-4-\alpha' t-\alpha' u-\alpha' s} \int dy_4\, |y_4|^{2\alpha' k_1\cdot k_4} |1-y_4|^{2\alpha' k_2\cdot k_4} \\ &\propto \lim_{y\to\infty} \int dy_4\, |y_4|^{2\alpha' k_1\cdot k_4} |1-y_4|^{2\alpha' k_2\cdot k_4} \end{aligned} \quad [6.117]$$

where in the last line we have used (6.4.7), i.e. $\alpha'(s+t+u) = -4$. The y thus has disappeared and we can take the limit and recover (6.4.8).

6.33 p 180: Eq. (6.4.9) The Four Tachyon Open String Amplitude with Mandelstam Variables

We need to bring the integrals $\int_{-\infty}^{+\infty}$ into the form$\int_0^1$. As explained in the text the integral can be split into $\int_{-\infty}^0 + \int_0^1 + \int_1^{+\infty}$ and bring it to the desired form using Möbius transformations. Recall that a general Möbius transformation is of the form

$$y' = P(y) = \frac{\alpha y + \beta}{\beta y + \delta} \qquad [6.118]$$

with α, β, γ and δ real and satisfying $|\alpha\delta - \beta\gamma| = 1$.

Start with $\int_0^1$, corresponding to Fig. 6.2 (b). This is already in the right form. We can thus write

$$I_{6.2(b)} = \int_0^1 dy\,|y|^{-\alpha' u-2}|1-y|^{-\alpha' t-2} = \int_0^1 dy\, y^{-\alpha' u-2}(1-y)^{-\alpha' t-2} \qquad [6.119]$$

It is convenient to perform a Möbius transformation $y' = P(y) = -y+1$. The Jacobian is $J = \partial y/\partial y' = -1$ and so

$$\begin{aligned} I_{6.2(b)} &= \int_1^0 (-dy')\,(1-y')^{-\alpha' u-2}(y')^{-\alpha' t-2} = \int_0^1 dy'\,(y')^{-\alpha' t-2}(1-y')^{-\alpha' u-2} \\ &= I(t,u) \end{aligned} \qquad [6.120]$$

Take now $\int_1^\infty$ transforming to the form $\int_0^1$ using a Möbius transformation. This corresponds to the vertex ordering in Fig. 6.2 (c).

We can achieve the desired integration bounds if 1 remains unchanged and ∞ becomes 0 and so 0 becomes ∞. This is obviously achieved by the Möbius transformation

$$y' = P(y) = 1/y \qquad [6.121]$$

The Jacobian of this transformation is $J = \partial y/\partial y' = -1/y'^2$ and so we can write the integral as

$$\begin{aligned} I_{6.2(c)} &= \int_1^\infty dy\,|y|^{-\alpha' u-2}|1-y|^{-\alpha' t-2} = \int_1^\infty dy\, y^{-\alpha' u-2}(y-1)^{-\alpha' t-2} \\ &= \int_1^0 -\frac{dy'}{y'^2}\left(\frac{1}{y'}\right)^{-\alpha' u-2}\left(\frac{1}{y'}-1\right)^{-\alpha' t-2} \\ &= \int_0^1 dy'\,\left(y'\right)^{-2+\alpha' u+2+\alpha' t+2}\left(1-y'\right)^{-\alpha' t-2} \\ &= \int_0^1 dy'\,\left(y'\right)^{2+\alpha' u+\alpha' t}\left(1-y'\right)^{-\alpha' t-2} \end{aligned} \qquad [6.122]$$

From (6.4.7) we see that $\alpha' u + \alpha' t = -\alpha' s - 4$ and thus

$$I_{6.2(c)} = \int_0^1 dy' \, \left(y'\right)^{2-\alpha' s-4} \left(1-y'\right)^{-\alpha' t-2} = \int_0^1 dy' \, \left(y'\right)^{-2-\alpha' s} \left(1-y'\right)^{-\alpha' t-2}$$
$$= I(s,t) \qquad [6.123]$$

Finally, consider $\int_{-\infty}^0$ into $\int_0^1$. This corresponds to the vertex ordering in Fig. 6.2. (a). The appropriate Möbius transformation is

$$y' = P(y) = \frac{1}{1-y} \qquad [6.124]$$

We see that $P(-\infty) = 0$ and $P(0) = 1$. The inverse transformation is $y = -(1-y')/y'$ and the Jacobian is $J = \partial y / \partial y' = 1/y'^2$. Moreover $1 - y = 1/y'$. Thus

$$\begin{aligned}
I_{6.2(a)} &= \int_{-\infty}^0 dy \, |y|^{-\alpha' u-2} |1-y|^{-\alpha' t-2} = \int_{-\infty}^0 dy \, (-y)^{-\alpha' u-2} (1-y)^{-\alpha' t-2} \\
&= \int_0^1 \frac{dy'}{y'^2} \left(\frac{1-y'}{y'}\right)^{-\alpha' u-2} \left(\frac{1}{y'}\right)^{-\alpha' t-2} \\
&= \int_0^1 dy' (y')^{-2+\alpha' u+2+\alpha' t+2} (1-y')^{-2-\alpha' u} \\
&= \int_0^1 dy' (y')^{2+\alpha' u+\alpha' t} (1-y')^{-2-\alpha' u} = \int_0^1 dy' (y')^{-2-\alpha' s} (1-y')^{-2-\alpha' u} \\
&= I(s,u) = I(u,s) \qquad [6.125]
\end{aligned}$$

where again we have used $\alpha' u + \alpha' t = -\alpha' s - 4$. We have also used the fact that $I(a,b) = I(b,a)$, which can be easily checked by a change of integration variables $y \to 1-y$.

The vertex orderings for Fig. 6.2 (d), (e) and (f) can be obtained from these by interchanging k_2 with k_3. This corresponds to interchanging s with t and leaving u unchanged. We thus have immediately

$$\begin{aligned}
I_{6.2(d)} &= I_{6.2(a)}(s \leftrightarrow t) = I(u,t) = I(t,u) \\
I_{6.2(e)} &= I_{6.2(b)}(s \leftrightarrow t) = I(s,u) = I(u,s) \\
I_{6.2(f)} &= I_{6.2(c)}(s \leftrightarrow t) = I(t,s) = I(s,t) \qquad [6.126]
\end{aligned}$$

Bringing the six contributions together we recover

$$2\big[I(s,t) + I(t,u) + Iu,s)\big] \qquad [6.127]$$

which gives (6.4.9).

6.34 p 181: Eq. (6.4.11) The Divergence of the Amplitude at the Intermediate Tachyon State

$I(s,t)$ has potential divergences as we approach the two integration boundaries, $y \to 0$ and $y \to 1$. The integral is symmetric under $y \leftrightarrow 1-y$ so it is sufficient to look at the divergence at only one of the integration points, say $y = 0$. In order to avoid any divergences, we first replace the upper integration boundary 1 by r, work in the region where the integrand is convergent and then take the limit of $r \to 1$. This allows us to expand the integrand around $y = 0$ and that gives the factor $1/(\alpha' s + 1)$ and so the divergence at $s = (p_1 + p_2)^2 = -1/\alpha'$ corresponding to an intermediate tachyon state.

6.35 p 182: Eq. (6.4.14) The Four-Tachyon Open String Amplitude and Factorisation

Eq. (6.4.13) is an example of factorisation: the four-string amplitude is a combination of two three-string amplitude amplitudes with an intermediate state of all possible momenta. It is also reminiscent of the BCFW recursion formula for scattering amplitudes in QFT, albeit that in the latter the intermediate momenta are complex. Let us work out (6.4.14). From (6.4.13) and (6.4.4) we have

$$\begin{aligned} S_{D_2} =& i\int \frac{d^{26}k}{(2\pi)^{26}} \frac{\left[2ig_0^3 C_{D_2}(2\pi)^{26}\delta^{26}(k_1+k_2+k)\right]\left[2ig_0^3 C_{D_2}(2\pi)^{26}\delta^{26}(-k+k_3+k_4)\right]}{-k^2+\alpha'^{-1}+i\varepsilon} \\ &+ \text{terms analytic at } \ k^2 = 1/\alpha' \\ =& -\frac{4ig_0^6 C_{D_2}^2(2\pi)^{26}\delta^{26}(k_1+k_2+k_3)}{-(k_1+k_2)^2+\alpha'^{-1}+i\varepsilon} + \text{terms analytic at } \ k^2 = 1/\alpha' \end{aligned} \tag{6.128}$$

But we also have the expression for the four tachyon amplitude (6.4.9) and we know how it behaves as $k^2 = -s = 1/\alpha'$ from (6.4.11). We get not only such a term from $I(s,t)$ but also from $I(u,s)$. Thus

$$\begin{aligned} S_{D_2} =& 2ig_0^4 C_{D_2}(2\pi)^{26}\delta^{26}(k_1+k_2+k_3+k_4) \times \left(-\frac{2}{\alpha' s+1}\right) \\ &+ \text{terms analytic at } \ k^2 = 1/\alpha' \end{aligned} \tag{6.129}$$

Equating these two expressions we find

$$\frac{g_0^2 C_{D_2}}{s+\alpha'^{-1}} = \frac{1}{\alpha' s+1} \tag{6.130}$$

or

$$C_{D_2} = \frac{1}{\alpha' g_0^2} \tag{6.131}$$

which is (6.4.14)

6.36 p 182: Eq. (6.4.17) The Pole of $I(s,t)$ at $\alpha' s = 0$

Using $(1-y)^{-\alpha' t - 2} = 1 + (\alpha' t + 2)y + o(y^2)$ we have

$$I(s,t) = \int_0^1 dy\, y^{-\alpha' s - 2} + (\alpha' t + 2)\int_0^1 dy\, y^{-\alpha' s - 1} + \cdots \tag{6.132}$$

The first integral gives a term that diverges as $\alpha' s = -1$ and is analytic as $\alpha' s = 0$. So we get, performing the integration over the second term, in oirder to identify the pole as $\alpha' s = 0$,

$$I(s,t) \sim \lim_{\alpha' s \to 0} -\frac{\alpha' t + 2}{\alpha' s} y^{-\alpha' s}\Big|_0^1 = -\lim_{\alpha' s \to 0} \frac{\alpha' t + 2}{\alpha' s} \tag{6.133}$$

Using $s + t + u = -4/\alpha'$ we can write the numerator as

$$\begin{aligned} \alpha' t + 2 &= \frac{1}{2}\alpha' t + \frac{1}{2}(\alpha' t + 4) = \frac{1}{2}\alpha' t + \frac{1}{2}(-\alpha' s - \alpha' u) = \frac{1}{2}\alpha'(t - u - s) \\ &\underset{\alpha' s \to 0}{=} \frac{1}{2}\alpha'(t - u) \end{aligned} \tag{6.134}$$

and so indeed

$$I(s,t) = \frac{u - t}{2s} + \text{terms analytic at } \alpha' s = 0 \tag{6.135}$$

6.37 p 183: Eq. (6.4.17) The Pole of the Amplitude $\alpha' s = 0$ is Actually not There

From $I(s,t) = (u-t)/2s + \cdots$ we obtain immediately from replacing t with u that $I(s,u) = (t-u)/2s + \cdots$ and so the pole at $s = 0$ cancels and is not present in the amplitude. The comment about this not being valid for the more general open string amplitudes we will study later is related to teh fact that we will add Chan-Paton factors λ^a to the string boundaries. The amplitude will then have contributions from traces of the λ^a's in different order, and so the contributions will not cancel and the amplitude will have a pole at $\alpha' s = 0$.

6.38 p 183: Eq. (6.4.22) Relating the Beta and Gamma Functions

We use the definition of the Gamma function, $\Gamma(z) = \int_0^\infty dz\, e^{-z} x^{z-1}$. Multiplying (6.4.21) by $\int_0^\infty dw\, e^{-w}$ the LHS becomes

$$LHS = \int_0^\infty dw\, e^{-w} w^{a+b-1} B(a,b) = \Gamma(a+b) B(a,b) \tag{6.136}$$

The RHS becomes

$$\begin{aligned} RHS &= \int_0^\infty dw\, e^{-w} \int_0^\infty dv\, v^{a-1}(w-v)^{b-1} \\ &= \int_0^\infty dv\, e^{-v} e^{+v} v^{a-1} \int_0^\infty dw\, e^{-w}(w-v)^{b-1} \\ &= \int_0^\infty dv\, e^{-v} v^{a-1} \int_0^\infty dw\, e^{-(w-v)}(w-v)^{b-1} \\ &= \int_0^\infty dv\, e^{-v} v^{a-1} \int_0^\infty dy\, e^{-y} y^{b-1} = \Gamma(a)\Gamma(b) \end{aligned} \quad [6.137]$$

6.39 p 183: Eq. (6.4.23) The Veneziano Amplitude

This is straightforward. Use (6.4.9) and plug in the expression for C_{D_2} and the expression of I in terms of the Beta functions (6.4.20).

6.40 p 184: Eq. (6.4.27) The Center of Mass Frame Kinematics

In the center of mass frame the incoming particles have momenta $p_1 = (p_1^0, \boldsymbol{p}_i)$ and $p_2 = (p_2^0, -\boldsymbol{p}_i)$ for some three-vector $\boldsymbol{p}$. The mass shell condition is $-m^2 = p_1^2 = -(p_1^0)^2 + \boldsymbol{p}_i^2$ and $-m^2 = p_2^2 = -(p_2^0)^2 + \boldsymbol{p}_i^2$. From this it follows that $p_1^0 = p_2^0$, which we will call E_0. The total center of mass energy is thus $E = 2E_0$.

The outgoing particles have momenta $p_3 = (p_3^0, -\boldsymbol{p}_o)$ and $p_4 = (p_4^0, \boldsymbol{p}_o)$ for some three-vector $\boldsymbol{p}_o$. The mass shell condition once more implies that $p_3^0 = p_4^0$ and energy conservation implies that $p_1^0 + p_2^0 + p_3^0 + p_4^0$ or hence $p_3^0 = p_4^0 = -E_0$. Let us also call θ the angle between the three-momenta of particle one and particle three. All this is summarised in fig. 6.9.

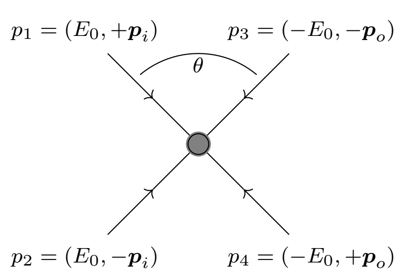


Figure 6.9: Center of mass frame kinematics four a four-string amplitude.

We then have

$$s = -(p_1 + p_2)^2 = -(2E_0, 0, 0, 0)^2 = 4E_0^2 = E^2 \qquad [6.138]$$

Similarly

$$\begin{aligned} t &= -(p_1 + p_3)^2 = -(0, \boldsymbol{p}_i - \boldsymbol{p}_o)^2 = 0 - (\boldsymbol{p}_i - \boldsymbol{p}_o)^2 \\ &= -\boldsymbol{p}_i^2 - \boldsymbol{p}_o^2 + 2\boldsymbol{p}_i{\cdot}\boldsymbol{p}_o \end{aligned} \qquad [6.139]$$

The mass shell condition is $-m^2 = -E_0^2 + \boldsymbol{p}_i^2 = -E_0^2 + \boldsymbol{p}_o^2$ and so $\boldsymbol{p}_i^2 = \boldsymbol{p}_o^2 = E_0^2 - m^2$ and $\boldsymbol{p}_i \cdot \boldsymbol{p}_o = |\boldsymbol{p}_i|\,|\boldsymbol{p}_o| \cos\theta = (E_0^2 - m^2)\cos\theta$. Thus

$$\begin{aligned} t &= -2(E_0^2 - m^2) + 2(E_0^2 - m^2)\cos\theta \\ &= -2(E_0^2 - m^2) + 2(E_0^2 - m^2)\left(1 - 2\sin^2\frac{\theta}{2}\right) \\ &= 4(m^2 - E_0^2)\sin^2\frac{\theta}{2} = (4m^2 - E^2)\sin^2\frac{\theta}{2} \end{aligned} \qquad [6.140]$$

We find, similarly,

$$\begin{aligned} u &= -(p_1 + p_4)^2 = -(0, \boldsymbol{p}_i + \boldsymbol{p}_o)^2 = -\boldsymbol{p}_i^2 - \boldsymbol{p}_o^2 - 2\boldsymbol{p}_i{\cdot}\boldsymbol{p}_o \\ &= -2(E_0^2 - m^2) - 2(E_0^2 - m^2)\cos\theta \\ &= -2(E_0^2 - m^2) - 2(E_0^2 - m^2)\left(2\cos^2\frac{\theta}{2} - 1\right) \\ &= 4(m^2 - E_0^2)\cos^2\frac{\theta}{2} = (4m^2 - E^2)\cos^2\frac{\theta}{2} \end{aligned} \qquad [6.141]$$

One easily checks that $s+t+u = 4m^2$ as it should for Mandelstam variables. The conclusion that having $s \to \infty$ at t fixed requires $\theta \to 0$, so that in that limit $t \to 0$, a fixed value indeed. Similarly, requiring t/s fixed means

$$\lim_{E\to\infty} \frac{4m^2 - E^2\sin^2\frac{\theta}{2}}{E^2} = 4m^2\sin^2\frac{\theta}{2} \qquad [6.142]$$

fixed, so looking at a fixed angle θ.

6.41 p 183: Eq. (6.4.28) The Regge Behaviour of the Veneziano Amplitude

From (6.4.23) we need to calculate

$$\mathfrak{B} = B(-\alpha' s - 1, -\alpha' t - 1) + B(-\alpha' s - 1, -\alpha' u - 1) + B(-\alpha' t - 1, -\alpha' u - 1) \qquad [6.143]$$

Let us start with using Stirling's formula[4] for large s keeping t fixed:

$$\begin{aligned}\mathfrak{B}_1 = B(-\alpha's-1,-\alpha't-1) &= \frac{\Gamma(-\alpha's-1)\Gamma(-\alpha't-1)}{\Gamma(-\alpha's-\alpha't-2)} = \frac{\Gamma(-\alpha's-2+1)\Gamma(-\alpha't-1)}{\Gamma(-\alpha's-\alpha't-3+1)} \\ &= \frac{(-\alpha's-2)^{-\alpha's-2}e^{\alpha's+2}\sqrt{-2/\pi(\alpha's+2)}}{(-\alpha's-\alpha't-3)^{-\alpha's-\alpha't-3}e^{\alpha's+\alpha't+3}\sqrt{-2/\pi(\alpha's+\alpha't+3)}}\Gamma(-\alpha't-1) \qquad [6.144]\end{aligned}$$

We now take the limit $s \to \infty$ holding t fixed. Thus

$$\begin{aligned}\mathfrak{B}_1 &\to \frac{(-\alpha's)^{-\alpha's-2}e^{\alpha's}\sqrt{-2/\pi\alpha's}}{(-\alpha's)^{-\alpha's-\alpha't-3}e^{\alpha's}\sqrt{-2/\pi\alpha's}}\Gamma(-\alpha't-1) \\ &\propto s^{\alpha't+1}\Gamma(-\alpha't-1) = s^{\alpha_o(t)}\Gamma(-\alpha_o(t)) \qquad [6.145]\end{aligned}$$

Let us now consider $\mathfrak{B}_2 = B(-\alpha's-1,-\alpha'u-1)$. From $s+t+u=-4/\alpha'$ We see that in the limit $s \to \infty$ with t fixed, $u \sim -s$ and thus

$$\begin{aligned}\mathfrak{B}_2 = B(-\alpha's-1,-\alpha'u-1) &= \frac{\Gamma(-\alpha's-1)\Gamma(-\alpha'u-1)}{\Gamma(-\alpha's-\alpha'u-2)} = \frac{\Gamma(-\alpha's-2+1)\Gamma(-\alpha'u-2+1)}{\Gamma(-\alpha's-\alpha'u-3+1)} \\ &= \frac{(-\alpha's-2)^{-\alpha's-2}e^{\alpha's+2}\sqrt{-2/\pi(\alpha's+2)}(-\alpha'u-2)^{-\alpha'u-2}e^{\alpha'u+2}\sqrt{-2/\pi(\alpha'u+2)}}{(-\alpha's-\alpha'u-3)^{-\alpha's-\alpha'u-3}e^{\alpha's+\alpha'u+3}\sqrt{-2/\pi(\alpha's+\alpha'u+3)}} \\ &= \frac{(-\alpha's-2)^{-\alpha's-2}e^{\alpha's+2}\sqrt{-2/\pi(\alpha's+2)}(+\alpha's-2)^{+\alpha's-2}e^{-\alpha's+2}\sqrt{-2/\pi(-\alpha's+2)}}{(-\alpha's+\alpha's-3)^{-\alpha's+\alpha's-3}e^{\alpha's-\alpha's+3}\sqrt{-2/\pi(\alpha's-\alpha's+3)}} \\ &\to (-\alpha's)^{-\alpha's-2}e^{\alpha's}s^{-1/2}(+\alpha's)^{+\alpha's-2}e^{-\alpha's}s^{-1/2} \propto s^{-5} \to 0 \qquad [6.146]\end{aligned}$$

Finally consider $\mathfrak{B}_3 = B(-\alpha't-1,-\alpha'u-1)$. As $u \sim -s$ in the limit we are considering, we have $\mathfrak{B}_3 \sim \mathfrak{B}_1 \propto s^{\alpha_o(t)}\Gamma(-\alpha_o(t))$. We thus conclude that $\mathfrak{B}_1+\mathfrak{B}_2+\mathfrak{B}_3 \propto s^{\alpha_o(t)}\Gamma(-\alpha_o(t))$, which is what we set out to show.

6.42 p 183: Eq. (6.4.29) The Hard Scattering Behaviour of the Veneziano Amplitude, I

This limit implies that all the Mandelstam variables becomes infinite:

$$s = E^2; \quad t \to -E^2\sin^2\frac{\theta}{2} = -s\sin^2\frac{\theta}{2}; \quad u \to -E^2\cos^2\frac{\theta}{2} = -s\cos^2\frac{\theta}{2} \qquad [6.147]$$

[4] There is a typo in Joe's book, as per his errata page. Stirling's formula should read $\Gamma(x+1) = x^x e^{-x}(2/\pi x)^{1/2}$.

We then have e.g.

$$\begin{aligned}
&B(-\alpha's-1,-\alpha'u-1)=\frac{\Gamma(-\alpha's-1)\Gamma(-\alpha'u-1)}{\Gamma(-\alpha's-\alpha'u-2)}=\frac{\Gamma(-\alpha's-2+1)\Gamma(-\alpha'u-2+1)}{\Gamma(-\alpha's-\alpha'u-3+1)}\\
&=\frac{(-\alpha's-2)^{-\alpha's-2}e^{\alpha's+2}\sqrt{-2/\pi(\alpha's+2)}(-\alpha'u-2)^{-\alpha'u-2}e^{\alpha'u+2}\sqrt{-2/\pi(\alpha'u+2)}}{(-\alpha's-\alpha'u-3)^{-\alpha's-\alpha'u-3}e^{\alpha's+\alpha'u+3}\sqrt{-2/\pi(\alpha's+\alpha'u+3)}}\\
&\to\frac{(\alpha's)^{-\alpha's-2}e^{\alpha's}s^{-1/2}(\alpha'u)^{-\alpha'u-2}e^{\alpha'u}u^{-1/2}}{(\alpha's+\alpha'u)^{-\alpha's-\alpha'u-3}e^{\alpha's+\alpha'u}(s+u)s^{-1/2}}\\
&\to\alpha'^{-\alpha's-\alpha'u-4+\alpha's+\alpha'u+3}e^{\alpha'(s+u-s-u)}s^{-\alpha's-2-1/2}u^{-\alpha'u-2-1/2}(s+u)^{+\alpha's+\alpha'u+3+1/2}\\
&\propto s^{-\alpha's-5/2}u^{-\alpha'u-5/2}(s+u)^{\alpha'(s+u)+7/2}\propto s^{-\alpha's}u^{-\alpha'u}t^{-\alpha't}
\end{aligned} \quad [6.148]$$

In the last line we have used $s+u=-t-4/\alpha'$ and we have also considered the limit where $s,t,u\to\infty$. The other Beta functions give the same result and so we do indeed find that

$$S_{D_2}\propto s^{-\alpha's}u^{-\alpha'u}t^{-\alpha't}=\exp\left[-\alpha'(s\ln s+t\ln t+u\ln u)\right] \quad [6.149]$$

6.43 p 183: Eq. (6.4.30) The Hard Scattering Behaviour of the Veneziano Amplitude, II

We have

$$\begin{aligned}
s\ln s+t\ln t+u\ln u&=s\left(\ln s-\cos^2\frac{\theta}{2}\ln s\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\ln s\sin^2\frac{\theta}{2}\right)\\
&=s\left(\ln s-\cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}\ln s-\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\ln s\right)\\
&=s\left[\left(1-\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\ln s-\cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}\right]\\
&=s\left(-\cos^2\frac{\theta}{2}\ln\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}\right)=sf(\theta)
\end{aligned} \quad [6.150]$$

which is (6.4.30).

We see that the amplitude of four tachyon scattering decays as an exponential decay $\exp[-\alpha'f(\theta)s]$. In particle based QFT a four particle scattering amplitude has a power decay. I seem to remember that this is a necessary condition for unitarity but I can't find it back in my notes. As an example the total tree level amplitude for the hard scattering process $e^+e^-\to\mu^+\mu^-$ is in the high-energy limit

$$\sigma=\frac{4\pi\alpha^2}{3s} \quad [6.151]$$

and any four-lepton scattering will have the $1/s$ behaviour in the high-energy limit, only the constant factor will differ, depending on the details of the particles involved.

6.44 p 185: The Hermiticity of the Chan-Paton Factors

The Chan-Paton degrees of freedom give quantum numbers that are measurable, hence these quantum numbers i, j in (6.5.1) should be real. Our theory being quantum mechanics these quantum numbers should be Eigenvalues of some operators. As there are n such quantum numbers on each end of the string the operators can be represented by $n \times n$ matrices that have to be Hermitian in order to have real Eigenvalues. A complete set of Hermitian $n \times n$ matrices is given by the n^2 matrices λ^a_{ij}. Here $a = 1, \cdots, n^2$ refers to the n^2 different matrices in the basis and $i, j = 1, \cdots, n$ are the elements of the $n \times n$ matrices.

6.45 p 185: Eq. (6.5.4) The Trace of Chan-Paton Factors

Consider a four open string tachyon scattering. Each endpoint of each string has a Chan-Paton factor associated to it. Each open string thus has a state $|N; k; a\rangle = \sum_{i,j=1}^n |N; k; ij\rangle \lambda^a_{ij}$ associated with it.

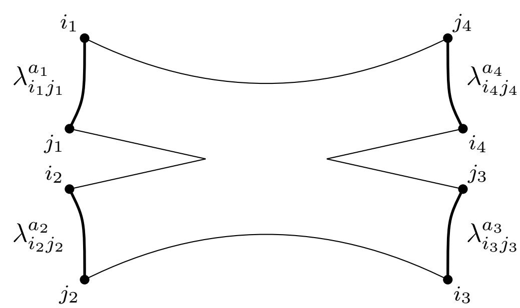


Figure 6.10: Open string Chan-Paton factors

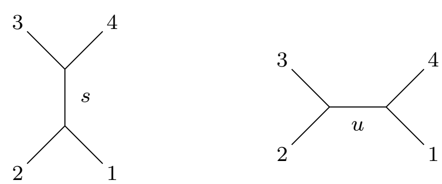
The matrices λ^a_{ij} don't evolve with time so we prescribe that the strings can interact, i.e. that two end points can coalesce, only if the Chan-Paton factors are the same. E.g. an endpoint with a Chan-Paton factor i can only coalesce with another string with endpoint j only if $i = j$. Each interaction thus introduces a δ_{ij}. The four-point amplitude in fig. 6.10 thus has a contribution

$$\begin{aligned} & \sum_{i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4 = 1}^{n} \lambda^{a_1}_{i_1 j_1} \delta_{j_1 i_2} \lambda^{a_2}_{i_2 j_2} \delta_{j_2 i_3} \lambda^{a_3}_{i_3 j_3} \delta_{j_3 i_4} \lambda^{a_4}_{i_4 j_4} \delta_{j_4 i_1} \\ = & \sum_{i,j,k,\ell=1}^{n} \lambda^{a_1}_{\ell i} \lambda^{a_2}_{ij} \lambda^{a_3}_{jk} \lambda^{a_4}_{k\ell} = \text{tr } \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \end{aligned} \quad [6.152]$$

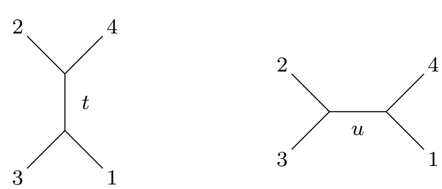
6.46 p 186: Eq. (6.5.6) The Four Tachyon Amplitude with Chan-Paton Factors

This requires a bit of work and thinking. We need to go back to the derivation (6.4.9) to track how the Chan-Paton factors occur. Recall that we arrived at this expression by splitting the integral in (6.4.8) from $\int_{-\infty}^{+\infty}$ into three integrals $\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{+\infty}$ and bring each of these into the form $\int_0^1$ using a Möbius transformation.

Let us start with the part $\int_{-\infty}^{0}$, corresponding to fig 6.2(a). This corresponds to the ordering 1423. We have seen in the derivation of (6.4.9) that this ordering gives $I(s,u)$, see [6.125]. This has a pole in the s-channel and in the u-channel as shown below:

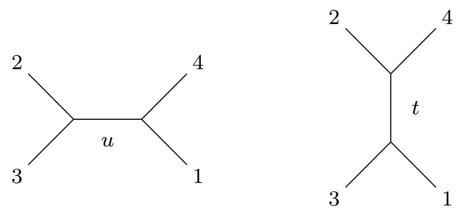


Looking at the diagram and connecting the Chan-Paton factors of the coalescing end-points, this means that $I(s,u)$ will come with a factor $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}$. Using $\mathrm{tr}\ abcd = \mathrm{tr}\ dcba$ and cyclicity of the trace we can rewrite this as $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}$. Now we also need to take into account the other cyclic ordering of the three vertices we have fixed. This means we need to consider the contribution of $k_2 \leftrightarrow k_3$, or equivalently $s \leftrightarrow t$. Hence the diagrams

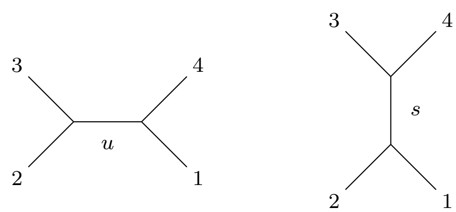


this gives a t-channel and a u-channel contribution hence $I(t,u)$. The ordering of the Chan-Paton factors is now $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}$ and so this trace will accompany $I(t,u)$.

Let us next go to the part $\int_0^1$. This corresponds to the vertex ordering 1423 and fig 6.2(b). From [6.125] we know that this corresponds to $I(u,t)$ with a pole in the t and in the u channels. The corresponding diagrams are:

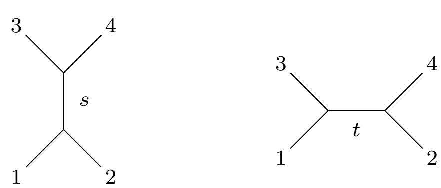


Connecting the Chan-Paton factor of the coalescing end-points we see that $I(t,u)$ will be accompanied by $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}$. The other cyclic ordering for this case will give $I(u,s)$ with diagrams

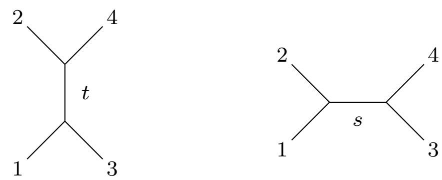


indeed giving a u and an s channel pole. Connecting the Chan-Paton factor of the coalescing end-points we see that $I(u,s)$ will be accompanied by $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}$.

Finally, consider $\int_1^\infty$.This corresponds to the vertex ordering 1243 and fig 6.2(c). From [6.123] we know that this corresponds to $I(s,t)$ with a pole in the s and in the t channels. The corresponding diagrams are



Thus $I(s,t)$ will come with a $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}$. The other cyclic ordering also gives $I(s,t)$ and has diagrams



and so this $I(s,t)$ will come with a $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}$.
Bringing everything together and recalling that $I(x,y) = I(y,x) = B(-\alpha_o(x), -\alpha_o(y))$ we find

$$\begin{aligned}
&(\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2})\, B(-\alpha_o(s), -\alpha_o(t)) \\
+&(\mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4})\, B(-\alpha_o(t), -\alpha_o(u)) \\
+&(\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2})\, B(-\alpha_o(s), -\alpha_o(u))
\end{aligned} \qquad [6.153]$$

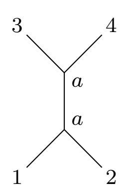
This is exactly (6.5.6), taking into account the errata on Joe's website.

6.47 p 186: Eq. (6.5.7-8) The Four Tachyon Amplitude and Unitarity

Extracting the terms in (6.5.6) that have an s-pole we find the combination of traces for the LHS of (6.4.13)

$$\begin{aligned}
&\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2} \\
=&\ \mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \mathrm{tr}\ \lambda^{a_2}\lambda^{a_1}\lambda^{a_3}\lambda^{a_4} + \mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \mathrm{tr}\ \lambda^{a_2}\lambda^{a_1}\lambda^{a_4}\lambda^{a_3} \\
=&\ \mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\{\lambda^{a_3}, \lambda^{a_4}\} + \mathrm{tr}\ \lambda^{a_2}\lambda^{a_1}\{\lambda^{a_3}, \lambda^{a_4}\} \\
=&\ \mathrm{tr}\ \{\lambda^{a_1}, \lambda^{a_2}\}\{\lambda^{a_3}, \lambda^{a_4}\}
\end{aligned} \qquad [6.154]$$

which is (6.5.7). Looking at (6.4.13) we have for the RHS of (6.4.13)



So the three-three tachyon vertices will give a contribution $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_2}\lambda^{a}$ and $\mathrm{tr}\ \lambda^{a_4}\lambda^{a_3}\lambda^{a}$ with a sum over all intermediate a's. But we need to take into account the other cyclic ordering as in (6.5.5) and this can be obtained by switching two of the vertices. Hence this gives a trace contribution

$$\begin{aligned} &\sum_a (\text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^a + \text{tr } \lambda^{a_2}\lambda^{a_1}\lambda^a)(\text{tr } \lambda^{a_4}\lambda^{a_3}\lambda^a + \text{tr } \lambda^{a_3}\lambda^{a_4}\lambda^a) \\ = &\sum_a \text{tr } \{\lambda^{a_1}, \lambda^{a_2}\}\lambda^a \text{ tr } \{\lambda^{a_3}, \lambda^{a_4}\}\lambda^a \end{aligned} \quad [6.155]$$

which is (6.5.8).

6.48 p 187: Eq. (6.5.9) Traces and the Completeness Relation

$$\text{tr } (A\lambda^a)\text{tr } (B\lambda^a) = A_{ij}\lambda^a_{ji}B_{k\ell}\lambda^a_{\ell k} = A_{ij}B_{k\ell}\lambda^a_{ji}\lambda^a_{\ell k} \quad [6.156]$$

As the λ^a form a complete basis, they satisfy

$$\lambda^a_{ji}\lambda^a_{\ell k} = \delta_{i\ell}\delta_{jk} \quad [6.157]$$

The normalisation can be easily checked by setting $\ell = i$ and $k = j$ and summing over them. The LHS becomes $\text{tr } \lambda^a\lambda^a$. Using (6.5.2) this is $\delta^{aa} = n^2$, the number of Hermitian matrices. The RHS is $\delta_{ii}\delta_{jj} = n^2$ as well. Thus

$$\text{tr } (A\lambda^a)\text{tr } (B\lambda^a) = A_{ij}B_{k\ell}\delta_{i\ell}\delta_{jk} = A_{ij}B_{ji} = \text{tr } AB \quad [6.158]$$

6.49 p 187: Eq. (6.5.10) One Gauge Boson and Two Tachyons, I

Recall the amplitude for three tachyons without Chan-Paton factors (6.4.1)

$$S_{D_2}(k_1, k_2, k_3) = g_0^3 e^{-\lambda} \left\langle {}^\times_\times c^1 e^{ik_1\cdot X}(y_1){}^\times_\times\, {}^\times_\times c^1 e^{ik_2\cdot X}(y_2){}^\times_\times\, {}^\times_\times c^1 e^{ik_3\cdot X}(y_3){}^\times_\times \right\rangle \quad [6.159]$$

If we now want to replace a tachyon, say the first one, by a gauge boson then we have to replace a tachyon vertex operator by a gauge boson operator. For an open string this is given by (3.6.26)

$$-\frac{ig_0}{\sqrt{2\alpha'}}\left[\dot{X}^\mu e^{ik\cdot X}\right]_r (y_1) \quad [6.160]$$

Recall that the vertex operator is on the boundary of the disk, i.e. on the real axis, so $\dot{X}^\mu(y) = \partial_y X^\mu(y)$. We have, of course, to integrate this over ∂M, which here is the real line, i.e. $\int_{-\infty}^{+\infty} dy_1$. But we know that we can fix the three coordinates by invariance under Möbius transformations, so we can ignore this integration over this coordinate y_1 and y_2 and y_3 as well. We do have to check that the final result does not depend on these coordinates. Adding the Chan-Paton factor and the other cyclic combination we get for the amplitude of a gauge boson with two tachyons

$$\begin{aligned} S_{D_2}(k_1, a_1, \epsilon_1; k_2, a_2; k_3, a_3) &= -ig_0^2 g_0' e^{-\lambda} \epsilon_1^\mu \\ &\times \left\langle {}^{\times}_{\times} c^1 \dot{X}^\mu e^{ik_1\cdot X}(y_1) {}^{\times}_{\times} {}^{\times}_{\times} c^1 e^{ik_2\cdot X}(y_2) {}^{\times}_{\times} {}^{\times}_{\times} c^1 e^{ik_3\cdot X}(y_3) {}^{\times}_{\times} \right\rangle \operatorname{tr} \lambda_{a_1}\lambda_{a_2}\lambda_{a_3} \\ &+ (k_2, a_2) \leftrightarrow (k_3, a_3) \end{aligned} \qquad [6.161]$$

Here $\epsilon_1 = \epsilon(k_1)$ is the polarisation vector of the gauge boson and we have replaced $g_0/\sqrt{2\alpha'}$ by g_0'.

6.50 p 187: Eq. (6.5.11) One Gauge Boson and Two Tachyons, II

This is just an application of (6.2.36) which is for one derivative and two exponentials

$$\begin{aligned} &\left\langle {}^{\times}_{\times} \dot{X}^\mu e^{ik_1\cdot X}(y_1) {}^{\times}_{\times} {}^{\times}_{\times} e^{ik_2\cdot X}(y_2) {}^{\times}_{\times} {}^{\times}_{\times} e^{ik_3\cdot X}(y_3) {}^{\times}_{\times} \right\rangle \\ &= iC_{D_2}^X (2\pi)^{26} \delta^{26}(k_1+k_2+k_3) |y_{12}|^{2\alpha' k_1\cdot k_2} |y_{13}|^{2\alpha' k_1\cdot k_3} |y_{23}|^{2\alpha' k_2\cdot k_3} \\ &\times \left[-2i\alpha' \left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}} \right) \right] \end{aligned} \qquad [6.162]$$

Because the $\dot{X}^\mu$ sits with $e^{ik\cdot X(y_1)}$ in a boundary normal ordering, no contraction has to be taken between these.[5]

6.51 p 187: Eq. (6.5.12) One Gauge Boson and Two Tachyons: Final Result

We now add the expectation value of the three ghost fields (6.3.4) and the Chan-Paton factors and find

$$\begin{aligned} S_{D_2}(k_1, a_1, \epsilon_1; k_2, a_2; k_3, a_3) &= -ig_0' g_0^2 e^{-\lambda} iC_{D_2}^X C_{D_2}^g (2\pi)^{26} \delta^{26}(\sum_i k_i)(-2i\alpha') \\ &\times \epsilon_1^\mu \left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}} \right) |y_{12}|^{2\alpha' k_1\cdot k_2+1} |y_{13}|^{2\alpha' k_1\cdot k_3+1} |y_{23}|^{2\alpha' k_2\cdot k_3+1} \operatorname{tr} \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\ &+ (k_2, a_2) \leftrightarrow (k_3, a_3) \end{aligned} \qquad [6.163]$$

[5]As that contraction would be divergent and the boundary normal ordering's job is precisely to regularize this divergence.

Let us first use (6.4.14), i.e. $C^X_{D_2} C^g_{D_2} e^{-\lambda} = C_{D_2} = 1/\alpha' g_0^2$. This gives

$$S_{D_2}(k_1, a_1, \epsilon_1; k_2, a_2; k_3, a_3) = -2ig'_0(2\pi)^{26}\delta^{26}(\sum_i k_i)\epsilon_1^\mu \left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}}\right) \\ \times |y_{12}|^{2\alpha' k_1\cdot k_2+1}|y_{13}|^{2\alpha' k_1\cdot k_3+1}|y_{23}|^{2\alpha' k_2\cdot k_3+1} \text{ tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\ - +(k_2, a_2) \leftrightarrow (k_3, a_3) \tag{6.164}$$

Next we use momentum conservation and the mass shell condition. The gauge boson is massless so $k_1^2 = 0$. The tachyons have $k_2^2 = k_3^2 = 1/\alpha'$. Thus

$$\frac{1}{\alpha'} = k_3^2 = (-k_1 - k_2)^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = 0 + \frac{1}{\alpha'} + 2k_1 \cdot k_2 \Rightarrow k_1 \cdot k_2 = 0 \tag{6.165}$$

Similarly $k_1 \cdot k_3 = 0$. Also

$$0 = k_1^2 = (-k_2 - k_3)^2 = k_2^2 + k_3^2 + 2k_2 \cdot k_3 = \frac{2}{\alpha'} + 2k_2 \cdot k_3 \Rightarrow 2\alpha' k_2 \cdot k_3 = -2 \tag{6.166}$$

Therefore

$$S_{D_2}(k_1, a_1\epsilon_1; k_2, a_2; k_3, a_3) = -2ig'_0(2\pi)^{26}\delta^{26}(\sum_i k_i)\epsilon_1^\mu \left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}}\right) y_{12}y_{13}y_{23}^{-1}\text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\ + (k_2, a_2) \leftrightarrow (k_3, a_3) \tag{6.167}$$

We have ignored signs in the last equation. Let us focus on the y-dependence. We use momentum conservation $k_3 = -k_1 - k_2$ and the fact the ϵ_1 is a polarisation vector of a massless boson, hence $\epsilon_1 \cdot k_1 = 0$:

$$\epsilon_1^\mu \left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}}\right) y_{12}y_{13}y_{23}^{-1} = \frac{1}{2}\epsilon_1^\mu \left(\frac{k_2^\mu - k_1^\mu - k_3^\mu}{y_{12}} + \frac{k_3^\mu - k_1^\mu - k_2^\mu}{y_{13}}\right) y_{12}y_{13}y_{23}^{-1} \\ = \frac{1}{2}\epsilon_1^\mu k_{23}^\mu (y_{13} - y_{12})y_{23}^{-1} = \frac{1}{2}\epsilon_1^\mu k_{23}^\mu y_{23}y_{23}^{-1} = \frac{1}{2}\epsilon_1 \cdot k_{23} \tag{6.168}$$

with $k_{ij} = k_i - k_j$. Thus

$$S_{D_2}(k_1, a_1\epsilon_1; k_2, a_2; k_3, a_3) = -ig'_0(2\pi)^{26}\delta^{26}(\sum_i k_i)\epsilon_1 \cdot k_{23}\text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + (k_2, a_2) \leftrightarrow (k_3, a_3) \\ = -ig'_0(2\pi)^{26}\delta^{26}(\sum_i k_i)\left(\epsilon_1 \cdot k_{23}\text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + \epsilon_1 \cdot k_{32}\text{tr } \lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\right) \\ = -ig'_0\epsilon_1 \cdot k_{23}(2\pi)^{26}\delta^{26}(\sum_i k_i) \text{ tr } \lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}] \tag{6.169}$$

which is (6.5.12).

6.52 p 188: Eq. (6.5.15) The Three Gauge Boson Amplitude

We first consider the matter part:

$$S_3^{X\mu\nu\sigma}\left\langle {}^\times_\times \dot{X}^\mu e^{ik_1\cdot X}(y_1){}^\times_\times\, {}^\times_\times \dot{X}^\nu e^{ik_2\cdot X}(y_2){}^\times_\times\, {}^\times_\times \dot{X}^\sigma e^{ik_3\cdot X}(y_3){}^\times_\times \right\rangle \tag{6.170}$$

From (6.2.36) we get

$$\begin{aligned} S_3^{X\mu\nu\sigma} =& iC^X_{D_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{13}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3}\Big[v^\mu(y_1)v^\nu(y_2)v^\sigma(y_3) \\ &+ v^\mu(y_1)\langle q^\nu(y_2)q^\sigma(y_3)\rangle + v^\nu(y_2)\langle q^\mu(y_1)q^\sigma(y_3)\rangle + v^\sigma(y_3)\langle q^\mu(y_1)q^\nu(y_2)\rangle\Big] \\ &\times \text{tr}\,\lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + (k_2,a_2,\epsilon_2)\leftrightarrow(k_3,a_3,\epsilon_3) \\ =& iC^X_{D_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{13}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3} \\ &\times\Bigg\{(-2i\alpha')^3\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right) \\ &+(-2\alpha')(-2i\alpha')\left[\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\frac{\eta^{\nu\sigma}}{y_{23}^2}+\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\frac{\eta^{\mu\nu}}{y_{13}^2}+\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right)\frac{\eta^{\mu\sigma}}{y_{12}^2}\right]\Bigg\} \\ &\times \text{tr}\,\lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + (k_2,a_2,\epsilon_2)\leftrightarrow(k_3,a_3,\epsilon_3) \end{aligned} \tag{6.171}$$

The first thing we note is that because of momentum conservation and mass-shell condition

$$0 = k_1^2 = (-k_2-k_3)^2 = k_2^2+k_3^2+2k_1\cdot k_2 \Rightarrow k_1\cdot k_2 = 0 \tag{6.172}$$

We now add the ghost contribution $\langle ccc\rangle = C^g_{D_2}y_{12}y_{13}y_{23}$, the polarisation vectors, the cosmological term $e^{-\lambda}$ and the normalisation of the boson vertex operators, $-ig_0/\sqrt{2\alpha'}$ see (3.6.26), and we find

$$\begin{aligned} S^{ggg}_{D_2} =& S_{D_2}(k_1,a_1,\epsilon_1;k_2,a_2,\epsilon_2;k_3,a_3,\epsilon_3) \\ =& \left(-\frac{ig_0}{\sqrt{2\alpha'}}\right)^3 4i\alpha'^2 C^X_{D_2}C^g_{D_2}e^{-\lambda}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\sigma}(2\pi)^{26}\delta^{26}(\sum_i k_i)y_{12}y_{13}y_{23} \\ &\times\Bigg[2\alpha'\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right) \\ &+\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\frac{\eta^{\nu\sigma}}{y_{23}^2}+\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\frac{\eta^{\mu\nu}}{y_{13}^2}+\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right)\frac{\eta^{\mu\sigma}}{y_{12}^2}\Bigg] \\ &\times \text{tr}\,\lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + (k_2,a_2,\epsilon_2)\leftrightarrow(k_3,a_3,\epsilon_3) \end{aligned} \tag{6.173}$$

Let us first simplify the pre-factor. Use (6.4.14), i.e. $C^X_{D_2}C^g_{D_2}e^{-\lambda} = C_{D_2} = 1/\alpha' g_0^2$ to write

$$\left(-\frac{ig_0}{\sqrt{2\alpha'}}\right)^3 4i\alpha'^2 C^X_{D_2}C^g_{D_2}e^{-\lambda} = -\frac{2\alpha' g_0^3}{\sqrt{2\alpha'}}C^X_{D_2}C^g_{D_2}e^{-\lambda} = -\frac{2g_0}{\sqrt{2\alpha'}} = -2g_0' \qquad [6.174]$$

in the last line we have used (6.5.14), i.e. $g_0' = g_0/\sqrt{2\alpha'}$. The three boson amplitude thus becomes

$$\begin{aligned} S^{ggg}_{D_2} =& S_{D_2}(k_1,a_1,\epsilon_1;k_2,a_2,\epsilon_2;k_3,a_3,\epsilon_3) \\ =& -2g_0'\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\sigma}(2\pi)^{26}\delta^{26}(\sum_i k_i)y_{12}y_{13}y_{23}\left[2\alpha'\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right)\right. \\ &\left.+\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\frac{\eta^{\nu\sigma}}{y_{23}^2}+\left(\frac{k_1^\nu}{y_{21}}+\frac{k_3^\nu}{y_{23}}\right)\frac{\eta^{\mu\nu}}{y_{13}^2}+\left(\frac{k_1^\sigma}{y_{31}}+\frac{k_2^\sigma}{y_{32}}\right)\frac{\eta^{\mu\sigma}}{y_{12}^2}\right] \\ &\times \operatorname{tr}\lambda^{a_1}\lambda^{a_2}\lambda^{a_3} + (k_2,a_2,\epsilon_2)\leftrightarrow(k_3,a_3,\epsilon_3) \end{aligned} \qquad [6.175]$$

In the remainder of the calculation we will repeatedly use the fact that $k_i\cdot\epsilon_i = 0$ for $i = 1,2,3$. This actually means that we can simply ignore any terms that have a k_1^μ, k_2^ν or a k_3^σ as these will be contracted with the polarisation vectors $\epsilon_{1\mu},\epsilon_{2\nu}$ and $\epsilon_{3\sigma}$ respectively. Consider first the terms linear in momentum and take the first such terms

$$\begin{aligned} &y_{12}y_{13}y_{23}\left(\frac{k_2^\mu}{y_{12}}+\frac{k_3^\mu}{y_{13}}\right)\frac{\eta^{\nu\sigma}}{y_{23}^2} = \frac{\eta^{\nu\sigma}}{y_{23}}(k_2^\mu y_{13}+k_3^\mu y_{12}) \\ =&\frac{\eta^{\nu\sigma}}{2y_{23}}\left[k_2^\mu y_{13}+(-k_1^\mu-k_3^\mu)y_{13}+k_3^\mu y_{12}+(-k_1^\mu-k_2^\mu)y_{12}\right] \\ =&\frac{\eta^{\nu\sigma}}{2y_{23}}(k_{23}^\mu y_{13}+k_{32}^\mu y_{12}) = \frac{k_{23}^\mu\eta^{\nu\sigma}}{2y_{23}}(y_{13}-y_{12}) = \frac{k_{23}^\mu\eta^{\nu\sigma}}{2y_{23}}y_{23} = \frac{1}{2}k_{23}^\mu\eta^{\nu\sigma} \end{aligned} \qquad [6.176]$$

Contracting with the polarisation vectors $\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\sigma$ this gives a contribution $\frac{1}{2}(\epsilon_1\cdot k_{23})(\epsilon_2\cdot\epsilon_3)$. The two other terms linear in the momenta give similar contributions so that we can write

$$\begin{aligned} S^{ggg[1]}_{D_2} =& -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i)\times\left[(\epsilon_1\cdot k_{23})(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot k_{31})(\epsilon_3\cdot\epsilon_1)+(\epsilon_3\cdot k_{12})(\epsilon_1\cdot\epsilon_2)\right] \\ &\times\operatorname{tr}\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}+(k_2,a_2,\epsilon_2)\leftrightarrow(k_3,a_3,\epsilon_3) \\ =& -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i)\Big\{\left[(\epsilon_1\cdot k_{23})(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot k_{31})(\epsilon_3\cdot\epsilon_1)\right. \\ &\left.+(\epsilon_3\cdot k_{12})(\epsilon_1\cdot\epsilon_2)\right]\operatorname{tr}\lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\ &+\left[(\epsilon_1\cdot k_{32})(\epsilon_3\cdot\epsilon_2)+(\epsilon_3\cdot k_{21})(\epsilon_2\cdot\epsilon_1)+(\epsilon_2\cdot k_{13})(\epsilon_1\cdot\epsilon_3)\right]\operatorname{tr}\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\Big\} \end{aligned} \qquad [6.177]$$

and finally for the linear term

$$S_{D_2}^{ggg[1]} = -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i) \times \left[(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot \epsilon_3) + (\epsilon_2 \cdot k_{31})(\epsilon_3 \cdot \epsilon_1) + (\epsilon_3 \cdot k_{12})(\epsilon_1 \cdot \epsilon_2)\right] \text{tr } \lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}] \quad [6.178]$$

Let us now focus on the term cubic in the momenta. We have

$$\begin{aligned}
&\left(\frac{k_2^\mu}{y_{12}} + \frac{k_3^\mu}{y_{13}}\right)\left(\frac{k_1^\nu}{y_{21}} + \frac{k_3^\nu}{y_{23}}\right)\left(\frac{k_1^\sigma}{y_{31}} + \frac{k_2^\sigma}{y_{32}}\right) \\
=&\frac{k_2^\mu y_{13} + k_3^\mu y_{12}}{y_{12}y_{13}} \frac{k_1^\nu y_{23} + k_3^\nu y_{21}}{y_{21}y_{23}} \frac{k_1^\sigma y_{32} + k_2^\sigma y_{31}}{y_{31}y_{32}} \\
=&\frac{[k_2^\mu y_{13} + (-k_1^\mu - k_2^\mu)y_{12}][k_1^\nu y_{23} + (-k_1^\nu - k_2^\nu)y_{21}][k_1^\sigma y_{32} + (-k_1^\sigma - k_3^\sigma)y_{31}]}{y_{12}y_{13}y_{21}y_{23}y_{31}y_{32}} \\
=&-\frac{k_2^\mu(y_{13} - y_{12})k_1^\nu(y_{23} - y_{21})(k_1^\sigma(y_{32} - y_{31})}{y_{12}^2 y_{13}^2 y_{23}^2} \\
=&\frac{k_2^\mu k_1^\nu k_1^\sigma y_{23}y_{13}y_{12}}{y_{12}^2 y_{13}^2 y_{23}^2} = \frac{k_2^\mu k_1^\nu k_1^\sigma}{y_{12}y_{13}y_{23}} \qquad [6.179]
\end{aligned}$$

We have repeatedly used momentum conservation and the fact that $\epsilon_1 \cdot k_1 = \epsilon_2 \cdot k_2 = \epsilon_3 \cdot k_3 = 0$. We now rewrite

$$\begin{aligned}
&k_2^\mu k_1^\nu k_1^\sigma = \left(\frac{1}{2}\right)^3 [k_2^\mu + (-k_1^\mu - k_3^\mu)][k_1^\nu + (-k_2^\nu - k_3^\nu)][k_1^\sigma + (-k_2^\sigma - k_3^\sigma)] \\
&= \frac{1}{8} k_{23}^\mu k_{13}^\nu k_{12}^\sigma \qquad [6.180]
\end{aligned}$$

Bringing it all together, we find for the cubic terms

$$\begin{aligned}
S_{D_2}^{ggg[3]} =& -2g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i) y_{12}y_{13}y_{23}\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\sigma 2\alpha' \frac{k_{23}^\mu k_{13}^\nu k_{12}^\sigma}{8y_{12}y_{13}y_{23}} \text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\
&+ (k_2, a_2, \epsilon_2) \leftrightarrow (k_3, a_3, \epsilon_3) \\
=& -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{\alpha'}{2}(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot k_{13})(\epsilon_3 \cdot k_{12}) \text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\
&+ (k_2, a_2, \epsilon_2) \leftrightarrow (k_3, a_3, \epsilon_3) \\
=& -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{\alpha'}{2}\Big[(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot k_{13})(\epsilon_3 \cdot k_{12}) \text{tr } \lambda^{a_1}\lambda^{a_2}\lambda^{a_3} \\
&+ (\epsilon_1 \cdot k_{32})(\epsilon_3 \cdot k_{12})(\epsilon_2 \cdot k_{13}) \text{tr } \lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\Big] \\
=& -g_0'(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{\alpha'}{2}(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot k_{13})(\epsilon_3 \cdot k_{12}) \text{tr } \lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}] \qquad [6.181]
\end{aligned}$$

Bringing the cubic and the linear terms together we thus find

$$S_{D_2}(k_1, a_1, \epsilon_1; k_2, a_2, \epsilon_2; k_3, a_3, \epsilon_3) = -g'_0(2\pi)^{26}\delta^{26}(\sum_i k_i) \times \Big[(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot \epsilon_3) + (\epsilon_2 \cdot k_{31})(\epsilon_3 \cdot \epsilon_1) + (\epsilon_3 \cdot k_{12})(\epsilon_1 \cdot \epsilon_2) + \frac{\alpha'}{2}(\epsilon_1 \cdot k_{23})(\epsilon_2 \cdot k_{13})(\epsilon_3 \cdot k_{12})\Big] \operatorname{tr} \lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}] \quad [6.182]$$

which is (6.5.15), up to a factor $-i$, but this might just be a normalisation issue of the boson vertex operator as $-i = i^3$.

6.53 p 188: Eq. (6.5.16) The Yang-Mills Effective Field Theory

I realise that I am going on a limb with the following comment, but he who dares .... We will not show that the action (6.5.16) gives the amplitudes we found if restricted to first order momenta. In a way this should not come as a surprise. Indeed it is a general fact in field theory that a theory of a massless boson that is Lorentz invariant, has a Lagrangian with at most two derivatives and whose energy is bounded by below, necessarily has gauge invariance. We will argue this to be the case for a spin one particle with no Chan-Paton factor. This will then lead to non-Abelian gauge theories.

MASSIVE SIN ONE PARTICLES

Let us write down the most general Lorentz invariant Lagrangian for a non-interacting massive spin one field A^μ ($\mu = 1, \cdots, D = 4$) with maximum two derivatives[6] . It must be of the form

$$\mathcal{L} = \frac{1}{2}aA^\mu \Box A_\mu + \frac{1}{2}bA^\mu \partial_\mu \partial_\nu A^\nu + \frac{1}{2}m^2 A^\mu A_\mu \quad [6.183]$$

with some coefficients a, b and m. The equations of motion are

$$a\Box A_\mu + b\partial_\mu \partial_\nu A^\nu + m^2 A_\mu = 0 \quad [6.184]$$

[6]A higher number of derivatives leads to non-local theories and these have unitarity problems. As an example a Lagrangian with a term of the form $\alpha\varphi\Box^2\varphi - \beta\varphi\Box\varphi$ would lead to a propagator proportional to $1/(\alpha k^4 - \beta k^2)$. We can rewrite this as $\frac{1}{\beta}\left[1/k^2 - \alpha/(\alpha k^2 - \beta)\right]$. We can thus view this as the sum of two propagating particles, but with opposite signs in the propagator. These lead to particles with opposite norm and hence violate unitarity. We thus need to have $\alpha = 0$. Note also that we are restricting ourselves to four dimensions here. This will make the argument for positive energy easy.

Let us take the divergence ∂^μ of this

$$\left[(a+b)\Box + m^2\right]\left(\partial_\mu A^\mu\right) = 0 \tag{6.185}$$

If $a+b=0$ then it follows that $\partial_\mu A^\mu = 0$. This is a Lorentz invariant condition and so it removes one degree of freedom. Let us chose $a=1$ and $b=-1$. The Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu \tag{6.186}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian is known as the ***Proca Lagrangian***. The equations of motion are

$$(\Box + m^2)A^\mu = 0 \qquad \text{and} \qquad \partial_\mu A^\mu = 0 \tag{6.187}$$

One can easily check that this model has positive energy, bounded by zero. Indeed, the energy-momentum tensor for this Lagrangian is

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\sigma)}\partial_\nu A^\sigma - g_{\mu\nu}\mathcal{L} = -F_{\mu\sigma}\partial_\nu A^\sigma + g_{\mu\nu}\left(\frac{1}{4}F^{\sigma\rho}F_{\sigma\rho} - \frac{1}{2}m^2 A^\sigma A_\sigma\right) \tag{6.188}$$

It is then a straightforward exercise to show that the energy density $\mathcal{E} = T_{00}$ can be written as

$$\begin{aligned}\mathcal{E} =& \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \frac{1}{2}m^2(A_0^2 + \boldsymbol{A}^2) + A_0\partial_0(\partial_\mu A^\mu)\\ &- A_0(\Box + m^2)A_0 + \partial_i(A_0 F_{0i})\end{aligned} \tag{6.189}$$

Where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field respectively and $\boldsymbol{A} = (A_1, A_2, A_3)$ are the gauge field space coordinates. The first two terms are manifestly positive, but the latter three are not. The third and the fourth term, however, vanish by the equations of motion and the last term is a total spatial derivative. If we take the conserved charge $E = \int d^3x\, \mathcal{E}$ then this term does not contribute. So the total energy of the Proca Lagrangian is indeed positive definite.

Had we chosen $a+b \neq 0$ then one easily sees that the energy would be unbounded from below. This would mean that after quantisation every state could move to a state with a lower energy ad infinitum and the theory is unstable. Hence our requirement for positive definite energy.

Masless Spin One Particles

Let us now take the massless limit of the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{6.190}$$

We, of course, know that this Lagrangian is invariant under transformations

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x) \qquad [6.191]$$

for an arbitrary function $\alpha(x)$, i.e. this Lagrangian has $U(1)$ gauge invariance.

Adding Chan-Paton factors amounts to replacing A^μ by $\mathsf{A}^\mu = A^\mu_a \tau^a$ with τ^a the generator of a symmetry group. The same reasoning then leads to a non-abelian gauge theory.

In a similar vein the tachyon is just a Lorentz scalar. The Chan-Paton factors just mean that we know have n^2 such tachyons living in the adjoint representation of the symmetry group.

It should thus not come as a surprised that the amplitudes of open strings, when limited to lowest order in momentum, are the same as the amplitudes for the corresponding particles, with the lowest possible number of derivatives and that for a spin one massless particle, this corresponding effective theory is a gauge theory.

6.54 p 189: Eq. (6.5.18) From a Global Worldsheet Symmetry to a Local Spacetime Symmetry

It may sound very deep that by introducing Chan-Paton factors, which leads to a global worldsheet symmetry, we suddenly have a theory with a spacetime gauge symmetry. This miracle is quickly demystified if we realize that the global worldsheet symmetry $\lambda^a \to U\lambda^a U^\dagger$ can be defined at each spacetime point X^μ for a different set of λ's. Thus this is indeed a local symmetry from the spacetime point of view.

6.55 p 190: Eq. (6.5.21) Worldsheet Parity for the Open String

$|N;k\rangle$ is a state of the form

$$(\alpha_{-i_1})^{n_{i_1}} \cdots (\alpha_{-i_\ell})^{n_{i_\ell}} |0;k\rangle \quad \text{with} \quad \sum_{m=1}^{\ell} n_{i_m} i_m = N \qquad [6.192]$$

Let us take the simplest such state $\alpha_{-N}|0;k\rangle$. Then

$$\Omega\alpha_{-N}|0;k\rangle = \left(\Omega\alpha_{-N}\Omega^{-1}\right)\Omega|0;k\rangle = (-1)^N \alpha_{-N}|0;k\rangle \qquad [6.193]$$

where we have used (6.5.19) and the fact that the ground state is invariant under the worldsheet parity operator. Moreover from (4.3.22) we have a relation between the level n of the matter excitations and its mass: $N = \alpha' m^2 + 1$. This then gives

$$\Omega\alpha_{-N}|0;k\rangle = (-1)^{1+\alpha' m^2} \alpha_{-N}|0;k\rangle \qquad [6.194]$$

For a more general state $(\alpha_{-i_1})^{n_{i_1}} \cdots (\alpha_{-i_\ell})^{n_{i_\ell}} |0;k\rangle$ we just insert a $\Omega^{-1}\Omega$ between each excitations and we immediately obtain the same result.

6.56 p 190: Eq. (6.5.23) Unoriented Open Strings with Chan-Paton factors

This is trivial, but sometimes it is useful to show the trivial.

$$\Omega\,|N;k;a\rangle = \Omega\,|N;k;ij\rangle = \omega_N\,|N;k;ji\rangle = \omega_N s^a\,|N;k;ij\rangle = \omega_N s^a\,|N;k;a\rangle \qquad [6.195]$$

The spin one particles have $\omega_N = (-1)^{1+\alpha' m^2} = -1$ and so the states with symmetric Chan-Paton factors, $s^a = +1$, vanish, whilst only the states with antisymmetric Chan-Paton factors, $s^a = -1$, are non-zero. The $U(n)$ symmetry of the oriented string is thus reduced to $SO(n)$ for the unoriented string. Since the Chan-Paton matrices λ are $n \times n$ matrices, they transform into the adjoint representation of $SO(n)$.

6.57 p 191: Eq. (6.5.26) The Orientation Reversing Symmetries of the Oriented String, I

$$\begin{aligned}\Omega_\gamma^2\,|N;k;ij\rangle &= \Omega_\gamma \omega_N \gamma_{jj'}\,|N;k;j'i'\rangle\,\gamma^{-1}_{i'i} = \omega_N^2 \gamma_{jj'}\gamma_{i'i''}\,|N;k;i''j''\rangle\,\gamma^{-1}_{j''j'}\gamma^{-1}_{i'i}\\ &= (\gamma^T)^{-1}_{ii'}\gamma_{i'i''}\,|N;k;i''j''\rangle\,\gamma^{-1}_{j''j'}\gamma^T_{j'j}\\ &= [(\gamma^T)^{-1}\gamma]_{ii''}\,|N;k;i''j''\rangle\,[\gamma^{-1}\gamma^T]j''j \end{aligned} \qquad [6.196]$$

which is (6.5.25) taking into account the errata on Joe's website.

6.58 p 191: Eq. (6.5.27) The Orientation Reversing Symmetries of the Oriented String, II

Setting $\Gamma = (\gamma^T)^{-1}\gamma$ we have $\Gamma^{-1} = \gamma^{-1}\gamma^T$ and so we can write (6.5.26) as

$$|N;k;ij\rangle = \Gamma_{ii'}\,|N;k;i'j'\rangle\,\Gamma^{-1}_{j'j} \qquad [6.197]$$

Multiplying to the right by Γ we find that, in matrix notation, $|N;k\rangle\,\Gamma = \Gamma\,|N;k\rangle$. As $|N;k\rangle$ contains a Chan-Paton factor λ^a_{ij}, this must hold for any of the n^2 unitary $n \times n$ matrices. Only multiples of the identity $n \times n$ matrix commutes with all other $n \times n$ matrices and so $\Gamma \propto \mathbb{1}_{n\times n}$. Without loss of generality we can normalise this to ± 1 and so we have

$$\pm 1 = \Gamma = (\gamma^T)^{-1}\gamma \Rightarrow \gamma = \pm\gamma^T \qquad [6.198]$$

6.59 p 191: Eq. (6.5.31), The Orientation Reversing Symmetries of the Oriented String, III

First we note that

$$M^2 = i^2 \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = - \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \mathbb{1} \qquad [6.199]$$

where $\mathbb{1} = \mathbb{1}_{k\times k}$ is the $k \times k$ identity matrix with $2k = N$. So $M^{-1} = M$. Let us now write a state $|N; k, ij\rangle$ as a matrix $|N; k\rangle \otimes \lambda^a_{ij}$. We then have from (6.5.25) and (6.5.30)

$$\begin{aligned}
\Omega_\gamma |N; k; ij\rangle &= \Omega_\gamma |N; k\rangle \otimes \lambda^a_{ij} = \omega_N |N; k\rangle \otimes \gamma_{jj'} \lambda^a_{j'i'} \gamma^{-1}_{i'i} \\
&= \omega_N |N; k\rangle \otimes M_{jj'} \lambda^{a,T}_{i'j'} M^{-1}_{i'i} = \omega_N |N; k\rangle \otimes M_{jj'} \lambda^{a,T}_{i'j'} M_{i'i} \\
&= \omega_N |N; k\rangle \otimes M^T_{j'j} \lambda^{a,T}_{i'j'} M^T_{ii'} = \omega_N |N; k\rangle \otimes (-M_{j'j}) \lambda^{a,T}_{i'j'} (-M_{ii'}) \\
&= \omega_N |N; k\rangle \otimes M_{ii'} \lambda^{a,T}_{i'j'} M_{j'j} = \omega_N |N; k\rangle \otimes (M \lambda^{a,T} M)_{ij} \\
&= \omega_N |N; k\rangle \otimes s^{a'} \lambda^a_{ij} = \omega_N s^{a'} |N; k\rangle \otimes \lambda^a_{ij} = \omega_N s^{a'} |N; k; ij\rangle \qquad [6.200]
\end{aligned}$$

and so $\omega_\gamma = \omega_N s^{a'}$. We therefore have

$$\Omega_\gamma |N; k; ij\rangle = (-1)^{1+\alpha' m} s^{a'} |N; k; ij\rangle \qquad [6.201]$$

and therefore to project out $\omega_\gamma = 1$ for the unoriented string we need $s^{a'} = -1$ if $\alpha' m$ is even. This means that for $\alpha' m$ even, which includes the spin one particles, we need $M(\lambda^a)^T M = s^{a'} \lambda^a = -\lambda^a$, which means that λ^a are in the adjoint representation of $Sp(k)$.

By introducing Chan-Paton factors and projecting out spin one particles in different ways we can thus have unoriented open strings with a spacetime $SO(n)$ or a spacetime $Sp(n/2)$, for n even, gauge symmetry. Recall also that the oriented open string the Chan-Paton factors give a $U(n)$ gauge symmetry.

6.60 p 192: Eq. (6.6.2) The Three Tachyon Tree Amplitude for Closed Strings

This is just a duplication on the open string three tachyon amplitude, so there is no point elaborating on it.

6.61 p 193: Eq. (6.6.4) The Four Tachyon Tree Amplitude for Closed Strings

First we note that the amplitude is calculated for the normal ordering $:e^{ik\cdot X}:$ and not for $\left[e^{ik\cdot X}\right]_r$. As explained in the paragraph under (6.2.31) we can obtain the former from the latter by setting the conformal factor $\omega = 0$, i.e. pushing the curvature to infinity. We can then use (6.2.17) with $\omega = 0$ for the expectation value of the matter part and (6.3.4) for the expectation of the ghost part:

$$\begin{aligned} S_{S_2}(k_1;k_2;k_3;k_4) =& g_c^4 e^{-2\lambda} i C^X_{S_2} C^g_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i) \\ & \times \int_{\mathbb{C}} d^2 z_4\, z_{12} z_{13} z_{23} \bar{z}_{12}\bar{z}_{13}\bar{z}_{23} \prod_{i<j=1}^{4} |z_{ij}|^{\alpha' k_i\cdot k_j} \end{aligned} \quad [6.202]$$

We set $z_1 = 0, z_2 = 1$ and $z_3 = \infty$. Just as for the open string case (6.4.5) the $z_3 \to \infty$ does not cause a problem and we are left with

$$S_{S_2}(k_1;k_2;k_3;k_4) = i g_c^4 C_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i) \int_{\mathbb{C}} d^2 z_4\, |z_4|^{\alpha' k_1\cdot k_4} |1-z_4|^{\alpha' k_2\cdot k_4} \quad [6.203]$$

We now have, introducing Mandelstam variables,

$$\begin{aligned} u =& -(k_1+k_4)^2 = -k_1^2 - k_4^2 - 2k_1\cdot k_4 = -\frac{8}{\alpha'} - 2k_1\cdot k_4 \\ & \Rightarrow \alpha' k_1\cdot k_4 = -\frac{u}{2} - 4 \end{aligned} \quad [6.204]$$

and similarly

$$\begin{aligned} t =& -(k_1+k_3)^2 = -(k_2+k_4)^2 = -k_2^2 - k_4^2 - 2k_2\cdot k_4 = -\frac{8}{\alpha'} - 2k_2\cdot k_4 \\ & \Rightarrow \alpha' k_2\cdot k_4 = -\frac{t}{2} - 4 \end{aligned} \quad [6.205]$$

where we have used the mass-shell condition for closed string tachyons $k_i^2 = -m_i^2 = 4/\alpha'$. Thus

$$S_{S_2}(k_1;k_2;k_3;k_4) = i g_c^4 C_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i) \int_{\mathbb{C}} d^2 z_4\, |z_4|^{-\alpha' u/2-4} |1-z_4|^{-\alpha' t/2-4} \quad [6.206]$$

6.62 p 193: Eq. (6.6.7) The Pole at $\alpha' s = -4$

From $\alpha'(s+t+u) = -16$ we find $-\alpha'(u+t)/2 - 8 = \alpha' s/2$ and so if we look at the behaviour of the amplitude for very large z_4, i.e. say $|z_4| > 1/\varepsilon$ we find

$$\begin{aligned} S_{S_2}(k_1;k_2;k_3;k_4) =& i g_c^4 C_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i) \int_{|z_4|>1/\varepsilon} d^2 z_4\, |z_4|^{-\alpha' u/2-4-\alpha' t/2-4} \\ =& i g_c^4 C_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i) \int_{|z_4|>1/\varepsilon} d^2 z_4\, |z_4|^{\alpha' s/2} \end{aligned} \quad [6.207]$$

We go to radial coordinates $z = re^{i\theta}$, use $d^2z = 2dxdxy = 2rdrd\theta$, and find

$$\begin{aligned} S_{S_2}(k_1;k_2;k_3;k_4) &= ig_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)2\int_0^{2\pi} d\theta \int_{r>1/\varepsilon} rdr\, r^{\alpha' s/2} \\ &= 2ig_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)2\pi \int_{r>1/\varepsilon} dr\, r^{\alpha' s/2+1} \\ &= 4\pi i g_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{1}{\alpha' s/2+2} r^{\alpha' s/2+2}\Big|_{r>1/\varepsilon} \\ &= 8\pi i g_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{1}{\alpha' s+4} r^{\alpha' s/2+2}\Big|_{r>1/\varepsilon} \end{aligned} \quad [6.208]$$

and we indeed see the poles at $\alpha' s + 4 = 0$ for very large z_4.

6.63 p 193: Eq. (6.6.8) The Four-Tachyon Closed String Amplitude and Factorisation

A consequence of unitarity is factorisation of the four-string amplitude is a combination of two three-string amplitude amplitudes with an intermediate state of all possible momenta, see (6.4.13). Let us work this out for the closed string. From (6.4.13) and (6.6.2) we have

$$\begin{aligned} S_{S_2} =& i\int \frac{d^{26}k}{(2\pi)^{26}} \frac{\left(ig_c^3 C_{S_2}(2\pi)^{26}\delta^{26}(k_1+k_2+k)\right)\left(ig_c^3 C_{S_2}(2\pi)^{26}\delta^{26}(-k+k_3+k_4)\right)}{-k^2+4\alpha'^{-1}+i\varepsilon} \\ &+ \text{terms analytic at } \ k^2 = 1/\alpha' \\ =& -\frac{ig_c^6 C_{S_2}^2 (2\pi)^{26}\delta^{26}(k_1+k_2+k_3)}{-(k_1+k_2)^2+4\alpha'^{-1}+i\varepsilon} + \text{terms analytic at } \ k^2 = 4/\alpha' \end{aligned} \quad [6.209]$$

Note that the intermediate string has a pole at the closed string tachyon mass-shell condition. But we also have the expression for the four tachyon amplitude (6.6.4) and we know how it behaves as $k^2 = -s = 4/\alpha'$ from (6.6.7):

$$\begin{aligned} S_{S_2} =& 8i\pi g_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)\frac{1}{\alpha' s+4} r^{\alpha' s/2+2}\Big|_{r>1/\varepsilon} \\ &+ \text{terms analytic at } \ k^2 = 4/\alpha' \end{aligned} \quad [6.210]$$

Equating these two expressions we find

$$\frac{ig_c^6 C_{S_2}^2}{s+4\alpha'^{-1}} = \frac{8i\pi g_c^4 C_{S_2}}{\alpha' s+4} \quad [6.211]$$

or

$$C_{S_2} = \frac{8\pi}{\alpha' g_c^2} \quad [6.212]$$

which is (6.6.8)

6.64 p 193: Eq. (6.6.10) The Virasoro-Shapiro Amplitude

We first work out

$$I(a,b) = \int d^2z\, |z|^{-2a}|1-z|^{-2b} \tag{6.213}$$

Using the representation

$$|z|^{-2a} = \Gamma(a)^{-1}\int_0^\infty dt\, t^{a-1}\exp(-tz\bar{z}) \tag{6.214}$$

that is given as part of exercise 6.10[7]
we have, writing $z = x + iy$,

$$\begin{aligned} I(a,b) &= \frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty dt\, t^{a-1}\int_0^\infty du\, u^{b-1}\int_{-\infty}^{+\infty} 2dxdy\, e^{-t(x^2+y^2)}e^{-u[(1-x)^2+y^2]} \\ &= \frac{2}{\Gamma(a)\Gamma(b)}\int_0^\infty dt\int_0^\infty du\, t^{a-1}u^{b-1}\int_{-\infty}^{+\infty} dxdy\, e^{-(t+u)x^2+2ux}e^{-(t+u)y^2}e^{-u} \end{aligned} \tag{6.215}$$

We can now use the Gaussian integral $\int_{-\infty}^{+\infty} e^{-ax^2} = \sqrt{\pi/a}$ to write

$$\int_{-\infty}^{+\infty} dy\, e^{-(t+u)y^2} = \sqrt{\frac{\pi}{t+u}} \tag{6.216}$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} dx\, e^{-(t+u)x^2+2ux} &= \int_{-\infty}^{+\infty} dx\, e^{-(t+u)\{[x-u/(t+u)]^2-u^2/(t+u)^2\}} \\ &= \sqrt{\frac{\pi}{t+u}}e^{u^2/(t+u)} \end{aligned} \tag{6.217}$$

and thus

$$\begin{aligned} I(a,b) &= \frac{2}{\Gamma(a)\Gamma(b)}\int_0^\infty dt\int_0^\infty du\, t^{a-1}u^{b-1}\frac{\pi}{t+u}e^{u^2/(t+u)}e^{-u} \\ &= \frac{2\pi}{\Gamma(a)\Gamma(b)}\int_0^\infty dt\int_0^\infty du\, \frac{t^{a-1}u^{b-1}}{t+u}e^{-tu/(t+u)} \end{aligned} \tag{6.218}$$

[7] This follows immediately from the definition of the Gamma function $\Gamma(a) = \int_0^\infty dx\, x^{a-1}e^{-x}$. Set $x = t|z|^2$ and find

$$\Gamma(a) = \int_0^\infty dt\, |z|^2 t^{a-1}|z|^{2a-2}e^{-t|z|^2} = |z|^{2a}\int_0^\infty dt\, t^{a-1}e^{-t|z|^2}$$

We now make a change of variables

$$x = \frac{ut}{t+u}; \qquad \lambda = \frac{u}{t+u} \tag{6.219}$$

The inverse transformation is

$$t = \frac{x}{\lambda}; \qquad u = \frac{x}{1-\lambda} \tag{6.220}$$

The Jacobian is

$$J = |\frac{\partial(t,u)}{\partial(x,\lambda)}| = \det \begin{pmatrix} 1/\lambda & -x\lambda^2 \\ 1/(1-\lambda) & x/(1-\lambda)^2 \end{pmatrix} = \frac{x}{\lambda^2(1-\lambda^2)} \tag{6.221}$$

Thus

$$\begin{aligned} I(a,b) &= \frac{2}{\Gamma(a)\Gamma(b)} \int_0^1 d\lambda \int_0^\infty dx \, \frac{x}{\lambda^2(1-\lambda)^2} \frac{\left(\frac{x}{\lambda}\right)^{a-1} \left(\frac{x}{1-\lambda}\right)^{b-1}}{\frac{x}{\lambda(1-\lambda)}} e^{-x} \\ &= \frac{2}{\Gamma(a)\Gamma(b)} \int_0^1 d\lambda \int_0^\infty dx \, x^{1+a-1+b-1-1} \lambda^{-a+1-2+1} (1-\lambda)^{-b+1-2+1} e^{-x} \\ &= \frac{2}{\Gamma(a)\Gamma(b)} \int_0^1 d\lambda \lambda^{-a} (1-\lambda)^{-b} \int_0^\infty dx \, x^{a+b-2} e^{-x} \end{aligned} \tag{6.222}$$

We now use, once more, the definition of the Gamma function

$$\Gamma(z) = \int_0^\infty dx \, x^{z-1} e^{-x} \tag{6.223}$$

to write this as

$$\begin{aligned} I(a,b) &= \frac{2}{\Gamma(a)\Gamma(b)} \Gamma(a+b-1) \int_0^1 d\lambda \, \lambda^{-a} (1-\lambda)^{-b} \\ &= \frac{2\pi\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)} B(-a+1, -b+1) \end{aligned} \tag{6.224}$$

With the Euler Beta function defined as

$$B(x,y) = \int_0^1 dt \, t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{6.225}$$

So we get our final result

$$\begin{aligned} I(a,b) &= \frac{2\pi\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(-a+1)\Gamma(-b+1)}{\Gamma(-a+b+2)} \\ &= 2\pi B(1-a, 1-b, a+b-1) \end{aligned} \tag{6.226}$$

where

$$B(x,y,z) = \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y)\Gamma(x+z)\Gamma(y+z)} \qquad [6.227]$$

This means that we can write the four-tachyon amplitude (6.6.4) as, using (6.6.8),

$$\begin{aligned}
S_{S_2} &= ig_c^4 C_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)\int_{\mathbb{C}} d^2z|z|^{-\alpha' u/2-4}|1-z|^{-\alpha' t-4} \\
&= \frac{8\pi i g_c^2}{\alpha'}(2\pi)^{26}\delta^{26}(\sum_i k_i) I(\alpha' u/4+2, \alpha' t+2) \\
&= \frac{8\pi i g_c^2}{\alpha'}(2\pi)^{26}\delta^{26}(\sum_i k_i) 2\pi B(-1-\alpha' u/4, -1-\alpha' t/4, \alpha' u/4+\alpha' t/4+3) \\
&= \frac{8\pi i g_c^2}{\alpha'}(2\pi)^{26}\delta^{26}(\sum_i k_i) 2\pi B(-\alpha_c(u), -\alpha_c(t), 1+\alpha_c(u)+\alpha_c(t)) \\
&= \frac{8\pi i g_c^2}{\alpha'}(2\pi)^{26}\delta^{26}(\sum_i k_i) C(-\alpha_c(u), -\alpha_c(t))
\end{aligned} \qquad [6.228]$$

with $\alpha_c(x) = \alpha' x/4 + 1$ and

$$C(x,y) = 2\pi B(x, y, 1-x-y) \qquad [6.229]$$

This gives us (6.6.10).

6.65 p 193: Eq. (6.6.12) The Regge Limit of the Virasoro-Shapiro Amplitude

The Regge limit is $s \to \infty$ with t fixed. This also implies $u \propto -s$, see our discussion of p183 (6.4.28). We will use Stirlings formula again: $\Gamma(x+1) \propto x^x e^{-x}\sqrt{2/\pi x}$. We need the Regge behaviour of

$$C(-\alpha_c(t), -\alpha_c(u)) \propto \frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))\Gamma(1+\alpha_c(t)+\alpha_c(u))}{\Gamma(-\alpha_c(t)-\alpha_c(u))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} \qquad [6.230]$$

We first rewrite this in a more symmetric form between the Mandelstam variables. We note that

$$1+\alpha_c(t)+\alpha_c(u) = 3+\frac{\alpha' t}{4}+\frac{\alpha' u}{4} = 3-4-\frac{\alpha' s}{4} = -1-\frac{\alpha' s}{4} = -\alpha_c(s) \qquad [6.231]$$

and

$$-\alpha_c(t)-\alpha_c(u) = -2-\frac{\alpha' t}{4}-\frac{\alpha' u}{4} = -2+4+\frac{\alpha' s}{4} = 2+\frac{\alpha' 1}{4} = 1+\alpha_c(s) \qquad [6.232]$$

Therefore

$$C(-\alpha_c(t), -\alpha_c(u)) \propto \frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))\Gamma(-\alpha_c(s))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} \tag{6.233}$$

Let us first consider our limit for

$$\frac{\Gamma(-\alpha_c(u))}{\Gamma(1+\alpha_c(u))} \to \frac{(-\alpha_c(u))^{-\alpha_c(u)}e^{\alpha_c(u)}\sqrt{2\pi(-\alpha_c(u))}}{(1+\alpha_c(u))^{1+\alpha_c(u)}e^{-1-\alpha_c(u)}\sqrt{2/\pi(1+\alpha_c(u))}}$$
$$\propto u^{-2\alpha_c(u)-1}e^{2\alpha_c(u)} \tag{6.234}$$

where we have used that for $u \to \infty$ we have $\alpha_c(u) \to u$. There is a similar expression for the factors depending on s and therefore, in the Regge limit

$$C(-\alpha_c(t), -\alpha_c(u)) \propto u^{-2\alpha_c(u)-1}e^{2\alpha_c(u)}s^{-2\alpha_c(s)-1}e^{2\alpha_c(s)}\frac{\Gamma(-\alpha_c(t))}{\Gamma(1+\alpha_c(t))} \tag{6.235}$$

Using $u \propto s$ in the Regge limit we get

$$C(-\alpha_c(t), -\alpha_c(u)) \propto s^{-2\alpha_c(u)-2\alpha_c(s)-2}e^{2\alpha_c(u)+2\alpha_c(s)}\frac{\Gamma(-\alpha_c(t))}{\Gamma(1+\alpha_c(t))}$$
$$\propto s^{2\alpha_c(t)}e^{2-2\alpha_c(t)}\frac{\Gamma(-\alpha_c(t))}{\Gamma(1+\alpha_c(t))} \propto s^{2\alpha_c(t)}\frac{\Gamma(-\alpha_c(t))}{\Gamma(1+\alpha_c(t))} \tag{6.236}$$

where we have again used $\alpha_c(s)+\alpha_c(t)+\alpha_c(u) = 1$ and the fact that $\alpha_c(t)$ is kept constant in the Regge limit.

6.66 p 194: Eq. (6.6.13) The Hard Scattering Limit of the Virasoro-Shapiro Amplitude

Recall that the hard scattering limit is the limit of all Mandelstam variables becoming very large, while keeping s/t fixed. We thus have, using [6.233] together with the limit [6.234] and $\alpha_c(x) = 1 + \alpha' x/4$

$$S_{S_2}(k_1;k_2;k_3;k_4) \propto u^{-2\alpha_c(u)-1}e^{2\alpha_c(u)}s^{-2\alpha_c(s)-1}e^{2\alpha_c(s)}t^{-2\alpha_c(t)-1}e^{2\alpha_c(t)}$$
$$= u^{-3-\alpha' u/2}s^{-3-\alpha' s/2}t^{-3-\alpha' t/2}e^{2+\alpha' u/2+2+\alpha' s/2+2+\alpha' t/2}$$
$$\propto u^{-\alpha' u/2}s^{-\alpha' s/2}t^{-\alpha' t/2} \tag{6.237}$$

in the last line we are taking the limit of infinite[8] s, t and u and have used $\alpha'(s+t+u) = -16$. Thus

$$S_{S_2}(k_1; k_2; k_3; k_4) \propto \exp \ln u^{-\alpha' u/2} \exp \ln s^{-\alpha' s/2} \exp \ln t^{-\alpha' t/2} \\ = \exp\left[-\frac{\alpha'}{2}\left(s \ln s + t \ln t + u \ln u\right)\right] \quad [6.238]$$

6.67 p 194: Eq. (6.6.14) The Amplitude for a Massless Closed String and Two Closed String Tachyons

We have a vertex operator for each of the tachyons and a vertex operator for the massless excitations. The two-sphere has zero moduli and six conformal Killing vectors. We thus have six c-ghost insertions and no b-ghost insertions. The Euler number of the two-sphere satisfies $3\chi = \kappa - \mu = 6 = 0 = 6$ and thus $\chi = 2$. This thus gives a contribution to the action of $e^{-\lambda\chi} = e^{-2\lambda}$. This gives us indeed

$$S_{s_2}(k_1, \epsilon_1; k_2; k_3) = g_c^2 g_c' e^{-2\lambda} \epsilon^1_{\mu\nu} \Big\langle :\tilde{c}c \partial X^\mu \bar{\partial} X^\nu e^{ik_1\cdot X} : (z_1, \bar{z}_1) \\ \times :\tilde{c}ce^{ik_2\cdot X} : (z_2, \bar{z}_2) : \tilde{c}ce^{ik_3\cdot X} : (z_3, \bar{z}_3)\Big\rangle \quad [6.239]$$

The expectation value of the matter part is just a special example of (6.2.19), with, of course, $\omega = 0$. We find

$$S^X_{s_2}(k_1, \epsilon_1; k_2; k_3) = \Big\langle : \partial X^\mu \bar{\partial} X^\nu e^{ik_1\cdot X} : (z_1, \bar{z}_1) : e^{ik_2\cdot X} : (z_2, \bar{z}_2) : e^{ik_3\cdot X} : (z_3, \bar{z}_3)\Big\rangle \\ = iC^X_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)|z_{12}|^{\alpha' k_1\cdot k_2}|z_{13}|^{\alpha' k_1\cdot k_3}|z_{23}|^{\alpha' k_2\cdot k_3} \\ \times \left[-\frac{i\alpha'}{2}\left(\frac{k_2^\mu}{z_1 - z_2} + \frac{k_3^\mu}{z_1 - z_3}\right)\right]\left[-\frac{i\alpha'}{2}\left(\frac{k_2^\nu}{\bar{z}_1 - \bar{z}_2} + \frac{k_3^\nu}{\bar{z}_1 - \bar{z}_3}\right)\right] \quad [6.240]$$

The calculation now proceeds in the same way as for the calculation of the amplitude of one massless and to tachyons states of the open string in (6.5.2), expect, of course, that there are no Chan-Paton factors. We just repeat it here for convenience and completeness. Adding the ghost contributions we have

$$S_{s_2}(k_1, \epsilon_1; k_2; k_3) = -\frac{i\alpha'^2}{4} g_c^2 g_c' e^{-2\lambda} C^X_{S_2} C^g_{S_2} (2\pi)^{26}\delta^{26}(\sum_i k_i)\epsilon^1_{\mu\nu} \\ \times |z_{12}|^{\alpha' k_1\cdot k_2 + 2}|z_{13}|^{\alpha' k_1\cdot k_3 + 2}|z_{23}|^{\alpha' k_2\cdot k_3 + 2}\left(\frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}}\right)\left(\frac{k_2^\nu}{\bar{z}_{12}} + \frac{k_3^\nu}{\bar{z}_{13}}\right) \quad [6.241]$$

[8]Note that s is strictly positive and t and u are negative, see [6.109]. The condition $\alpha'(s+t+u) = -16$ and $\lim_{s,t\to\infty} s/t$ fixed, then implies that $\lim_{u,t\to\infty} u/t$ is also fixed and $\lim_{s,t,u\to\infty}(s/t + u/t + 1) = 0$.

To start, we have $C_{S_2} = e^{-2\lambda} C^X_{S_2} C^g_{S_2}$. Next we use momentum conservation and the mass shell condition. The gauge boson is massless so $k_1^2 = 0$. The tachyons have $k_2^2 = k_3^2 = 4/\alpha'$. Thus

$$\frac{4}{\alpha'} = k_3^2 = (-k_1 - k_2)^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = 0 + \frac{4}{\alpha'} + 2k_1 \cdot k_2 \Rightarrow k_1 \cdot k_2 = 0 \qquad [6.242]$$

Similarly $k_1 \cdot k_3 = 0$. Also

$$0 = k_1^2 = (-k_2 - k_3)^2 = k_2^2 + k_3^2 + 2k_2 \cdot k_3 = \frac{8}{\alpha'} + 2k_2 \cdot k_3 \Rightarrow \alpha' k_2 \cdot k_3 = -4 \qquad [6.243]$$

Therefore

$$\begin{aligned} S_{s_2}(k_1, \epsilon_1; k_2; k_3) = & -\frac{i\alpha'^2}{4} g_c^2 g_c' C_{S_2} (2\pi)^{26} \delta^{26}(\sum_i k_i) \epsilon^1_{\mu\nu} \\ & \times |z_{12}|^2 |z_{13}|^2 |z_{23}|^{-2} \left(\frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}} \right) \left(\frac{k_2^\nu}{\bar{z}_{12}} + \frac{k_3^\nu}{\bar{z}_{13}} \right) \end{aligned} \qquad [6.244]$$

We use momentum conservation $k_3 = -k_1 - k_2$ and the fact the ϵ_1 is a polarisation vector of a massless boson, hence $\epsilon_{1\mu\nu} \cdot k_1^\mu = \epsilon_{1\mu\nu} \cdot k_1^\nu = 0$:

$$\begin{aligned} \epsilon_{1\mu\nu} \left(\frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}} \right) z_{12} z_{13} z_{23}^{-1} = & \frac{1}{2} \epsilon_1^{\mu\nu} \left(\frac{k_2^\mu - k_1^\mu - k_3^\mu}{z_{12}} + \frac{k_3^\mu - k_1^\mu - k_2^\mu}{z_{13}} \right) z_{12} z_{13} z_{23}^{-1} \\ = & \frac{1}{2} \epsilon_{1\mu\nu} k_{23}^\mu (z_{13} - z_{12}) z_{23}^{-1} = \frac{1}{2} \epsilon_{1\mu\nu} k_{23}^\mu z_{23} z_{23}^{-1} = \frac{1}{2} \epsilon_{1\mu\nu} k_{23}^\mu \end{aligned} \qquad [6.245]$$

with $k_{ij} = k_i - k_j$. We have similarly

$$\epsilon_{1\mu\nu} \left(\frac{k_2^\nu}{\bar{z}_{12}} + \frac{k_3^\nu}{\bar{z}_{13}} \right) \bar{z}_{12} \bar{z}_{13} \bar{z}_{23}^{-1} = \frac{1}{2} \epsilon_{1\mu\nu} k_{23}^\nu \qquad [6.246]$$

Therefore

$$S_{s_2}(k_1, \epsilon_1; k_2; k_3) = -\frac{i\alpha'^2}{16} g_c^2 g_c' C_{S_2} (2\pi)^{26} \delta^{26}(\sum_i k_i) \epsilon^1_{\mu\nu} k_{23}^\mu k_{23}^\nu \qquad [6.247]$$

Finally we use the relation (6.6.8), i.e. $C_{S_2} = 8\pi/\alpha' g_c^2$ to obtain

$$S_{s_2}(k_1, \epsilon_1; k_2; k_3) = -\frac{i\alpha'\pi}{2} g_c' (2\pi)^{26} \delta^{26}(\sum_i k_i) \epsilon^1_{\mu\nu} k_{23}^\mu k_{23}^\nu \qquad [6.248]$$

which is (6.6.14).

6.68 p 194: Eq. (6.6.15) The Relation between the Coupling Constant of Tachyonic and Massless Closed Strings

From factorisation we know that we can write a scattering of four tachyonic closed string as a combination of three-string amplitudes with all possible intermediate states. We used this in (6.6.8) already to derive the relation between C_{S_2} and the coupling constant g_c for the closed string tachyon vertex operator. From (6.6.6) we know that the four tachyon amplitude for closed strings has poles at $\alpha' s, \alpha' t, \alpha' u = -4, 0, 4, 8, \cdots$, i.e. at the mass-squared of all the intermediate states. The next pole is for the massless intermediate state. Thus, by looking at the behaviour of the factorisation as $s \to 0$ we should be able to link the four tachyon closed string amplitude to two amplitudes of a massless closed string with two tachyonic closed strings, and hence also their vertex couplings.

As $s \to 0$ we should thus have, in analogy with (6.4.13),

$$\lim_{s\to 0} S_{S_2}(k_1; k_2; k_3; k_4) = i \lim_{s\to 0} \sum_{\epsilon} \int \frac{d^{26}k}{(2\pi)^{26}} \frac{S_{S_2}(k, \epsilon; k_1; k_2) S_{S_2}(-k, \epsilon; k_3; k_4)}{-k^2 + i\varepsilon} \qquad [6.249]$$

Note that we do not only have to integrate over all momenta of the intermediate state, but also over all possible polarisation vectors. Let us focus first on the s-pole of the four string amplitude. We use (6.6.10)

$$S_{S_2}(k_1; k_2; k_3; k_4) = \frac{8\pi i g_c^2}{\alpha'} (2\pi)^{26} \delta^{26}(\sum_i k_i) C(-\alpha_c(t), -\alpha_c(u)) \qquad [6.250]$$

with C given by the symmetric form we worked out in [6.233], i.e.

$$C(-\alpha_c(t), -\alpha_c(u)) = 2\pi \frac{\Gamma(-\alpha_c(s))\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} \qquad [6.251]$$

We work out

$$\begin{aligned} \lim_{s\to 0} \frac{\Gamma(-\alpha_c(s)))}{\Gamma(1+\alpha_c(s))} &= \lim_{s\to 0} \frac{\Gamma(-\alpha' s/4 - 1)}{\Gamma(\alpha' s/4 + 2)} = \lim_{s\to 0} \frac{\Gamma(-\alpha' s/4)}{(-\alpha' s/4 - 1)\Gamma(2)} \\ &= -\lim_{s\to 0} \Gamma(-\alpha' s/4) = \frac{4}{\alpha' s} \end{aligned} \qquad [6.252]$$

We have used $\Gamma(x) = (x-1)\Gamma(x-1)$ and $\lim_{\varepsilon\to 0} \Gamma(\epsilon) = \epsilon^{-1} + \text{finite}$ a relation that should be well known if you have studied regularisation of QFTs. From $\alpha'(s+t+u) = -16$ we have that in this limit – and going further we will always assume we are working in this limit – we have $\alpha' t = -\alpha' u - 16$ and thus likewise $\alpha_c(t) = \alpha' t/4 + 1 = -\alpha' u/4 - 4 + 1 =$

$-\alpha' u/4 - 1 - 2 = -\alpha_c(u) - 2$. Therefore

$$\begin{aligned}\frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))}{\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} &= \frac{\Gamma(\alpha_c(u)+2)\Gamma(-\alpha_c(u))}{\Gamma(-\alpha_c(u)-1)\Gamma(1+\alpha_c(u))} \\ &= \frac{(\alpha_c(u)+1)\Gamma(\alpha_c(u)+1)(-\alpha_c(u)-1)\Gamma(-\alpha_c(u)-1)}{\Gamma(-\alpha_c(u)-1)\Gamma(1+\alpha_c(u))} \\ &= (\alpha_c(u)+1)(-\alpha_c(u)-1) = -(\alpha_c(u)+1)^2 \\ &= -\left(\alpha' u/4+2\right)^2 \end{aligned} \qquad [6.253]$$

Now recall form our little excursion on Mandelstam variables that we established that $s \geq |t|$, see [6.112]. If $s \to 0$, this thus implies that $t \to 0$ as well and hence from $\alpha'(s+t+u) = -16$ that $\alpha' u = -16$ and so

$$\frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))}{\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} = -(-4+2)^2 = -4 \qquad [6.254]$$

Bringing all this together we find that

$$\lim_{s\to 0} C(-\alpha_c(t), -\alpha_c(u)) = -2\pi 4 \frac{4}{\alpha' s} = -\frac{32\pi}{\alpha' s} \qquad [6.255]$$

and that the s-pole of the four tachyon amplitude is

$$LHS = \lim_{s\to 0} S_{S_2}(k_1;k_2;k_3;k_4) = -\frac{256\pi^2 i g_c^2}{\alpha'^2}(2\pi)^{26}\delta^{26}(\sum_i k_i) \times \frac{1}{s} \qquad [6.256]$$

Let us now look at the s-pole of the expression with the intermediate massless closed string. The tachyon-tachyon-massless closed string amplitude is given by (6.6.14) so

$$\begin{aligned} RHS &= i \lim_{s\to 0} \sum_{\epsilon} \int \frac{d^{26}k}{(2\pi)^{26}} \frac{S_{S_2}(k,\epsilon;k_1;k_2) S_{S_2}(-k,\epsilon;k_3;k_4)}{-k^2+i\varepsilon} \\ &= i \lim_{s\to 0} \sum_{\epsilon} \int \frac{d^{26}k}{(2\pi)^{26}} \left[-\frac{\pi i \alpha'}{2} g_c' \epsilon_{\mu\nu} k_{12}^{\mu} k_{12}^{\nu} (2\pi)^{26} \delta^{26}(k_1+k_2+k) \right] \\ &\quad \times \left[-\frac{\pi i \alpha'}{2} g_c' \epsilon_{\rho\sigma} k_{34}^{\rho} k_{34}^{\sigma} (2\pi)^{26} \delta^{26}(k_3+k_4-k) \right] \times \frac{1}{-k^2+i\varepsilon} \\ &= -\lim_{s\to 0} \sum_{\epsilon} \frac{\pi^2 i \alpha'^2 g_c'^2}{4} (2\pi)^{26} \delta^{26}(k_1+k_2+k_3+k_4) \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{k_{12}^{\mu} k_{12}^{\nu} k_{34}^{\rho} k_{34}^{\sigma}}{-(k_3+k_4)^2+i\varepsilon} \end{aligned} \qquad [6.257]$$

Since $s = -(k_1+k_2)^2 = -(k_3+k_4)^2$, we can write this as

$$RHS = -\lim_{s\to 0} \sum_{\epsilon} \frac{\pi^2 i \alpha'^2 g_c'^2}{4} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} k_{12}^{\mu} k_{12}^{\nu} k_{34}^{\rho} k_{34}^{\sigma} (2\pi)^{26} \delta^{26}(\sum_i k_i) \times \frac{1}{s} \qquad [6.258]$$

Identifying the pole at $s = 0$ of the LHS with the RHS thus gives

$$\frac{256\pi^2 i g_c^2}{\alpha'^2} = \frac{\pi^2 i\alpha'^2 g_c'^2}{4}\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma \qquad [6.259]$$

or

$$g_c^2 = \frac{\alpha'^4 g_c'^2}{1064}\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma \qquad [6.260]$$

Let us now work out $\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma$. First we note that we can write $k_1 - k_2 = k_1 + k_2 - 2k_2 = k - 2k_2$ and similarly $k_3 - k_4 = k_3 + k_4 - 2k_4 = -k - 2k_4$ and since $\epsilon_{\mu\nu}k^\mu = \epsilon_{\mu\nu}k^\nu = 0$, we have

$$\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma = 16\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_2^\mu k_2^\nu k_4^\rho k_4^\sigma \qquad [6.261]$$

We now use the completeness relation for polarisation vectors familiar from QFT. For a photon in QED it reads $\sum_\epsilon \epsilon^\mu\epsilon^\nu = -g^{\mu\nu}$, so in our case it is $\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma} = g^{\mu\rho}g^{\nu\sigma}$. Therefore

$$\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma = 16(k_2\cdot k_4)^2 \qquad [6.262]$$

But $t = -(k_1+k_3)^2 = -(k_2+k_4)^2 = k_2^2 + k_4^2 + 2k_2\cdot k_4 = 8/\alpha' + 2k_2\cdot k_4$. We have seen that in the limit of $s\to 0$ we also have $t \to 0$ so that $k_2\cdot k_4 = -4/\alpha'$ and thus

$$\sum_\epsilon \epsilon_{\mu\nu}\epsilon_{\rho\sigma}k_{12}^\mu k_{12}^\nu k_{34}^\rho k_{34}^\sigma = \frac{256}{\alpha'^2}. \qquad [6.263]$$

Plugging this in [6.260] we find

$$g_c^2 = \frac{\alpha'^4 g_c'^2}{1064}\frac{256}{\alpha'^2} \quad\Rightarrow\quad g_c^2 = \frac{\alpha'^2}{4}g_c'^2 \qquad [6.264]$$

This means that

$$g_c' = \frac{2}{\alpha'}g_c \qquad [6.265]$$

which is (6.6.15)

6.69 p 194: Eq. (6.6.19) The Amplitude for Three Massless Closed Strings

Eq.(6.6.19) should be clear following the derivation of (6.5.15) i.e. the amplitude for three massless open strings. In particular, we saw there that the terms linear in momentum gives a contribution proportional to $k_{23}^{\mu}\eta^{\alpha\gamma} + k_{31}^{\alpha}\eta^{\mu\gamma} + k_{12}\gamma^{\mu\alpha}$ and the terms cubic in the momenta gives a contribution proportional to $k_{23}^{\mu}k_{31}^{\alpha}k_{12}^{\gamma}$. It is now just a matter of doubling this with the anti-holomorphic sector, contracting with the appropriate polarisation tensors and checking out that the coefficients match. We will not bother doing this.

We will however point out that we see that the amplitude of three massless closed strings is more or less equal to the square of the amplitude of three massless open strings. The massless closed string particles, or at least the symmetric part of the field, are gravitons and the massless open string particles are gauge bosons (and non-abelian ones once we attach Chan-Paton factors). This is a first, albeit very heuristic, sign that

$$\text{gravity} = (\text{gauge theory})^2 \qquad [6.266]$$

6.70 p 195: Eq. (6.6.21) The Relation between $I(x, y, z)$ and $I(x, y)$

From the derivation of (6.6.12) we already showed, see [6.233], that we can write

$$J(s,t,u;\alpha') = C(-\alpha_c(t), -\alpha_c(u)) = 2\pi\frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))\Gamma(-\alpha_c(s))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} \qquad [6.267]$$

From (6.4.20) and ((6.4.24) we also have

$$I(s,t;\alpha') = B\big(-\alpha_o(s), -\alpha_o(t)\big) = \frac{\Gamma(-\alpha_o(s))\Gamma(-\alpha_o(t))}{\Gamma(-\alpha_o(s)-\alpha_o(t))} \qquad [6.268]$$

Let us now calculate

$$\begin{aligned}
\frac{J(s,t,u;\alpha')}{I(s,t;4\alpha')I(t,u;4\alpha')} =& 2\pi\frac{\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))\Gamma(-\alpha_c(s))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))} \\
&\times \frac{\Gamma(-\alpha_c(s)-\alpha_c(t))\Gamma(-\alpha_c(t)-\alpha_c(u))}{\Gamma(-\alpha_c(s))\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(t))\Gamma(-\alpha_c(u))} \\
&= 2\pi\frac{\Gamma(-\alpha_c(s)-\alpha_c(t))\Gamma(-\alpha_c(t)-\alpha_c(u))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(t))\Gamma(1+\alpha_c(u))\Gamma(-\alpha_c(t))} \\
&= 2\pi\frac{\Gamma(1+\alpha_c(u))\Gamma(1+\alpha_c(s))}{\Gamma(1+\alpha_c(s))\Gamma(1+\alpha_c(u))[\Gamma(-\alpha_c(t))\Gamma(1+\alpha_c(t))]} \\
&= 2\pi\frac{1}{\pi/\sin(-\pi\alpha_c(t))} = -2\sin\pi\alpha_c(t)
\end{aligned} \qquad [6.269]$$

We have used $\alpha_c(s) + \alpha_c(t) + \alpha_c(u) + 1 = 0$. This is (6.6.21).

6.71 p 195: Eq. (6.6.23) The Relation between Closed and Open String Four-Point Amplitudes

We first define for any amplitude $S = (2\pi)^{26}\delta^{26}(\sum_i k_i)A$. I don't remember this being mentioned in Joe's book. We thus have from (6.6.4) and (6.6.8)

$$\begin{aligned} A_c(s,t,u;\alpha',g_c) &= \frac{8\pi i g_c^2}{\alpha'}J(s,t,u) \\ &= -\frac{16\pi i g_c^2}{\alpha'}\sin\pi\alpha_c(t)I(s,t;g_0;\alpha'/4)I(t,u;g_0;\alpha'/4) \end{aligned} \quad [6.270]$$

where we have used (6.6.21). We can now rewrite the open string four tachyon amplitude for just one of the six cyclic permutations of (6.4.9), using (6.4.14), as

$$A_o(s,t;\alpha'/4,g_o) = \frac{1}{2}\frac{2ig_o^2}{\alpha'/4}I(s,t) = \frac{4ig_o^2}{\alpha'}I(s,t) \quad [6.271]$$

Note the extra factor $1/2$ because every $I(x,y)$ appears twice in the six cyclic permutations and also note that α' is replaced by $\alpha'/4$. Thus

$$\begin{aligned} A_c(s,t,u;\alpha',g_c) &= -\frac{16\pi i g_c^2}{\alpha'}\sin\pi\alpha_c(t)\left(\frac{\alpha'}{4ig_o^2}A_o(s,t;\alpha'/4,g_o)\right)\left(\frac{\alpha'}{4ig_o^2}A_o(t,u;\alpha'/4,g_o)\right) \\ &= \frac{\pi i g_c^2\alpha'}{g_o^4}A_o(s,t;\alpha'/4,g_o)A_o(t,u;\alpha'/4,g_o) \end{aligned} \quad [6.272]$$

I am not sure why the second A_o is conjugated in Joe's book. As $I(x,y)$ is manifestly positive, this would just change a sign, which is not needed. It may be due to his sign error just below (6.6.21).

6.72 p 195: Eq. (6.6.24-25) The OPE of Two Tachyon Vertex Operators and its Poles

(6.5.24) is just a Taylor Expansion of (2.2.13)

$$\begin{aligned} :e^{ik_1\cdot X(z_1,\bar z_1)}::e^{ik_4\cdot X(z_4,\bar z_4)}: &:= |z_{14}|^{\alpha' k_1\cdot k_4}:e^{ik_1\cdot X(z_1,\bar z_1)}e^{ik_4\cdot X(z_4,\bar z_4)}: \\ &= |z_{14}|^{\alpha' k_1\cdot k_4}:\Big(1 + iz_{14}k_1\cdot\partial X + i\bar z_{14}k_1\cdot\bar\partial X \\ &\qquad - z_{14}\bar z_{14}(k_1\cdot\partial X)(k_1\cdot\bar\partial X)+\cdots\Big)e^{i(k_1+k_4)\cdot X}(z_4,\bar z_4): \end{aligned} \quad [6.273]$$

Let us take the first term, consider the integration over z_{14} and write it in radial coordinates $z_{14} = re^{i\theta}$. This gives a contribution proportional to

$$\int_0^{2\pi} d\theta\int rdr\, r^{\alpha' k_1\cdot k_4} \propto \frac{1}{\alpha' k_1\cdot k_4 + 2} \quad [6.274]$$

and so indeed a pole when $\alpha' k_1 \cdot k_4 = -2$ and convergence for $\alpha' k_1 \cdot k_4 > -2$. We also have $(k_1 + k_4)^2 = k_1^2 + k_4^2 + 2k_1 \cdot k_4 = 8/\alpha' + 2k_1 \cdot k_4$ so that the pole corresponds to

$$-2 = \alpha' k_1 \cdot k_4 + 2 = \frac{\alpha'}{2}(k_1 + k_4)^2 - 4 \Rightarrow -u = (k_1 + k_4)^2 = \frac{4}{\alpha'} \tag{6.275}$$

and so the pole occurs as a on-shell tachyon is created in the u-channel. There will of course be similar poles in the s-channel (from the contractions of the 1 and 2 tachyon vertex operators) and in the t-channel (from the contractions of the 1 and 3 tachyon vertex operators).

The next term in the OPE gives an integration

$$\int_0^{2\pi} d\theta \int r dr \, r^{\alpha' k_1 \cdot k_4} r e^{i\theta} = 0 \tag{6.276}$$

due to the θ integration. Every term in the OPE with a different number of z_{14} and $\bar{z}_{14}$ will similarly give zero due to the integration over θ and it is just terms with equal amount of z_{14} and $\bar{z}_{14}$, or equivalently ∂X and $\bar{\partial} X$ that do not vanish and give poles, corresponding to intermediate particles becoming on-shell.

6.73 p 198: Eq. (6.7.3) The One-Point Function from the Möbius Group

Let us first remins ourselves of some preliminaries. We will focus on the holomorphic side, the anti-holomorphic side being just a copy of this. We are choosing operators $\mathcal{A}(z)$ that are Eigenstates under a rigid rescaling by a complex parameter $z \to z' = \gamma z$, see (2.4.9), i.e. operators that transform as (2.4.13)

$$\mathcal{A}'(z') = \gamma^{-h} \mathcal{A}(z) \tag{6.277}$$

Now, under a general infinitesimal conformal transformation $z \to z' = z + v(z)$ an operator $\mathcal{A}(z)$ transforms as (2.4.12)

$$\delta \mathcal{A}(z) = -\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n v(z) \mathcal{A}^n(z) \tag{6.278}$$

where $\mathcal{A}^n(z)$ are the coefficients of the poles of the OPE of the energy-momentum tensor with the relevant operator, see (2.4.11),

$$T(z)\mathcal{A}(0) \sim \sum_{n=0}^{\infty} \frac{\mathcal{A}^n(0)}{z^{n+1}} \tag{6.279}$$

We have absorbed the infinitesimal parameter ε in $v(z)$ for convenience. Eigenstates under rigid transformation then have the OPE given in (2.4.14)

$$T(z)\mathcal{A}(0) \sim \cdots + \frac{h\mathcal{A}(0)}{z^2} + \frac{\partial\mathcal{A}(0)}{z} \qquad [6.280]$$

From this we read that $\mathcal{A}^1 = h\mathcal{A}$ and $\mathcal{A}^0 = \partial\mathcal{A}$. The infinitesimal rigid conformal transformation is $z \to z + v(z) = \gamma z = (1+\varepsilon)z$, i.e. $v(z) = \varepsilon z$. The only non-vanishing derivative of $v(z)$ is $\partial v(z) = \varepsilon$. Thus

$$\delta\mathcal{A}(z) = \mathcal{A}'(z) - \mathcal{A}(z) = -\left[\partial v(z)\mathcal{A}^1(z) + v(z)\mathcal{A}^0(z)\right] = -\varepsilon h\mathcal{A}(z) - \varepsilon z\partial\mathcal{A}(z) \qquad [6.281]$$

and thus

$$\begin{aligned}\mathcal{A}'(z') &= \mathcal{A}'(z+\varepsilon z) = \mathcal{A}'(z) + \varepsilon z\partial z\mathcal{A}'(z) = \mathcal{A}(z) - \varepsilon h\mathcal{A}(z) - \varepsilon z\partial z\mathcal{A}(z) + \varepsilon z\partial z\mathcal{A}(z)\\ &= \left[1+\varepsilon(-h)\right]\mathcal{A}(z) = (1+\varepsilon)^{-h}\mathcal{A}(z) = \gamma^{-h}\mathcal{A}(z)\end{aligned} \qquad [6.282]$$

Which recovers [6.277] as it should.

We can now establish (6.7.3). Indeed, on the sphere we have three complex Killing vectors, so we can always fix the coordinate to $z = 0$. We then have the transformation $z = 0 \to z' = \gamma z = 0$ and so $\mathcal{A}'(0) = \gamma^{-h}\mathcal{A}(0)$. This gives (6.7.3).

6.74 p 198: Eq. (6.7.4) The Two-Point Function from the Möbius Group

We first perform a Möbius transformation $z \to z - z_2$. This gives $v(z) = -z_2 = c^{\text{te}}$ and so all derivatives of v vanish. Thus[9]

$$\delta\mathcal{A}(z) = \mathcal{A}'(z) - \mathcal{A}(z) = -v(z)\mathcal{A}^0(z) = z_2\partial A(z) \qquad [6.283]$$

and

$$\mathcal{A}'(z') = \mathcal{A}'(z-z_2) = \mathcal{A}'(z) - z_2\partial A(z) = \mathcal{A}(z) + z_2\partial A(z) - z_2\partial A(z) = \mathcal{A}(z) \qquad [6.284]$$

which leads to

$$\langle\mathcal{A}_i(z_1,\bar{z}_1)A_j(z_2,\bar{z}_2)\rangle = \langle\mathcal{A}_i(z_{12},\bar{z}_{12})A_j(0,0)\rangle \qquad [6.285]$$

This is, of course, nothing else but translation invariance of the two-point function.

[9] This is only valide for infinitesimal transformations, but we integrate over all these infinitesimal transformations to end up with a finite transformation.

We now perform a rigid transformation $z \to z' = z_{12}^{-1} z$, i.e. $\gamma = z_{12}$ in (6.7.3). This gives immediately the behaviour

$$\langle \mathcal{A}_i(z_1,\bar{z}_1) A_j(z_2,\bar{z}_2)\rangle = z_{12}^{-h_i-h_j}\,\bar{z}_{12}^{-\tilde{h}_i-\tilde{h}_j}\,\langle \mathcal{A}_i(1,1)A_j(0,0)\rangle \qquad [6.286]$$

Don't be confused by the fact that we have the weights h_i and h_j but z_{ij} only appears in $\mathcal{A}_j$. The scale factor comes from the rescaling of both the operators $\mathcal{A}_i$ and $\mathcal{A}_j$ with the points $z_{12} \to \gamma z = z_{12}^{-1} z_{12} = 1$ and $0 \to \gamma\, 0 = 0$.

Let us now require this two-point function to be single-valued. We replace z_1 and z_2 by $e^{2\pi} z_1$ and $e^{2\pi i} z_2$ respectively and find

$$\begin{aligned}\langle \mathcal{A}_i(z_1,\bar{z}_1) A_j(z_2,\bar{z}_2)\rangle &= e^{-2\pi i(h_i-h_j)}\, z_{12}^{-h_i-h_j}\, e^{+2\pi i(\tilde{h}_i-\tilde{h}_j)}\,\bar{z}_{12}^{-\tilde{h}_i-\tilde{h}_j}\,\langle \mathcal{A}_i(1,1)A_j(0,0)\rangle\\ &= e^{-2\pi i(J_j+J_j)}\,\langle \mathcal{A}_i(1,1)A_j(0,0)\rangle\end{aligned} \qquad [6.287]$$

where $J_i = h_i - \tilde{h}_i$ and $J_j = h_j - \tilde{h}_j$. Single-valuedness indeed requires $J_j + J_j \in \mathbb{Z}$.

6.75 p 199: Eq. (6.7.5) The Two-Point Function Of Tensor Fields

Tensor fields, aka primary fields, have OPE with the energy-momentum tensor

$$T(z)\mathcal{O}(0) \sim \frac{h\mathcal{O}(0)}{z^2} + \frac{\partial\mathcal{O}(0)}{z} \qquad [6.288]$$

i.e. the highest order pole is of degree two and we have $\mathcal{O}^1(z) = h\mathcal{O}(z)$ and $\mathcal{O}^0(z) = \partial\mathcal{O}(z)$.We now consider the conformal transformation $z \to z' = z + \varepsilon(z-z_1)(z-z_2)$, i.e.

$$v(z) = (z-z_1)(z-z_2) \quad ; \quad \partial v(z) = 2z - (z_1+z_2) \quad ; \quad \partial^2 v(z) = 2 \qquad [6.289]$$

Because we are restricting ourselves to primary fields $\partial^2 v(z)$ doesn't contribute and we have

$$\begin{aligned}\delta\mathcal{O}(z) &= -\,\varepsilon\big[\partial v(z)\mathcal{O}^1(z) + v(z)\mathcal{O}^0(z)\big]\\ &= -\,\varepsilon\big[(2z-z_1-z_2)h\mathcal{O}(z) + (z-z_1)(z-z_2)\partial O(z)\big]\end{aligned} \qquad [6.290]$$

Therefore

$$\begin{aligned}\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle &= \langle \mathcal{O}'_i(z_1)\mathcal{O}'_j(z_2)\rangle = \big\langle (\mathcal{O}_i(z_1)+\delta\mathcal{O}_i(z_1))(\mathcal{O}_j(z_2)+\delta\mathcal{O}_j(z_2))\big\rangle\\ &= \Big\langle \big[\mathcal{O}_i(z_1) - \varepsilon(z_1-z_2)h_i\mathcal{O}_i(z_1)\big]\big[\mathcal{O}_j(z_2) + \varepsilon(z_1-z_2)h_j\mathcal{O}_j(z_2)\big]\Big\rangle\\ &= \langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle - \varepsilon(z_1-z_2)(h_i-h_j)\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle\end{aligned} \qquad [6.291]$$

Which implies that

$$(z_1-z_2)(h_i-h_j)\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle = 0 \qquad [6.292]$$

and thus if $h_i \neq h_j$, we necessarily have $\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle = 0$.

6.76 p 199: Eq. (6.7.6) The Three-Point Function Of Tensor Fields

We start by using translational invariance to write

$$\mathfrak{G}_3 = \langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\rangle = \langle \mathcal{O}_i(z_{13})\mathcal{O}_j(z_{23})\mathcal{O}_k(0)\rangle \tag{6.293}$$

In order to proceed we will need to work out how a primary field transforms under a finite Möbius transformation. But this is easily done as a Möbius transformation is also a conformal transformation. Under an infinitesimal conformal transformation $z \to z' = z + \varepsilon v(z)$ a primary field $\mathcal{O}(z)$ of weight h transforms as

$$\delta\mathcal{O}(z) = \mathcal{O}'(z) - \mathcal{O}(z) = -\varepsilon[h\partial v(z)\mathcal{O}(z) + v(z)\partial\mathcal{O}(z)] \tag{6.294}$$

The finite form of this transformation is

$$(\partial z')^h \mathcal{O}'(z') = \mathcal{O}(z) \tag{6.295}$$

as is easily checked by looking at the infinitesimal form $z' = z + \varepsilon v(z)$:

$$\begin{aligned}(\partial z')^h \mathcal{O}'(z') &= [1 + \varepsilon\partial v(z)]^h[\mathcal{O}'(z) + \varepsilon v(z)\partial\mathcal{O}(z)] \\ &= \mathcal{O}'(z) + h\varepsilon\partial v(z)\mathcal{O}(z) + \varepsilon v(z)\partial\mathcal{O}(z)\end{aligned} \tag{6.296}$$

Using this in [6.295] does indeed give [6.294]

We now need the Möbius transformations that transforms the triplet $(z_{13}, z_{23}, 0)$ into $(\Lambda, 1, 0)$. Arguably we should take $\Lambda \to \infty$ but if we do that now we will see that it leads to complications, which are related to the fact that for the point at infinity we actually would have to go to the $u = 1/z$ patch first. But we can fix the coordinates as three points we want, so we can as well keep Λ finite.

The Möbius transformation that achieves this is

$$z \to z' = f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{6.297}$$

with

$$\begin{aligned}\alpha &= -\frac{\Lambda z_{12}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{13}z_{23}}} \\ \beta &= 0 \\ \gamma &= \frac{\Lambda z_{23} - z_{13}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{13}z_{23}}} \\ \delta &= -\frac{(\Lambda-1)z_{13}z_{23}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{13}z_{23}}}\end{aligned} \tag{6.298}$$

It is straightforward algebra to check that $f(0) = 0$, $f(z_{23}) = 1$ and $f(z_{13}) = \Lambda$. It is also straightforward algebra to check that $\alpha\delta - \beta\gamma = 1$ so that this is indeed a Möbius transformation. Another straightforward calculation gives

$$\partial z' = \frac{\partial f}{\partial z} = -\frac{\Lambda(\Lambda-1)z_{12}z_{13}z_{23}}{\left[(\Lambda-1)z_{13}z_{23} + (z_{13} - \Lambda z_{23})z\right]^2} \qquad [6.299]$$

and from this we find that

$$\begin{aligned} \partial z'\Big|_{z=0} &= \frac{\Lambda z_{12}}{(\Lambda-1)z_{13}z_{23}} \\ \partial z'\Big|_{z=z_{23}} &= \frac{(\Lambda-1)z_{13}}{\Lambda z_{12}z_{23}} \\ \partial z'\Big|_{z=z_{13}} &= \frac{\Lambda(\Lambda-1)z_{23}}{z_{12}z_{13}} \end{aligned} \qquad [6.300]$$

These calculation are most easily done in Mathematica. The code for this is shown in fig.6.11. Note that we can take the limit $\Lambda \to \infty$ for the first two derivatives, but not for the last one, that scales as Λ^2.

We can now work out the three-point function

$$\begin{aligned} \mathfrak{G}_3 &= \langle \mathcal{O}_i(z_{13})\mathcal{O}_j(z_{23})\mathcal{O}_k(0)\rangle \\ &= \left\langle \left(\partial z'\Big|_{z=z_{13}}\right)^{h_i} \mathcal{O}_i(\Lambda) \left(\partial z'\Big|_{z=z_{23}}\right)^{h_j} \mathcal{O}_j(1) \left(\partial z'\Big|_{z=0}\right)^{h_k} \mathcal{O}_k(0) \right\rangle \\ &= \left(\frac{\Lambda(\Lambda-1)z_{23}}{z_{12}z_{13}}\right)^{h_i} \left(\frac{(\Lambda-1)z_{13}}{\Lambda z_{12}z_{23}}\right)^{h_j} \left(\frac{\Lambda z_{12}}{(\Lambda-1)z_{13}z_{23}}\right)^{h_k} \langle \mathcal{O}_i(\Lambda)\mathcal{O}_j(1)\mathcal{O}_k(0)\rangle \\ &= z_{12}^{h_k-h_i-h_j}\, z_{13}^{h_j-h_i-h_k}\, z_{23}^{h_i-h_j-h_k}\, \Lambda^{h_i-h_j+h_k}(\Lambda-1)^{h_i+h_j-h_k} \langle \mathcal{O}_i(\Lambda)\mathcal{O}_j(1)\mathcal{O}_k(0)\rangle \\ &= C_{ijk}\, z_{12}^{h_k-h_i-h_j}\, z_{13}^{h_j-h_i-h_k}\, z_{23}^{h_i-h_j-h_k} \end{aligned} \qquad [6.301]$$

with C_{ijk} a constant, independent of the position of the vertex operators z_1, z_2 and z_3. This is (6.7.6).

```
In[43]:= ClearAll [f, a, b, c, d, z1, z2, z3, x, y]
f[z_] := ( a * z + b) / (c * z + d)
a = L * (y - x) / (L * y - x) * c;
b = 0;
d = (1 - L) * x * y / (L * y - x) * c;
c = (L * y - x) / Sqrt [ L * (1 - L) * x * y * (y - x)];
x = z1 - z3;
y = z2 - z3;
Print ["Transformed  Coordinates "]
{f13 = Simplify [f[z1 - z3]], f23 = Simplify [f[z2 - z3]], f33 = Simplify [f[z3 - z3]]}
Print ["Derivatives "]
{df13  = Simplify [D[f[z], z] /. z → z1 - z3], df23  = Simplify [D[f[z], z] /. z → z2 - z3],
df33  = Simplify [D[f[z], z] /. z → z3 - z3]}
Print ["Derivatives  in L→Infinity  Limit "]
{Ldf13  = Simplify [Limit [df13 / L ^ 2, L → Infinity ]],
 Ldf23  = Simplify [Limit [df23 , L → Infinity ]],
Ldf33  = Simplify [Limit [df33 , L → Infinity ]]}
Print ["Jacobian "]
(Ldf13 ^ hi) * (Ldf23 ^ hj) * (Ldf33 ^ hk) /. {z1 - z2 → z12 , z1 - z3 → z13 , z2 - z3 → z23}

Transformed  Coordinates
Out[52]= {L, 1, 0}
Derivatives
```

Out[54]= $\left\{\frac{(-1+L)\,L\,(z2-z3)}{(z1-z2)(z1-z3)}, \frac{(-1+L)(z1-z3)}{L\,(z1-z2)(z2-z3)}, \frac{L\,(-z1+z2)}{(-1+L)(z1-z3)(-z2+z3)}\right\}$

Derivatives in L→Infinity Limit

Out[56]= $\left\{\frac{z2-z3}{(z1-z2)(z1-z3)}, \frac{z1-z3}{(z1-z2)(z2-z3)}, \frac{z1-z2}{(z1-z3)(z2-z3)}\right\}$

Jacobian

Out[58]= $\left(\frac{z12}{z13\ z23}\right)^{hk}\left(\frac{z13}{z12\ z23}\right)^{hj}\left(\frac{z23}{z12\ z13}\right)^{hi}$

Figure 6.11: Mathematica code for the three-point function from a Möbius transformation.

6.77 p 199: Eq. (6.7.7) The Four-Point Function Of Tensor Fields

We follow the same procedure as for the three-point function. First we use translational invariance

$$\mathfrak{G}_4 = \langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)O_\ell(z_4)\rangle = \langle \mathcal{O}_i(z_{14})\mathcal{O}_j(z_{24})\mathcal{O}_k(z_{34})\mathcal{O}_\ell(0)\rangle \qquad [6.302]$$

Next we use the same Möbius transformation as for the three-point function to bring z_{14} to Λ, z_{24} to 1. This is just [6.303] with z_3 replaced by z_4, i.e.

$$\begin{aligned}\alpha &= -\frac{\Lambda z_{12}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{14}z_{24}}}\\ \beta &= 0\\ \gamma &= \frac{\Lambda z_{24}-z_{14}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{14}z_{24}}}\\ \delta &= -\frac{(\Lambda-1)z_{14}z_{24}}{\sqrt{\Lambda(\Lambda-1)z_{12}z_{14}z_{24}}}\end{aligned} \qquad [6.303]$$

Under that Möbius transformation one then finds that

$$z'_{34} = f(z_{34}) = \frac{\Lambda z_{12}z_{34}}{(\Lambda-1)(z_1z_2+z_3z_4)-\Lambda(z_2z_3+z_1z_4)+z_1z_3+z_2z_4} \qquad [6.304]$$

For $\Lambda \to \infty$ this becomes

$$\lim_{\Lambda\to\infty} z'_{34} = \frac{z_{12}z_{34}}{z_{13}z_{24}} = x \qquad [6.305]$$

The partial derivatives are now

$$\begin{aligned}\partial z'\Big|_{z=z_{14}} &= \frac{(\Lambda-1)\Lambda z_{24}}{z_{12}z_{14}}\\ \partial z'\Big|_{z=z_{24}} &= \frac{(\Lambda-1)z_{14}}{\Lambda z_{12}z_{24}}\\ \partial z'\Big|_{z=z_{34}} &= \frac{\Lambda(\Lambda-1)z_{12}z_{14}z_{24}}{[(\Lambda-1)(z_1z_2+z_3z_4)-\Lambda(z_2z_3+z_1z_4)+z_1z_3+z_2z_4]^2}\\ \partial z'\Big|_{z=0} &= \frac{\Lambda z_{12}}{(\Lambda-1)z_{14}z_{24}}\end{aligned} \qquad [6.306]$$

and for $\Lambda \to \infty$ this is

$$\begin{aligned}\lim_{\Lambda\to\infty}\partial z'\Big|_{z=z_{14}} &= \frac{z_{24}}{z_{12}z_{14}}\Lambda^2\\ \lim_{\Lambda\to\infty}\partial z'\Big|_{z=z_{24}} &= \frac{z_{14}}{z_{12}z_{24}}\\ \lim_{\Lambda\to\infty}\partial z'\Big|_{z=z_{34}} &= \frac{z_{12}z_{14}}{z_{13}^2z_{24}}\\ \lim_{\Lambda\to\infty}\partial z'\Big|_{z=0} &= \frac{z_{12}}{z_{14}z_{24}}\end{aligned} \qquad [6.307]$$

Just as for the three-point function, only one of the derivatives does not have a finite limit as $\Lambda - \to \infty$, but scales as Λ^2. These calculations are, once again, best done in Mathematica; the code for this is shown in fig.6.12.

```
ClearAll [f, a, b, c, d, z1, z2, z3, z4, x, y]
f[z_] := ( a * z + b) / (c * z + d)
a = L * (y - x) / (L * y - x) * c;
b = 0;
d = (1 - L) * x * y / (L * y - x) * c;
c = (L * y - x) / Sqrt[ L * (1 - L) * x * y * (y - x)];
x = z1 - z4;
y = z2 - z4;
Print["Transformed  Coordinates "]
{f14 = Simplify [f[z1 - z4]], f24 = Simplify [f[z2 - z4]],
f34 = Simplify [f[z3 - z4]], f44 = Simplify [f[z4 - z4]]}
Print["Derivatives "]
{df14 = Simplify [D[f[z], z] /. z → z1 - z4], df24 = Simplify [D[f[z], z] /. z → z2 - z4],
df34 = Simplify [D[f[z], z] /. z → z3 - z4], df44 = Simplify [D[f[z], z] /. z → z4 - z4]}
Print["Derivatives  in L->Infinity  Limit "]
{Ldf14 = Simplify [Limit[df14 / L ^ 2, L → Infinity ]],
 Ldf24 = Simplify [Limit[df24 , L → Infinity ]],
Ldf34 = Simplify [Limit[df34 , L → Infinity ]], Ldf44 = Simplify [Limit[df44 , L → Infinity ]]}
Print["Jacobian "]
(Ldf14 ^ hi) * (Ldf24 ^ hj) * (Ldf34 ^ hk) * (Ldf44 ^ hl) /.
 {z1 - z2 → z12 , z1 - z3 → z13 , z1 - z4 → z14 , z2 - z3 → z23 , z2 - z4 → z24 , z3 - z4 → z34}
```

Transformed Coordinates

Out[10]= $\left\{L,\ 1,\ \frac{L\,(z1-z2)\,(z3-z4)}{(z2-z3)\,z4+L\,z3\,(-z2+z4)+z1\,((-1+L)\,z2+z3-L\,z4)},\ 0\right\}$

Derivatives

Out[12]= $\left\{\frac{(-1+L)\,L\,(z2-z4)}{(z1-z2)\,(z1-z4)},\ \frac{(-1+L)\,(z1-z4)}{L\,(z1-z2)\,(z2-z4)},\ \frac{(-1+L)\,L\,(z1-z2)\,(z1-z4)\,(z2-z4)}{((z2-z3)\,z4+L\,z3\,(-z2+z4)+z1\,((-1+L)\,z2+z3-L\,z4))^2},\ \frac{L\,(-z1+z2)}{(-1+L)\,(z1-z4)\,(-z2+z4)}\right\}$

Derivatives in L->Infinity Limit

Out[14]= $\left\{\frac{z2-z4}{(z1-z2)\,(z1-z4)},\ \frac{z1-z4}{(z1-z2)\,(z2-z4)},\ \frac{(z1-z2)\,(z1-z4)}{(z1-z3)^2\,(z2-z4)},\ \frac{z1-z2}{(z1-z4)\,(z2-z4)}\right\}$

Jacobian

Out[16]= $\left(\frac{z12}{z14\ z24}\right)^{hl}\left(\frac{z14}{z12\ z24}\right)^{hj}\left(\frac{z12\ z14}{z13^2\ z24}\right)^{hk}\left(\frac{z24}{z12\ z14}\right)^{hi}$

Figure 6.12: Mathematica code for the four-point function from a Möbius transformation.

Filling in all the details, we find for the four point function

$$\begin{aligned}\mathfrak{G}_4 &= \left(\frac{z_{24}}{z_{12}z_{14}}\right)^{h_i}\left(\frac{z_{14}}{z_{12}z_{24}}\right)^{h_j}\left(\frac{z_{12}z_{14}}{z_{13}^2z_{24}}\right)^{h_k}\left(\frac{z_{12}}{z_{14}z_{24}}\right)^{h_\ell}\Lambda^{2h_2}\left\langle\mathcal{O}_i(\Lambda)\mathcal{O}_j(1)\mathcal{O}_k(x)O_\ell(0)\right\rangle \\ &= z_{12}^{-h_i-h_j+h_k+h_\ell}z_{13}^{-2h_k}z_{14}^{-h_i+h_j+h_k-h_\ell}z_{24}^{+h_i-h_j-h_k-h_\ell}\Lambda^{2h_i}\left\langle\mathcal{O}_i(\Lambda)\mathcal{O}_j(1)\mathcal{O}_k(x)O_\ell(0)\right\rangle\end{aligned} \quad [6.308]$$

At first sight this does look nothing like (6.7.7). But let us rewrite the z-factors as

$$\begin{aligned}\mathfrak{z} &= (z_{12}z_{34})^{-h_i-h_j-h_k-h_\ell}z_{12}^{-h_i-h_j+h_k+h_\ell}z_{13}^{-2h_k}z_{14}^{-h_i+h_j+h_k-h_\ell}z_{24}^{+h_i-h_j-h_k-h_\ell} \\ &= z_{12}^{-2h_i-2h_j}z_{13}^{-2h_k}z_{14}^{-h_i+h_j+h_k-h_\ell}z_{24}^{+h_i-h_j-h_k-h_\ell}z_{34}^{-h_i-h_j-h_k-h_\ell} \\ &= z_{12}^{-h_i-h_j}z_{13}^{-h_i-h_k}z_{14}^{-h_i-h_\ell}z_{23}^{-h_j-h_k}z_{24}^{-h_j-h_\ell}z_{34}^{-h_k-h_\ell} \\ &\qquad\times z_{12}^{-h_i-h_j}z_{13}^{+h_i-h_k}z_{14}^{+h_j+h_k}z_{23}^{+h_j+h_k}z_{24}^{+h_i-h_k}z_{34}^{-h_i-h_j} \\ &= z_{12}^{-h_i-h_j}z_{13}^{-h_i-h_k}z_{14}^{-h_i-h_\ell}z_{23}^{-h_j-h_k}z_{24}^{-h_j-h_\ell}z_{34}^{-h_k-h_\ell} \\ &\qquad\times\left(\frac{z_{13}z_{24}}{z_{12}z_{34}}\right)^{h_i}\left(\frac{z_{14}z_{23}}{z_{12}z_{34}}\right)^{h_j}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{h_k}\end{aligned} \quad [6.309]$$

It turns out that we can express the three fractions in terms of x. Indeed we have $x = z_{12}z_{34}/z_{13}z_{24}$, from which it follows that

$$1-x = \frac{z_{14}z_{23}}{z_{13}z_{24}} \quad ; \quad \frac{1-x}{x} = \frac{z_{14}z_{23}}{z_{12}z_{34}} \quad [6.310]$$

and thus

$$\begin{aligned}\mathfrak{z} &= x^{-h_i}\left(\frac{1-x}{x}\right)^{h_j}(1-x)^{h_k}z_{12}^{-h_i-h_j}z_{13}^{-h_i-h_k}z_{14}^{-h_i-h_\ell}z_{23}^{-h_j-h_k}z_{24}^{-h_j-h_\ell}z_{34}^{-h_k-h_\ell} \\ &= x^{-h_i-h_j}(1-x)^{h_j+h_k}\prod_{\substack{i,j=1\\i<j}}^{4}z_{ij}^{-h_i-h_j}\end{aligned} \quad [6.311]$$

The products is a slight abuse of notation, for which I hope I will be forgiven. We thus get for the four-point function

$$\begin{aligned}\mathfrak{G}_4 &= (z_{12}z_{34})^h\left(\prod_{\substack{i,j=1\\i<j}}^{4}z_{ij}^{-h_i-h_j}\right)x^{-h_i-h_j}(1-x)^{h_j+h_k}\Lambda^{2h_i}\left\langle\mathcal{O}_i(\Lambda)\mathcal{O}_j(1)\mathcal{O}_k(x)O_\ell(0)\right\rangle \\ &= C_{ijk\ell}(x)(z_{12}z_{34})^h\prod_{\substack{i,j=1\\i<j}}^{4}z_{ij}^{-h_i-h_j}\end{aligned} \quad [6.312]$$

where $h = h_i + h_j + h_k + h_\ell$. Adding the anti-holomorphic part gives (6.7.7).

One last word about the ratio x, which is known as the anharmonic ratio. This ratio is an invariant under Möbious transformations, in the sense that if calculated for the original coordinates or for the transformed coordinate the result is the same:

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \frac{(z_1' - z_2')(z_3' - z_4')}{(z_1' - z_3')(z_2' - z_4')} \qquad [6.313]$$

as can be checked by direct calculation.

The four-point function [6.312] is thus determined by Möbius invariance up to some function of the anharmonic ratio x. This function is not arbitrary but needs to satisfy some constraints, set by more general conformal transformations. In some cases these constraints allow to determine this function completely and hence we can have an exact expression for the four-point function. This process is known as the conformal bootstrap. This is linked to the associativity of three operators in (2.9.3) and in our fig.2.9.

6.78 p 200: Eq. (6.7.9) The Operator-State Mapping for the Two-Point Function

At this point it could be useful to review the state operator mapping around (2.8.17) or my sections 2.73 and 2.74.

The operator $\mathcal{A}_j$ at $z = 0$ is mapped to the state $\Psi_{\mathcal{A}_j}[\phi_b(z)]$ taken at the unit circle $z = 1$; likewise, the operator $\mathcal{A}_i$ at $u = 0$, or equivalently $z = \infty$ is mapped to the state $\Psi_{\mathcal{A}_i}[\phi_b(u)]$ taken at the unit circle $z = 1$. But we need to take into account that $\mathcal{A}_i$ is taken in the u-patch and we need to write this in the z-patch. Now $z = e^{-i\sigma+\tau}$, so $u = 1/z = e^{+i\sigma-\tau}$. We are working on the unit circle so $z\bar{z} = u\bar{u} = 1$ and thus $e^{2\tau} = 1$ which means that τ is the same in both patches, but going from u to z we have σ going to $-\sigma$. This coordinate, however, is periodic and defined between 0 and 2π, so to bring it back into the range of its definition we see that transforming u to z is the same as transforming σ to $2\pi - \sigma$. All this then leads to the two-point function in the operator-state mapping

$$\left\langle \mathcal{A}_i'(\infty)\mathcal{A}_j(0)\right\rangle_{S_2} = \int [d\phi_b]\Psi_{\mathcal{A}_i}[\phi_b^\Omega]\Psi_{\mathcal{A}_j}[\phi_b] \qquad [6.314]$$

where the integration is over the fields on the boundary, i.e. the unit circle and $\phi_b^\Omega(\sigma) = \phi_b(2\pi - \sigma)$ reflects the transformation from u to z. The impact of moving the vertex operator $\mathcal{A}_i'(\infty)$ from $z = \infty$ to the unit circle is thus to replace σ by $2\pi - \sigma$.

This is of the form $\int dt f(t-a)g(t)$ and thus looks like an inner product, but with the functions taken at different times, $t - a$ and t, i.e. a convolution.

6.79 p 200: Eq. (6.7.11) The Two-Point Function of Primary Fields with Zamolodchikov's Inner Product

We quickly repeat the analysis of the two-point function for operators $\mathcal{A}_i(z_1)$ and $\mathcal{A}_i(0)$ we did for (6.7.4). The difference here is that we wish to take z_1 to infinity rather than to one. We achieve this by a global re-scaling $z \to z' = \Lambda z_1^{-1} z$ and let $\Lambda \to \infty$ This gives immediately

$$\langle \mathcal{A}_i(z_1) A_j(0) \rangle = \left(\frac{z_1}{\Lambda}\right)^{-h_i-h_j} \langle \mathcal{A}_i(\infty) A_j(0) \rangle = z_1^{-h_i-h_j} \Lambda^{h_i+h_j} \langle \mathcal{A}_i(\infty) A_j(0) \rangle \qquad [6.315]$$

For primary operators $\mathcal{O}_i$ and $\mathcal{O}_j$ we have

$$\mathcal{O}_i(z)\mathcal{O}_j(0) = \frac{[\mathcal{O}_i\mathcal{O}_j]_{h_i+h_j}}{z^{h_i+h_j}} + \cdots \qquad [6.316]$$

where $\cdots$ denote lower order poles. In an expectation value on the two-sphere this becomes

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(0) \rangle = \frac{[\mathcal{O}_i\mathcal{O}_j]_{h_i+h_j}}{z^{h_i+h_j}} \langle \mathbb{1} \rangle_{S_2} + \cdots \qquad [6.317]$$

From (6.7.10) we can also write this as

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(0) \rangle = z_1^{-h_i-h_j} \Lambda^{h_i+h_j} \langle\langle i|j \rangle \qquad [6.318]$$

Thus

$$[\mathcal{O}_i\mathcal{O}_j]_{h_i+h_j} \langle \mathbb{1} \rangle_{S_2} = \Lambda^{h_i+h_j} \langle\langle i|j \rangle \qquad [6.319]$$

and thus

$$\mathcal{O}_i(z)\mathcal{O}_j(0) = \frac{\Lambda^{h_i+h_j} \langle\langle i|j \rangle}{z^{h_i+h_j} \langle \mathbb{1} \rangle_{S_2}} + \cdots \qquad [6.320]$$

I am not sure how to get rid of the Λ-factor.

Let us, just for the fun of it, derive [6.315] for a general Möbius transformation that changes z_1 into Λ and keeps 0 fixed. It is easily checkes that the most general Möbius transformation achieving this is $z \to z' = f(z) = (\alpha z + \beta)/(\gamma z + \delta)$ with α a free parameter and

$$\beta = 0 \quad ; \quad \gamma = \frac{\alpha}{\Lambda} - \frac{1}{\alpha z_1} \quad ; \quad \delta = \frac{1}{\alpha} \qquad [6.321]$$

One easily checks that $f(0) = 0$ and $f(z_1) = \Lambda$. We also know from [6.295] that primary fields transform under finite conformal transformations as $(\partial z')^h \mathcal{O}'(z') = \mathcal{O}(z)$ and thus we need the derivatives at z_1 and 0 and these turn out to be

$$\partial z'\Big|_{z=0} = \alpha^2 \quad ; \quad \partial z'\Big|_{z=z_1} \frac{\Lambda^2}{a^2 z_1^2} \tag{6.322}$$

and so we find

$$\begin{aligned} \langle \mathcal{O}_i(z)\mathcal{O}_j(0)\rangle &= \left(\frac{\Lambda^2}{a^2 z_1^2}\right)^{h_i} \left(a^2\right)^{h_j} \langle \mathcal{O}_i(\infty)\mathcal{O}_j(0)\rangle \\ &= z_1^{-2h_i} a^{2(h_j-h_i)} \Lambda^{2h_i} \langle \mathcal{O}_i(\infty)\mathcal{O}_j(0)\rangle \end{aligned} \tag{6.323}$$

This does seem different from [6.318] until we remember from (6.7.5) that the two-point function of primary fields is zero unless both fields have the same conformal weight and we do indeed recover [6.318].

6.80 p 200: Eq. (6.7.14) The Three-Point Function of Primary Fields as a Function of the OPE Coefficients, I

The OPE for a general pair of operators is given by (2.4.20)

$$\mathcal{A}_i(z_1)\mathcal{A}_j(z_2) = \sum_k \frac{c^k_{ij}\mathcal{A}_k(z_2)}{z_{12}^{h_i+h_j-h_k}} \tag{6.324}$$

Setting $z_1 = 1$ and $z_2 = 0$ we get

$$\begin{aligned} \left\langle \mathcal{A}'_i(\infty)\mathcal{A}_k(1)\mathcal{A}_j(0)\right\rangle &= \left\langle \mathcal{A}'_i(\infty)\sum_\ell c^\ell{}_{kj}\mathcal{A}_\ell(0)\right\rangle = \sum_\ell c^\ell{}_{kj}\left\langle \mathcal{A}'_i(\infty)\mathcal{A}_\ell(0)\right\rangle \\ &= \sum_\ell c^\ell{}_{kj}\mathcal{G}_{i\ell} = c_{ikj} \end{aligned} \tag{6.325}$$

6.81 p 200: Eq. (6.7.15) The Three-Point Function of Primary Fields as a Function of the OPE Coefficients, II

We first rescale z as $z \to z' = z_1^{-1} z$ to obtain

$$\left\langle \mathcal{A}'_i(\infty)\mathcal{A}_k(z_1)\mathcal{A}_j(0)\right\rangle = z_1^{h_i-h_k-h_j}\left\langle \mathcal{A}'_i(\infty)\mathcal{A}_k(1)\mathcal{A}_j(0)\right\rangle \tag{6.326}$$

Recall that for $\mathcal{A}'(\infty)$ we are in the u-patch and $z = 1/u$ hence the positive sign of h_i. We then just apply (6.7.14).

6.82 p 201: Eq. (6.7.18) The Four-Point Function of Primary Fields as a Function of the OPE Coefficients

Four a general four-point function where two of the coordinates are unfixed, when we go to the state operator mapping in the Hilbert space formalism we need to define the ordering of the operators.[10] In our case equal-time on the worldsheet means a circle of fixed radius in the complex plane so time ordering becomes radial ordering.Thus, we start from (6.7.16). We now assume $|z_1| > |z_2|$ and insert the complete set – recall that we have assumed that the $\mathcal{A}_i$ form a complete set.

$$\mathfrak{G}_4 = \langle\!\langle i|\, \mathcal{A}_k(z_1)\mathcal{A}_\ell(z_2)\, |j\rangle = \sum_{m,n} \langle\!\langle i|\, \mathcal{A}_k(z_1)\, |m\rangle\, \mathcal{G}^{mn} \langle\!\langle n|\, \mathcal{A}_\ell(z_2)\, |j\rangle \tag{6.327}$$

We now use (6.715) twice

$$\mathfrak{G}_4 = \sum_{m,n} z_1^{h_i - h_k - h_m} c_{ikm} \mathcal{G}^{mn} z_2^{h_n - h_\ell - h_j} c_{n\ell j} \tag{6.328}$$

Now Joe used the $\sum_n \mathcal{G}^{mn}$ to raise $c_{n\ell j}$ to $c^m_{\ell j}$, but there is also a $z_2^{h_2}$ so I am not sure how this exactly works, unless of course $\sum_n \mathcal{G}^{mn} z^{h_n} c_{n\ell j} = z^{h_m} c^m_{\ell j}$.

Let us now check the other way to calculate this that is mentioned in Joe's book. We first perform a translation $z \to z - z_1$ and get

$$\mathfrak{G}_4 = \left\langle \mathcal{A}_i'(\infty)\mathcal{A}_k(0)\mathcal{A}_\ell(z_2 - z_1)\mathcal{A}_j(-z_1)\right\rangle \tag{6.329}$$

In the case of two point z_1 and z_2 that satisfy both $|z_1| > |z_2|$ and $|z_1 - z_2| > |z_2|$, the time ordering gives in the Hilbert space formalism[11]

$$\mathfrak{G}_4 = \langle\!\langle i|\, \mathcal{A}_\ell(z_2 - z_1)\mathcal{A}_k(-z_1)\, |j\rangle \tag{6.330}$$

We now introduce a complete set and get

$$\mathfrak{G}_4 = \sum_m z_{21}^{h_i - h_\ell - h_m} (-z_1)^{h_m - h_k - h_j} c_{i\ell m} c^m_{kj} \tag{6.331}$$

Equating both expression for $\mathfrak{G}_4$ in the region of overlap gives

$$\sum_{m,n} z_1^{h_i - h_k - h_m} z_2^{h_m - h_\ell - h_j} c_{ikm} c^m_{\ell j} = \sum_m z_{21}^{h_i - h_\ell - h_m} (-z_1)^{h_m - h_k - h_j} c_{i\ell m} c^m_{kj} \tag{6.332}$$

[10]Recall form QFT that time-ordering is needed in the Hilbert space mechanism, but that this is automatically included in the path-integral formalism.

[11]Naively one might thinks this is not possible if you think about the radii of circles r_1 and r_2 as we cannot both have $r_1 > r_2$ and $r_1 - r_2 > r_1$, as this would imply $r_2 < 0$. But we are, of course talking about complex number. So an example is $z_1 = 6 + 8i$ and $z_2 = 3 - 5i$. This gives $|z_1| = 10 > |z_2| = \sqrt{34} \approx 5.83$ and $|z_1 - z_2| = \sqrt{178} \approx 13.34 > |z_2| = \sqrt{34} \approx 5.83$.

6.83 p 201: Eq. (6.7.19-22) The Four-Point Function from the Hilbert Space Expression, I

Eq. (2.7.11) states that

$$: X^\mu(z,\bar{z}) X^\nu(z,\bar{z}') :=\, \circ\!\!\circ X^\mu(z,\bar{z}) X^\nu(z,\bar{z}') \circ\!\!\circ \qquad [6.333]$$

Here : are the (conformal) normal ordering symbols subtracting the singular part of the product

$$: X^\mu(z,\bar{z}) X^\nu(z,\bar{z}') := X^\mu(z,\bar{z}) X^\nu(z,\bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu} \ln |z - z'|^2 \qquad [6.334]$$

and $\circ\!\!\circ\ \circ\!\!\circ$ are the creation-annihilation normal ordering symbols, instructing to put all annihilation operators to the right of the creation operators in the normal ordered product of the mode expansions. (2.7.11) just states that for the free theory consisting of the fields X^μ both normal ordering procedures are the same.

The remainder of these equations is just a split of the field X^μ in a creation and annihilation part and a rewriting of the four-point function in the Hilbert space formalism, Eq (6.7.22). Recall from the paragraph above (2.7.10) that p^μ is included in the lowering, i.e. annihilation, operators and x^μ is included in the raising, i.e. creation, operators.

6.84 p 202: Eq. (6.7.19-23) The Four-Point Function from the Campbell-Baker-Hausdorff Formula

More commonly known as the Baker-Campbell-Hausdorff (BCH) formula it is actually given by

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y[X,Y]]+\cdots} \qquad [6.335]$$

In our case $X = ik_1 \cdot X_{1A}$ and $Y = ik_2 \cdot X_{2C}$ and we have that $[X, Y]$ is a c-number so that the BCH formula reduces to

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \qquad [6.336]$$

Thus also, trivially,

$$e^Y e^X = e^{Y+X+\frac{1}{2}[Y,X]} = e^{X+Y-\frac{1}{2}[X,Y]} \qquad [6.337]$$

and

$$\left(e^Y e^X\right)^{-1} = e^{-(X+Y)+\frac{1}{2}[X,Y]} \qquad [6.338]$$

Therefore

$$\left(e^Y e^X\right)^{-1} e^X e^Y = e^{-(X+Y)+\frac{1}{2}[X,Y]} e^{X+Y+\frac{1}{2}[X,Y]} \qquad [6.339]$$

We can once more apply BCH to the RHS and since $[X,[X,Y]] = [Y,[X,Y]] = 0$ we have

$$\left(e^Y e^X\right)^{-1} e^X e^Y = e^{-(X+Y)+\frac{1}{2}[X,Y]+X+Y+\frac{1}{2}[X,Y]} = e^{[X,Y]} \qquad [6.340]$$

Multiplying both sides on the left with $e^Y e^X$ gives

$$e^X e^Y = e^Y e^X e^{[X,Y]} \qquad [6.341]$$

and thus

$$e^{ik_1\cdot X_{1A}} e^{ik_2\cdot X_{2C}} = e^{ik_2\cdot X_{2C}} e^{ik_1\cdot X_{1A}} e^{-[k_1\cdot X_{1A}, k_2\cdot X_{2C}]} \qquad [6.342]$$

Let us now work out the commutator, using (2.75), i.e. $[x^\mu, p^\nu] = i\eta^{\mu\nu}$ and $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$. We consider the holomoprhic side first as the anti-holomorphic side will follow from there easily

$$\begin{aligned}
[X^\mu_{1A}, X^\nu_{2C}]_{\text{hol}} =& [X^\mu_{A\text{ hol}}(z_1), X^\nu_{C\text{ hol}}(z_2)] \\
=& \Big[-\frac{i\alpha'}{2} p^\mu \ln z_1 + i\sqrt{\frac{\alpha'}{2}} \sum_{m=1}^\infty \frac{1}{m} \frac{\alpha^\mu_m}{z_1^m}, x^\nu - i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^\infty \frac{1}{n} \alpha^\nu_{-n} z_2^n \Big] \\
=& -\frac{i\alpha'}{2} \ln z_1 [p^\mu, x^\nu] + \frac{\alpha'}{2} \sum_{m,n=1}^\infty \frac{z_1^{-m} z_2^n}{mn} [\alpha^\mu_m, \alpha^\nu_{-n}] \\
=& -\frac{\alpha'}{2} \eta^{\mu\nu} \ln z_1 + \frac{\alpha'}{2} \sum_{m,n=1}^\infty \frac{z_1^{-m} z_2^n}{mn} m\delta_{m-n}\eta^{\mu\nu} \\
=& -\frac{\alpha'}{2} \eta^{\mu\nu} \ln z_1 + \frac{\alpha'}{2} \eta^{\mu\nu} \sum_{m=1}^\infty \frac{(z_2/z_1)^m}{m} \\
=& -\frac{\alpha'}{2} \eta^{\mu\nu} \ln z_1 - \frac{\alpha'}{2} \eta^{\mu\nu} \ln\left(1 - \frac{z_2}{z_1}\right) \\
=& -\frac{\alpha'}{2} \eta^{\mu\nu} \left(\ln z_1 + \ln \frac{z_1 - z_2}{z_1}\right) = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln z_{12}
\end{aligned} \qquad [6.343]$$

Adding the anti-holomorphic side, which is just a copy we find thus

$$[X^\mu_{1A}, X^\nu_{2C}] = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln z_{12} - \frac{\alpha'}{2} \eta^{\mu\nu} \ln \bar{z}_{12} = -\alpha' \ln |z_{12}| \qquad [6.344]$$

and thus

$$e^{-[k_1\cdot X_{1A}, k_2\cdot X_{2C}]} = e^{\alpha' k_1\cdot k_2 \ln|z_{12}|} = |z_{12}|^{\alpha' k_1\cdot k_2} \qquad [6.345]$$

Plugging this into [6.342] gives the required (6.7.23)

6.85 p 202: Eq. (6.7.24) The Four-Point Function from the Hilbert Space Expression, II

We first use (6.7.23) and then recall that all $\alpha^\mu_{m\geq 1}\,|0;k_3\rangle = 0$ and $\langle\langle 0;k_4|\,\alpha^\mu_{m\leq -1} = 0$. This gives

$$\begin{aligned}\mathfrak{G}_4 &= \langle\langle 0;k_4|\, e^{ik_1\cdot X_{1C}} e^{ik_1\cdot X_{1A}} e^{ik_2\cdot X_{2C}} e^{ik_2\cdot X_{3C}}\,|0;k_3\rangle \\ &= |z_{12}|^{\alpha' k_1\cdot k_2} \langle\langle 0;k_4|\, e^{ik_1\cdot X_{1C}} e^{ik_2\cdot X_{2C}} e^{ik_1\cdot X_{1A}} e^{ik_2\cdot X_{3C}}\,|0;k_3\rangle \\ &= |z_{12}|^{\alpha' k_1\cdot k_2} \langle\langle 0;k_4|\, e^{ik_1\cdot x + ik_2\cdot x} e^{\frac{\alpha'}{2}(k_1\cdot p \ln|z_1|^2 + k_2\cdot p\ln|z_2|^2)}\,|0;k_3\rangle \\ &= |z_{12}|^{\alpha' k_1\cdot k_2} \langle\langle 0;k_4|\, e^{ik_1\cdot x + ik_2\cdot x} e^{\alpha'(k_1\cdot k_3 \ln|z_1| + k_2\cdot k_3\ln|z_2|)}\,|0;k_3\rangle \\ &= |z_{12}|^{\alpha' k_1\cdot k_2}|z_1|^{\alpha' k_1\cdot k_3}|z_2|^{\alpha' k_2\cdot k_3} \langle\langle 0;k_4|\, e^{i(k_1+k_2)\cdot x}\,|0;k_3\rangle \end{aligned} \qquad [6.346]$$

Now, the action of $e^{ik\cdot x}$ on a state is to give it a momentum boost of k. Indeed $p^\mu e^{ik\cdot x}\,|0;K\rangle = (k^\mu e^{ik\cdot x} + e^{ik\cdot x}p^\mu)\,|0;K\rangle = (k^\mu + K^\mu)e^{ik\cdot x}\,|0;K\rangle$. Thus $e^{i(k_1+k_2)\cdot x}\,|0;k_3\rangle \propto |0;k_1+k_2+k_3\rangle$ and we have

$$\mathfrak{G}_4 \propto |z_{12}|^{\alpha' k_1\cdot k_2}|z_1|^{\alpha' k_1\cdot k_3}|z_2|^{\alpha' k_2\cdot k_3}\langle\langle 0;k_4|0;k_1+k_2+k_3\rangle \qquad [6.347]$$

We now use the orthogonality condition (4.1.15), i.e. $\langle 0;k|0;k'\rangle = (2\pi)^D\delta^D(k-k')$ to obtain

$$\mathfrak{G}_4 = C^X_{S_2}|z_{12}|^{\alpha' k_1\cdot k_2}|z_1|^{\alpha' k_1\cdot k_3}|z_2|^{\alpha' k_2\cdot k_3}(2\pi)^D\delta^D(\sum_i k_i) \qquad [6.348]$$

The coefficient C^X_{S-2} is the contribution of the zero modes and the functional determinant, see (6.2.6). In the Polyakov string it combines with the similar ghost coefficient $C^g_{S_2}$ and the cosmological term to form $C_{S_2} = e^{-2\lambda}C^X_{S_2}C^g_{S_2}$. see just below (6.6.2) which in turn is linked to the closed string coupling constant $C_{S_2} = 8\pi/\alpha' g_c^2$, see (6.6.8).

This is indeed (6.2.31) for $n=4$

$$\begin{aligned} A^4_{S_2}(k) &\propto \delta^D(\sum_i k_i)\prod_{\substack{i,j=1\\ i<j}}^{4}|z_{ij}|^{\alpha' k_i\cdot k_j} \\ &= \delta^D(\sum_i k_i)|z_{12}|^{\alpha' k_1\cdot k_2}|z_{13}|^{\alpha' k_1\cdot k_3}|z_{14}|^{\alpha' k_1\cdot k_4}|z_{23}|^{\alpha' k_2\cdot k_3}|z_{24}|^{\alpha' k_2\cdot k_4}|z_{34}|^{\alpha' k_3\cdot k_4} \end{aligned} \qquad [6.349]$$

We set $z_3 = 0$ and take $z_4\to\infty$ to get

$$\begin{aligned} A^4_{S_2}(k) &\propto |z_{12}|^{\alpha' k_1\cdot k_2}|z_1|^{\alpha' k_1\cdot k_3}|z_2|^{\alpha' k_2\cdot k_3}|z_4|^{\alpha'(k_1+k_2+k_3)\cdot k_4}\delta^D(\sum_i k_i) \\ &= |z_{12}|^{\alpha' k_1\cdot k_2}|z_1|^{\alpha' k_1\cdot k_3}|z_2|^{\alpha' k_2\cdot k_3}|z_4|^{-\alpha' k_4^2}\delta^D(\sum_i k_i) \end{aligned} \qquad [6.350]$$

where we have used momentum conservation. If we wish to compare this to [6.348] we need to go to the u-patch with $u = 1/z_4$. As $e^{ik\cdot X}$ is a primary operator with weight $\alpha' k^2/4$ we can use the formula for the (finite) transformation of such an operator $(\partial u)^h (\bar{\partial}\bar{u})^{\tilde{h}} \mathcal{O}'(u, \bar{u}) = \mathcal{O}(z, \bar{z})$. Using $\partial u = -1/z^2$ and the fact that $e^{ik\cdot X}$ is worldsheet scalar so that $\mathcal{O} = \mathcal{O}'$, we find

$$z_4^{-\alpha' k_4^2/2} \bar{z}_4^{-\alpha' k_4^2/2} e^{ik_4\cdot X(u)} = e^{ik_4\cdot X(z_4)} \qquad [6.351]$$

We thus see that if we replace $e^{ik_4\cdot X(z_4)}$ by $e^{ik_4\cdot X(u)}$ we introduce an extra contribution $|z_4|^{+\alpha' k_4^2}$ that cancels the one in [6.350] so that it is indeed the same as [6.348].

Chapter 7

One-Loop Amplitudes

Open Questions

I have a number of unanswered points for this chapter. They are briefly mentioned here and more detail is given under the respective headings. Any help in resolving them can be sent to hepnotes@hotmail.com and is more than welcome.

- ♣ (7.3.8) I am not sure about this formula. It seems like Joe is saying that the ghosts cancel the contribution of two of the non-compact dimensions, and that all the other non-compact dimensions don't contribute to the partition function. That seems strange. I would have expected the following. The theory consists of d scalar fields X and a general CFT that has a Hilbert space $\mathcal{H}\perp$ with highest weights $(h_i, \tilde{h}_i)$. The total central charge of the matter sector plus the CFT is $d + c_{\text{CFT}} = 26$. The ghosts cancel the contribution to the transition function of two of the matter oscillators, but we are still left with $d-2$ matter oscillators that contribute a Dedekind function $|\eta(\tau)|^{2(d-2)}$. I would thus expect the partition function for this theory to be

$$Z_{T^2}\left[X^{[d]};\text{CFT}\right] = iV_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-d/2}|\eta(\tau)|^{2(d-2)} \sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{\tilde{h}_i-1}$$

- ♣ (7.4.5) If we consider the integral

$$I(\Lambda) = \int_0^\Lambda ds\, e^{\beta s} = \frac{1}{\beta}\left(e^{\beta\Lambda} - 1\right)$$

  then the analytic continuation is obtained by just ignoring the Λ dependence. It just means you ignore the divergence. I don't understand why this is an analytic continuation. I also don't understand why the second term gives a divergence $1/0$ as the corresponding integral is $\int_0^\infty ds$. The argument that this looks like a zero-momentum closed string propagator between two disks (i.e. open strings) makes sense and thus gives a $1/k^2$ divergence, but that is a purely heuristic argument.

- ♣ (7.4.11-7.4.13) Joe has completely lost me here. I understand the idea that you have a closed string propagating from a time $\sigma^1 = 0$ to a time $\sigma^1 = s$ and that the partition function is then

$$Z_{C_2} = \langle B|\, c_0 b_0 e^{-s(L_0+\tilde{L}_o)}\, |B\rangle \qquad [7.1]$$

with $|B\rangle$ denoting the closed string state at the boundary. The $b_0 c_0$ are the ghost insertions for the cylinder which has one modulus and one CKV. The central charge term$(c + \tilde{c})/24$ vanishes because in the critical dimension the total central charge of matter plus ghost sector is zero.

But, I don't understand that you can determine the boundary state $|B\rangle$ by requiring it to vanish under $\partial_1 X^\mu$, c^1 and b b_{12}. Why these components? Why not $\partial_2 X^\mu$, c^2 or b_{11}? And what happens with the anti-holomorphic side?

7.1 p 206: The Torus T^2

The Euler number is given by $\chi = 2g - 2$ for a closed oriented surface, and so the only such surface with Euler number zero has one handle, viz. the torus. We already know that the torus has two real moduli, reflected in τ. From the Riemann-Roch theorem $3\chi = \#\,\text{CKVs} - \#\,\text{moduli}$ and thus the number of CKVs must be two as well. Recall from (5.2.8) that the CKVs are the holomorphic vector fields, i.e. the fields satisfying $\bar{\partial}\delta z = \partial\delta\bar{z} = 0$. The only solutions to this that also satisfy the periodic boundary conditions are fixed translations in the two directions of the complex plane.

The sewing procedure in the w-space is illustrated in fig. **22**; the one in the z-space is illustrated in fig. 7.1C … y end of his book that describes all this in exc …

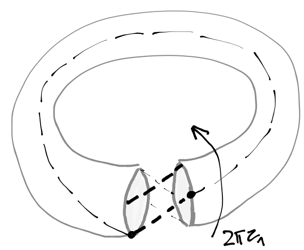


Figure 7.1: The sewing procedure for the torus in w-space

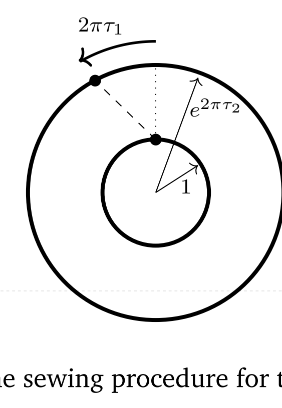


Figure 7.2: The sewing procedure for the torus in z-space

7.2 p 206: The Cylinder C_2

In the w-space the cylinder C_2 is a rolled up strip: the thick ends are identified. Taking $t \to 0$ gives a very fine cylinder, more like capellini; taking very large t makes it look more like cannelloni. The cylinder has one moduli t and so there is also only one CKV.

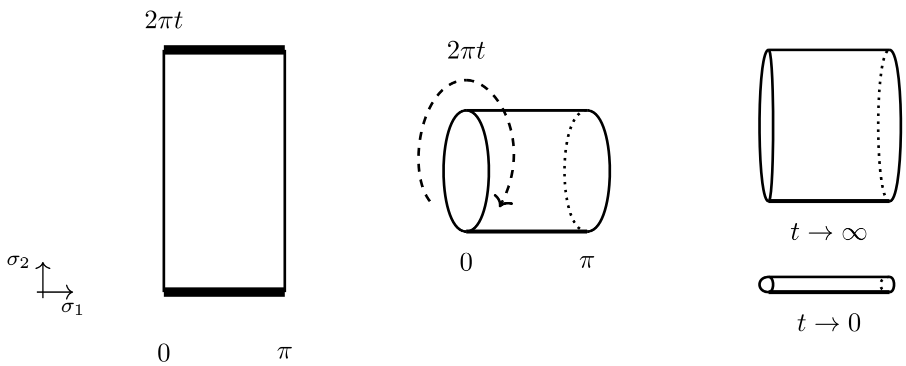


Figure 7.3: The cylinder in w-space

To obtain the cylinder from the torus we first take a torus with $\tau = it$, i.e. $\tau_1 = 2$ and $\tau_2 = t$ so there is no stretching, and identify $w \equiv -\bar{w}$. Writing $w = \sigma^1 + i\sigma^2$ this is indeed equivalent to setting $\sigma^1 \equiv -\sigma^1$ or identification under a reflection through the imaginary axis. Under this reflection $\sigma^1 = 0$ obviously remains fixed and $\sigma^1 = \pi$ goes to $-\pi$, but due to periodicity this is the same as π and so $\sigma^1 = \pi$ remains fixed as well. We thus have a surface with 2 boundaries and one periodic direction, i.e. a cylinder.

Recall the formula for the Euler number (3.5.6) for surface g handles, b boundaries and c cross-caps

$$\chi = 2 - 2g - b - c \qquad [7.2]$$

The cylinder has $g = 0$, $b = 2$ and $c = 0$, hence χ is indeed zero.

7.3 p 206: The Klein Bottle K_2

We start from a strip in the complex plane with base 2π and height $2\pi t$. We first make a cylinder by rolling up, then we perform a reflection around the imaginary axis and identify the boundary. This is illustrated in fig.7.5

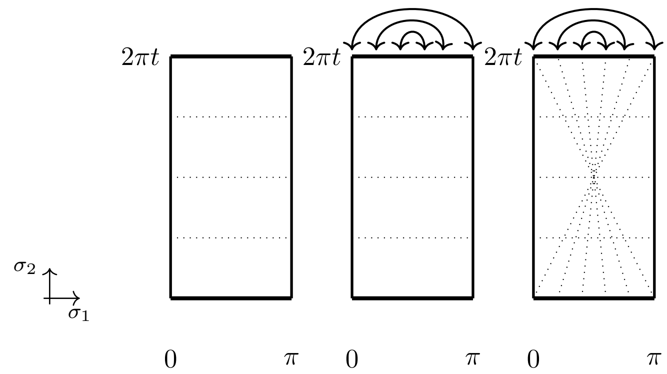
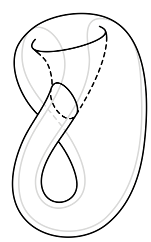


Figure 7.4: The Klein bottle in w-space. The points connected by a dotted line are identified with one another,

A more artistic representation of the Klein bottle is given in fig.3.33 of these notes, which we reproduce here for convenience.



The Klein bottle is a two-sphere with two cross-caps, no handles and no boundaries. From (3.5.6) we then have $\chi = 2 - 0 - 0 - 2 = 0$ and it is indeed an Euler number zero Riemann surface. There is one modulus, t, and hence also one CKV, translation in the σ_2 direction.

7.4 p 206: The Möbius Strip M_2

For the Möbius strip we identify $w \equiv -\bar{w} + \pi + 2\pi t$ or

$$\begin{aligned} \sigma^1 &\equiv -\sigma^2 1 + \pi \\ \sigma^2 &\equiv \sigma^2 + 2\pi t \end{aligned} \quad [7.3]$$

There is periodicity in the σ^2 direction but one of the ends of the strip is twisted

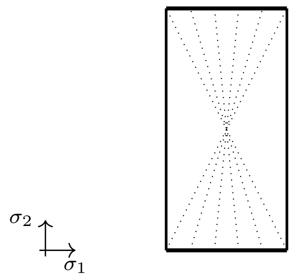


Figure 7.5: The Möbius strip in w-space. The points connected by a dotted line are identified with one another,

7.5 p 208: Eq. (7.2.1) The Equation for the Green's Function on the Torus T^2

Recall that in section 6.2 Joe worked out the partition function for the bosonic string on the sphere by expanding the filed X^μ in a complete set $X^\mu(\sigma) = \sum_I x_I^\mu \mathsf{X}_I$. When we then considered the partition function with source $Z[J] = \big\langle \exp\big[i \int d^2\sigma\, J(\sigma)\cdot X(\sigma)\big]\big\rangle$, all the integrals were Gaussian except for the zero-mode X_0, i.e. the mode that satisfies $\nabla^2 \mathsf{X}_0 = 0$. The integral over the zero was linear and resulted in momentum conservation. The Gaussian integrals could be performed and give a functional determinant and the usual Green's function contribution of the form $\exp\big[-\frac{1}{2}\int d^2\sigma\, d^2\sigma'\, J(\sigma)\cdot J(\sigma')\widetilde{G}'(\sigma,\sigma')\big]$. Here $\widetilde{G}'(\sigma,\sigma')$ is the Green's function with the zero mode excluded, (6.2.7)

$$\widetilde{G}'(\sigma,\sigma') = \sum_{I\neq 0} \frac{2\pi\alpha'}{\omega_I^2}\mathsf{X}_I(\sigma_1)\mathsf{X}_I(\sigma_2) \qquad [7.4]$$

where ω_I^2 are the Eigenvalues of the complete set, $\nabla^2\mathsf{X}_I = -\omega_I^2\mathsf{X}_I$. Because we are dividing the RHS by ω_I^2 we see why we had to exclude the zero mode. This Green's function satisfies the PDE (6.2.8)

$$-\frac{1}{2\pi\alpha'}\nabla^2\widetilde{G}'(\sigma_1,\sigma_2) = g^{-1/2}\delta^2(\sigma_1-\sigma_2) - \mathsf{X}_0^2 \qquad [7.5]$$

When discussing (6.2.8) we explained the appearance of the zero-mode in this equation as linked to the fact that on a compact surface without boundary, such as the two-sphere or torus, the Poisson equation $\nabla^2\phi = \delta^2(\sigma)$ has no solution. The physical explanation for this is the equation for an electric field create by a point charge at $\sigma = 0$, but on a compact surface without boundary these electric field lines have nowhere to end. We can solve this by adding a constant term to the Poisson equation $\nabla^2\phi = \delta^2(\sigma) - \kappa^{-1}$. This constant term can be viewed as a density charge over the surface, that cancels the point charge. As a final comment, we also noted that the zero mode is essentially the inverse surface of the manifold $\mathsf{X}_0 = \big(\int d^2\sigma\sqrt{g}\big)^{-1/2}$, see (6.2.5).

All of this is repetition only, but it helps us understand (7.2.1). Indeed, in the conformal gauge $g_{ab} = e^{2\omega}\delta_{ab}$ we have $\nabla^2 = 4e^{-2\omega}\partial\bar{\partial}$ and $\sqrt{g} = e^{2\omega}$; moreover using also $\delta^2(\sigma) = 2\delta^2(z)$, (6.2.8) becomes

$$\frac{2}{\alpha'}\partial\bar{\partial}G(w,\bar{w};w',\bar{w}') = -2\pi\delta^2(w-w') + \pi e^{2\omega}\mathsf{X}_0^2 \tag{7.6}$$

What remains to show is that

$$\pi e^{2\omega}\,\mathsf{X}_0^2 = \frac{1}{4\pi\tau_2} \tag{7.7}$$

or that for the torus

$$\pi e^{2\omega}\,\Big(\int d^2\sigma\sqrt{g}\Big)^{-1} = \frac{1}{4\pi\tau_2} \tag{7.8}$$

or finally

$$\int d^2\sigma\sqrt{g} = 4\pi^2 e^{2\omega}\tau_2 \tag{7.9}$$

Does this makes sense? We can represent the torus in the complex plane as a parallelogram with sides 2π and $2\pi\tau$, see fig.7.6, Withe the curvature pulled to infinity, i.e. $\omega = 0$ the area of the torus is exactly $4\pi^2\,\mathrm{Im}\,\tau = 4\pi^2\tau_2$ and thus indeed

$$\int d^2\sigma\sqrt{g} = 4\pi^2\tau_2 \tag{7.10}$$

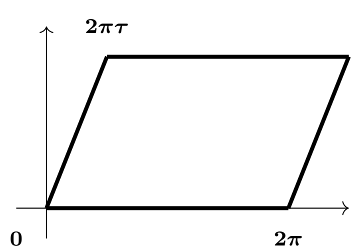


Figure 7.6: The torus as a parallelogram with sides 2π and $2\pi\tau$. The surface area of the torus is $4\pi^2\,\mathrm{Im}\,\tau$

7.6 p 208: Eq. (7.2.3) The Green's Function on the Torus

We need to show two things: (1) G' satisfies the equation (7.2.1) and (2) G' is doubly periodic, i.e. it is periodic under $w \to w+1$ and under $w \to w + 2\pi\tau$. We start with the latter property.

Obviously, if we add one or $2\pi\tau$ to both w and w' the Green's function remains the same, but we must also have periodicity if we transform only one of the coordinates. In order to avoid any confusion, let us call $\widetilde{G}'(w,\bar{w};w',\bar{w}')$ the Green's function in (7.2.2) and $G'(w,\bar{w};w',\bar{w}')$ the Green's function in (7.2.3). If we replace $w \to w+1$ in $\widetilde{G}'(w,\bar{w};w',\bar{w}')$ we get

$$\widetilde{G}'(w,\bar{w};w',\bar{w}') \to -\frac{\alpha'}{2}\left[\ln\vartheta_1\Big(\frac{w-w'}{2\pi}+1,\tau\Big) + \ln\overline{\vartheta_1\Big(\frac{w-w'}{2\pi}+1,\tau\Big)}\right] \qquad [7.11]$$

Now from (7.2.37d) $\vartheta_1(\nu,\tau) = -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau)$ and from (7.2.36) we have, using (7.2.32a),

$$\begin{aligned}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu+1,\tau) &= e^{\pi i a^2\tau + 2\pi i a(\nu+1+b)}\vartheta(\nu + a\tau + b + 1,\tau)\\ &= e^{2\pi i a}e^{\pi i a^2\tau+2\pi i a(\nu+b)}\vartheta(\nu+a\tau+b,\tau)\\ &= e^{2\pi i a}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau)\end{aligned} \qquad [7.12]$$

and thus

$$\vartheta_1(\nu+1,\tau) = -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu+1,\tau) = -e^{2\pi i a}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau) = e^{2\pi i a}\vartheta_1(\nu,\tau) \qquad [7.13]$$

Therefore

$$\begin{aligned}\widetilde{G}'(w,\bar{w};w',\bar{w}') &\to -\frac{\alpha'}{2}\left\{\ln\left[e^{-2\pi a}\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\right] + \ln\left[e^{2\pi a}\vartheta_1\Big(\frac{w-w'}{2\pi}\Big)\right]^*\right\}\\ &= -\frac{\alpha'}{2}\ln\left[e^{2\pi i a}\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)e^{-2\pi i a}\overline{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}\right]\\ &= -\frac{\alpha'}{2}\ln\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\overline{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)} = \widetilde{G}'(w,\bar{w};w',\bar{w}')\end{aligned} \qquad [7.14]$$

and so we indeed have periodicity under $w \to w+1$ for $\widetilde{G}'(w,\bar{w};w',\bar{w}')$. This clearly also implies periodicity of $G'(w,\bar{w};w',\bar{w}')$ as $\mathrm{Im}(w-w') \to \mathrm{Im}(w+1-w') = \mathrm{Im}(w-w')$.

Now, if we replace $w \to w + 2\pi\tau$, we get

$$\widetilde{G}'(w,\bar{w};w',\bar{w}') \to -\frac{\alpha'}{2}\left[\ln\vartheta_1\Big(\frac{w-w'}{2\pi}+\tau,\tau\Big) + \ln\overline{\vartheta_1\Big(\frac{w-w'}{2\pi}+\tau,\tau\Big)}\right] \qquad [7.15]$$

From (7.2.37d) $\vartheta_1(\nu,\tau) = -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau)$ and from (7.2.36) we have, using (7.2.32b),

$$\begin{aligned}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu+\tau,\tau) &= e^{\pi i a^2\tau + 2\pi i a(\nu+\tau+b)}\vartheta(\nu+a\tau+b+\tau,\tau)\\ &= e^{2\pi i a\tau}e^{\pi i a^2\tau+2\pi i a(\nu+b)}e^{-\pi i\tau - 2\pi i(\nu+a\tau+b)}\vartheta(\nu+a\tau+b,\tau)\\ &= e^{-\pi i\tau-2\pi i\nu-2\pi i b}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau)\end{aligned} \quad [7.16]$$

and thus

$$\begin{aligned}\vartheta_1(\nu+\tau,\tau) &= -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu+\tau,\tau) = -e^{-2\pi i\nu-\pi i\tau-\pi i}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau)\\ &= e^{-2\pi i\nu-\pi i\tau}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\nu,\tau) = -e^{-2\pi i\nu-\pi i\tau}\vartheta_1(\nu,\tau)\end{aligned} \quad [7.17]$$

and thus

$$\begin{aligned}\widetilde{G}'(w,\bar{w};w',\bar{w}') \to &-\frac{\alpha'}{2}\left\{\ln\left(-e^{-2\pi i\frac{w-w'}{2\pi}-\pi i\tau}\vartheta_1\left(\frac{w-w'}{2\pi},\tau\right)\right)\right.\\ &\left.+\ln\left(-e^{-2\pi i\frac{w-w'}{2\pi}-\pi i\tau}\vartheta_1\left(\frac{w-w'}{2\pi}\right)\right)^*\right\}\\ =&-\frac{\alpha'}{2}\ln\left[e^{-i(w-w')-\pi i\tau}\vartheta_1\left(\frac{w-w'}{2\pi},\tau\right)e^{+i(\bar{w}-\bar{w}')+\pi i\bar{\tau}}\overline{\vartheta_1\left(\frac{w-w'}{2\pi},\tau\right)}\right]\\ =&-\frac{\alpha'}{2}\ln e^{-i[(w-w')-(\bar{w}-\bar{w}')]-\pi i(\tau-\bar{\tau})} - \frac{\alpha'}{2}\ln\vartheta_1\left(\frac{w-w'}{2\pi},\tau\right)\overline{\vartheta_1\left(\frac{w-w'}{2\pi},\tau\right)}\\ =&-\frac{\alpha'}{2}\ln e^{2\mathrm{Im}(w-w')+2\pi\mathrm{Im}(\tau)}+\widetilde{G}'(w,\bar{w};w',\bar{w}')\\ =&-\alpha'\big[\mathrm{Im}(w-w')+\pi\tau_2\big]+\widetilde{G}'(w,\bar{w};w',\bar{w}')\end{aligned} \quad [7.18]$$

and so the periodicity condition indeed picks up a term $-\alpha'\big[\mathrm{Im}(w-w')+\pi\tau_2\big]$. On the other hand we have that under $w \to w+2\pi\tau$

$$\begin{aligned}[\mathrm{Im}(w-w')]^2 &\to [\mathrm{Im}(w+2\pi\tau-w')]^2 = [\mathrm{Im}(w-w')+2\pi\tau_2]^2\\ &= [\mathrm{Im}(w-w')]^2 + 4\pi\tau_2\mathrm{Im}(w-w') + 4\pi^2\tau_2^2\end{aligned} \quad [7.19]$$

and so

$$\frac{\alpha'[\mathrm{Im}(w-w')]^2}{4\pi\tau_2} \to \frac{\alpha'[\mathrm{Im}(w-w')]^2}{4\pi\tau_2} + \alpha'\big[\mathrm{Im}(w-w')+\pi\tau_2\big] \quad [7.20]$$

Combining the two we thus find that indeed $G'(w,\bar{w};w',\bar{w}')$ is also periodic under $w \to w+2\pi\tau$.

Let us now show that $G'(w,\bar{w};w',\bar{w}')$ also satisfies (7.2.1). Let us first consider the case of $\widetilde{G}'(w,\bar{w};w',\bar{w}')$ with $w \neq w'$. In that case we have $\widetilde{G}'(w,\bar{w};w',\bar{w}')$ is the sum of a holomorphic and an anti-holomorphic function of w and so clearly $\partial\bar{\partial}G'(w,\bar{w};w',\bar{w}')=0$ when $w\neq w'$. Just as for the case of the sphere, we need to be careful at coinciding points because we expect that this will generate the delta function. Now the $\vartheta_1(\nu,\tau)$ function has a Taylor expansion in ν and the constant term $\vartheta_1(0,\tau)$ is zero by (7.2.42). Therefore, near $\nu=0$,

$$\vartheta_1(\nu,\tau)=\nu\,\vartheta_1'(0,\tau)+o(\nu^2) \tag{7.21}$$

where $\vartheta_1'(\nu,\tau)=\partial_\nu\vartheta_1(\nu,\tau)$. We thus have near $w=w'$

$$\begin{aligned}
-\frac{\alpha'}{2}&\left[\ln\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)+\ln\overline{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}\right]\approx\\
&-\frac{\alpha'}{2}\left\{\ln\left[\frac{w-w'}{2\pi}\vartheta_1'(0,\tau)+o\big(w-w')^2\big)\right]+\ln\left[\frac{w-w'}{2\pi}\vartheta_1'(0,\tau)+o\big(w-w')^2\big)\right]^*\right\}\\
=&-\frac{\alpha'}{2}\left[\ln|w-w'|^2+\ln\left|\frac{\vartheta_1'(0,\tau)}{2\pi}\right|^2+o\big(w-w')^2\big)\right]
\end{aligned} \tag{7.22}$$

and thus as $w\to w'$ we have the same behaviour as on the sphere – which makes sense as at coinciding points the global topology should not play a role in this – and we indeed have

$$\frac{2}{\alpha'}\lim_{w\to w'}\partial\bar{\partial}\widetilde{G}'(w,\bar{w};w',\bar{w}')=-2\pi\delta^2(w-w') \tag{7.23}$$

Let us now check the second second term in $G'(w,\bar{w};w',\bar{w}')$:

$$\begin{aligned}
\frac{2}{\alpha'}\bar{\partial}\partial\left\{\alpha'\frac{\big[\mathrm{Im}(w-w')\big]^2}{4\pi\tau_2}\right\}&=\frac{1}{2\pi\tau_2}\bar{\partial}\partial\left(\frac{w-w'-\bar{w}+\bar{w}'}{2i}\right)^2\\
&=-\frac{1}{8\pi\tau_2}\bar{\partial}\partial(-2w\bar{w})=\frac{1}{4\pi\tau_2}
\end{aligned} \tag{7.24}$$

The function $k(\tau,\bar{\tau})$ obviously satisfies $\bar{\partial}\partial k(\tau,\bar{\tau})=0$ and we thus conclude that $G'(w,\bar{w};w',\bar{w}')$ indeed satisfies (7.2.1).

7.7 p 209: Eq. (7.2.4) The Expectation Value of Vertex Operators on the Torus

The derivation is entirely similar as for the expectation value of vertex operators on the sphere in the (6.2.13). We refer to that derivation for details. The first ting we did there was to extract the regularised part of the vertex operators [6.33]

$$\left[e^{ik_i\cdot X(\sigma_i)}\right]_r=\exp\left(-\frac{1}{2}\frac{\alpha' k_i^2}{2}\ln d^2(\sigma_i,\sigma_i)\right)e^{ik_i\cdot X(\sigma_i)} \tag{7.25}$$

Using this we obtained via a straightforward calculation that [6.39]

$$A^n_{S^2}(k,\sigma) = iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\left[-\sum_{i<j=1}^{n} k_i\cdot k_j G'(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i=1}^{n} k_i^2 G'_r(\sigma_i,\sigma_i)\right] \tag{7.26}$$

where we defined

$$G'_r(\sigma_i,\sigma_j) = G'(\sigma_i,\sigma_j) + \frac{\alpha'}{2}\ln d^2(\sigma_i,\sigma_j) \tag{7.27}$$

with d the geodesic distance, which at short distance is given by (3.6.9)

$$d^2(\sigma_1,\sigma_2) = (\sigma_1-\sigma_2)^2 e^{2\omega(\sigma)} = |z_{12}|^2 e^{2\omega(\sigma)} \tag{7.28}$$

We now need to work out G'_r at coinciding points:

$$\lim_{w'\to w} G'_r(w,\bar{w};w')\bar{w}' = \lim_{w'\to w} G'(w,\bar{w};w') + \frac{\alpha'}{2}\ln|w-w'|^2 + \alpha'\omega(w) \tag{7.29}$$

But we saw in [7.21] that as $\nu \to 0$

$$\vartheta_1(\nu,\tau) = \nu\,\vartheta_1'(0,\tau) + o(\nu^2) \tag{7.30}$$

and therefore

$$\lim_{w'\to w} G'(w,\bar{w};w') = -\frac{\alpha'}{2}\left[\ln|w-w'|^2 + \ln\left|\frac{\partial_\nu\vartheta_1(0,\tau)}{2\pi}\right|^2 + o\big(w-w')^2\big)\right] \tag{7.31}$$

so that

$$\lim_{w'\to w} G'_r(w,\bar{w};w') = -\frac{\alpha'}{2}\left[\ln\left|\frac{\partial_\nu\vartheta_1(0,\tau)}{2\pi}\right|^2 + \alpha'\omega(w)\right] \tag{7.32}$$

Bringing it all together we find

$$\begin{aligned}
A^n_{S^2}(k,\sigma) =& iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\Bigg\{ -\sum_{i<j=1}^n k_i\cdot k_j\Bigg(-\frac{\alpha'}{2}\ln\Big|\vartheta_1\Big(\frac{w_{ij}}{2\pi},\tau\Big)\Big|^2 \\
&+\frac{\alpha'(\text{Im}(w_{ij}))^2}{4\pi\tau_2}\Bigg) - \frac{1}{2}\sum_{i=1}^n k_i^2\left[-\frac{\alpha'}{2}\left(\ln\left|\frac{\partial_\nu\vartheta_1(0,\tau)}{2\pi}\right|^2 + \alpha'\omega(w_i)\right)\right]\Bigg\} \\
=& iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\Bigg\{\sum_{i<j=1}^n \alpha' k_i\cdot k_j\ln\left[\Big|\vartheta_1\Big(\frac{w_{ij}}{2\pi},\tau\Big)\Big|\exp\left(-\frac{\alpha'(\text{Im}w_{ij})^2}{4\pi\tau_2}\right)\right] \\
&\times -\frac{1}{2}\sum_{i=1}^n\alpha' k_i^2\omega(w_i) + \sum_{i=1}^n\frac{\alpha'}{2}k_i^2\ln\left|\frac{\partial_\nu\vartheta_1(0,\tau)}{2\pi}\right|\Bigg\} \\
=& iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\Bigg\{\sum_{i<j=1}^n \ln\left[\Big|\vartheta_1\Big(\frac{w_{ij}}{2\pi},\tau\Big)\Big|\exp\left(-\frac{\alpha'(\text{Im}w_{ij})^2}{4\pi\tau_2}\right)\right]^{\alpha' k_i\cdot k_j} \\
&-\frac{1}{2}\sum_{i=1}^n\alpha' k_i^2\omega(w_i) - \sum_{i=1}^n\ln\left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{\frac{\alpha'}{2}k_i^2}\Bigg\} \\
=& iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\left[-\frac{1}{2}\sum_{i=1}^n\alpha' k_i^2\omega(w_i)\right] \\
&\prod_{i<j=1}^n\left[\Big|\vartheta_1\Big(\frac{w_{ij}}{2\pi},\tau\Big)\Big|\exp\left(-\frac{\alpha'(\text{Im}w_{ij})^2}{4\pi\tau_2}\right)\right]^{\alpha' k_i\cdot k_j}\prod_{i=1}^n\left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{\frac{-\alpha'}{2}k_i^2}
\end{aligned} \qquad [7.33]$$

We can now rewrite the last product as follows. We use (6.44) with $f_i = 1$ for $i = 1, \cdots, n$

$$-2\sum_{i<j=1}^n k_i\cdot k_j - \sum_{i=1}^n k_i^2 = -\sum_{i=1}^n k_i\cdot\sum_{j=1}^n k_j \qquad [7.34]$$

By momentum conservation the RHS is zero so that

$$\sum_{i=1}^n k_i^2 = -2\sum_{i<j=1}^n k_i\cdot k_j \qquad [7.35]$$

Therefore

$$\begin{aligned}
\prod_{i=1}^n\left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{\frac{-\alpha'}{2}k_i^2} &= \left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{-\frac{\alpha'}{2}\sum_{i=1}^n k_i^2} = \left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{\alpha'\sum_{i<j=1}^n k_i\cdot k_j} \\
&= \prod_{i<j=1}^n\left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\right|^{\alpha' k_i\cdot k_j}
\end{aligned} \qquad [7.36]$$

This gives our final result

$$A^n_{S^2}(k,\sigma) = iC^X_{T^2}(2\pi)^D\delta^D(\sum_i k_i)\exp\left[-\frac{1}{2}\sum_i \alpha' k_i^2\omega(w_i)\right] \times \prod_{i<j=1}^{n}\left|\frac{2\pi}{\partial_\nu\vartheta_1(0,\tau)}\vartheta_1\Big(\frac{w_{ij}}{2\pi},\tau\Big)\exp\left(-\frac{\alpha'(\mathrm{Im}w_{ij})^2}{4\pi\tau_2}\right)\right|^{\alpha' k_i\cdot k_j} \qquad [7.37]$$

This is (7.2.4), if just as in (6.2.13) we set the conformal factor $\omega \equiv 0$ by taking a conformally flat metric in a region containing all vertex operators.

7.8 p 209: Eq. (7.2.5) The Scalar Partition Function on the Torus, I

We have one translation in "time" and one in "space". The translation is Euclidean time t is generated by the Hamiltonian , $e^{-\hat{H}t}$ and the translation in space is generated by the momentum operator $e^{i\hat{P}x}$. Think of the torus $w = \sigma^1 + i\sigma^2 \equiv w + 2\pi\tau$ as being a closed string of circumference $2\pi\tau_1$ that evolves and turns into itself after a time $2\pi\tau_2$. So we have a time evolution of $e^{-2\pi\tau_2 H}$ with H the Hamiltonian of the worldsheet field theory. We worked out in (2.6.10) that this is given by $H = L_0 + \tilde{L}_0 - (c+\tilde{c})/24$. But that is not all, when the closed string comes back to itself it has been "twisted" by an amount $2\pi\tau_1$, that is a translation of $e^{2piP\tau_1}$ with the translation operator given by $P = L_0 - \tilde{L}_0$.

To calculate the path integral without vertex operators $Z(\tau) = \langle \mathbb{1}\rangle_{T^2(\tau)}$ we thus start from any ground state, we let it evolve over time $2\pi\tau_2$ while at the same time we twist it over $2\pi\tau_1$ and then measure the overlap of that evolved state, with the original state. We need to do this for all possible intermediate states, so we need to sum over all possible states. Therefore

$$\begin{aligned} Z(\tau) = \langle \mathbb{1}\rangle_{T^2(\tau)} = \int_{T^2(\tau)}[dX]\, e^{-S_P} &= \sum_{\text{all states } |\psi\rangle} \langle\psi|\, e^{-2\pi\tau_2 H} e^{2\pi i\tau_1 P}\,|\psi\rangle \\ &= \mathrm{Tr}\,\exp(-2\pi\tau_2 H + 2\pi i\tau_1 P) \end{aligned} \qquad [7.38]$$

where the trace is over all states and S_P denotes the Polyakov action. Let us now work this out. We also use the fact that for the matter sector $c = \tilde{c} = d$, with d the number of scalar fields.

$$\begin{aligned} Z[\tau] &= \mathrm{Tr}\ \exp\left[-2\pi\tau_2\left(L_0 + \tilde{L}_0 - \frac{d}{12}\right) + 2\pi i\tau_1(L_0 - \tilde{L}_0)\right] \\ &= \mathrm{Tr}\ \exp\left[2\pi i(i\tau_2+\tau_1)L_0 - 2\pi i(-i\tau_2+\tau_1)\tilde{L}_0 + 4\pi\tau_2\frac{d}{24}\right] \\ &= \mathrm{Tr}\ \exp\left[2\pi i\tau L_0 - 2\pi i\bar{\tau}\tilde{L}_0 + 4\pi\tau_2\frac{d}{24}\right] \end{aligned} \qquad [7.39]$$

We now introduce $q = \exp 2\pi i\tau$ so that

$$q\bar{q} = \exp 2\pi i(\tau - \bar{\tau}) = \exp(2\pi i 2i\mathrm{Im}\tau) = \exp -4\pi\tau_2 \qquad [7.40]$$

and thus indeed

$$Z(\tau) = (q\bar{q})^{-d/24}\,\mathrm{Tr}\; q^{L_0}\bar{q}^{\tilde{L}_0} \qquad [7.41]$$

7.9 p 209: Eq. (7.2.6) The Scalar Partition Function on the Torus, II

The trace in the partition function is over all states in the Hilbert space. For the scalar field these states are characterised by their momenta k and their oscillator level. The oscillator level is given by $N_{\mu n}$ where $\mu = 0, \cdots, 25$ is the spacetime index and $n = 1, \cdots, \infty$ is the oscillator index. We saw in (4.3.31) and (4.3.32) that, focussing on the matter part only,[1]

$$L_0 = \frac{\alpha'}{4}(p^2 + m^2) = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} n \sum_{\mu=0}^{25} N_{\mu n} \qquad [7.42]$$

and the same of course for $\tilde{L}_0$. Let us spend a minute to make sure we understand this formula. Take as an example a state $(\alpha^0_{-3})^2\alpha^{17}_{-1}\left|0;k\right\rangle$. We have chosen a timelike and a transverse Lorentz index for convenience. This state has non-zero oscillator indices and levels $n = 1; N_{17,1} = 1$ and $n = 3; N_{0,3=3}$ with all other $N_{\mu k} = 0$. The weight of this state is thus

$$\begin{aligned} L_0(\alpha^0_{-3})^2\alpha^{17}_{-1}\left|0;k\right\rangle &= \left[\frac{\alpha'}{4}k^2 + (1 + 2\times 3)\right](\alpha^0_{-3})^2\alpha^{17}_{-1}\left|0;k\right\rangle \\ &= \left(\frac{\alpha'}{4}k^2 + \sum_{n=0}^{\infty} n\sum_{\mu=0}^{25} N_{n\mu}\right)(\alpha^0_{-3})^2\alpha^{17}_{-1}\left|0;k\right\rangle \qquad [7.43] \end{aligned}$$

To find the partition function we need to take the trace over all possible states

$$Z(\tau) = (q\bar{q})^{-d/24}\,\mathrm{Tr}\; q^{\frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} n\sum_{\mu=0}^{25} N_{\mu n}}\bar{q}^{\frac{\alpha'}{4}p^2 + \sum_{m=1}^{\infty} m\sum_{\mu=0}^{25}\tilde{N}_{\mu m}} \qquad [7.44]$$

To rewrite this is less simple than it is often inferred. Let us ignore the anti-holomorphic side and the Lorentz index for convenience. We thus wish to evaluate $\mathrm{Tr}\, q^{\sum_{n=1}^{\infty} nN_n}$. Let us work this out for the first few levels

[1] This means ignoring the ghost contributions N_{bn}, N_{cn} and $a^g = 1$, see (2.7.21). Recall that $a^X = 0$, see (2.7.9).

weight L_0	**states**	**nr of states**
0	$\mathbb{1}$	1
1	α_{-1}	1
2	$\alpha_{-2}, \alpha_{-1}^2$	2
3	$\alpha_{-3}, \alpha_{-2}\alpha_{-1}, \alpha_{-1}^3$	3
4	$\alpha_{-4}, \alpha_{-3}\alpha_{-1}, \alpha_{-2}\alpha_{-2}, \alpha_{-2}\alpha_{-1}^2, \alpha_{-1}^4$	5
5	$\alpha_{-5}, \alpha_{-4}\alpha_{-1}, \alpha_{-3}\alpha_{-2}, \alpha_{-3}\alpha_{-1}^2, \alpha_{-2}^2\alpha_{-1}, \alpha_{-2}\alpha_{-1}^3, \alpha_{-1}^5$	7

Table 7.1: First few states of the string, matter sector

One sees that for weight n one needs to find all partitions of n, denoted by $p(n)$ and we have

$$\operatorname{Tr} q^{\sum_{n=1}^{\infty} nN_n} = \sum_{k=0}^{\infty} p(k) q^k \tag{7.45}$$

There is no closed form expression for $p(n)$, but we can rewrite it as

$$\sum_{k=0}^{\infty} p(k) q^k = \prod_{\ell=1}^{\infty} \sum_{m=0}^{\infty} q^{\ell m} \tag{7.46}$$

One can easily work out the first few cases for oneself, but it quickly becomes very messy. The easiest way is to use common sense. If you consider the expression

$$\begin{aligned}&\Big(1+q+q^2+q^3+q^4+\cdots\Big)\times\Big(1+q^2+q^4+q^6+q^8+\cdots\Big)\times\Big(1+q^3+q^6+q^9+\cdots\Big)\\ &\times\Big(1+q^4+q^8+\cdots\Big)\times\Big(1+q^5+\cdots\Big)\times\cdots\end{aligned} \tag{7.47}$$

If we wish to extract e.g. how many times we find q^3 e.g. we have a contribution $q^3 + q^2 \times q + q \times q \times q$, so we recognise the partitions of three. This is easily seen to extend to all power of q which proves [7.46].

Adding the momenta, the Lorentz index and the anti-holomorphic part is straightforward and we get

$$Z(\tau) = (q\bar{q})^{-d/24}\, V_d \int \frac{d^d k}{(2\pi)^d} (q\bar{q})^{\alpha' k^2/4} \prod_{n=1}^{\infty} \prod_{\mu=0}^{25} \sum_{N_{\mu n}, \tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}} \bar{q}^{n\tilde{N}_{\mu n}} \tag{7.48}$$

Using, once more, $q\bar{q} = \exp -4\pi\tau_2$ this gives

$$Z(\tau) = (q\bar{q})^{-d/24}\, V_d \int \frac{d^d k}{(2\pi)^d} e^{-\pi\tau_2\alpha' k^2} \prod_{n,\mu} \sum_{N_{\mu n}, \tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}} \bar{q}^{n\tilde{N}_{\mu n}} \tag{7.49}$$

7.10 p 210: Eq. (7.2.8)-(7.2.9) The Scalar Partition Function on the Torus, III

The momentum integration is just a Gaussian, after we perform a Wick rotation of the spacetime momentum $k^0 \to ik^E$:

$$\int \frac{d^d k}{(2\pi)^d} e^{-\pi\tau_2\alpha' k^2} = i\left(\int \frac{d\kappa}{2\pi} e^{-\pi\tau_2\alpha'\kappa^2}\right)^d = i\left(\frac{1}{2\pi}\sqrt{\frac{\pi}{\pi\tau_2\alpha'}}\right)^d = i(4\pi^2\tau_2\alpha')^{-d/2} \qquad [7.50]$$

We thus get

$$\begin{aligned}
Z(\tau) &= (q\bar{q})^{-d/24}\, V_d i(4\pi^2\tau_2\alpha')^{-d/2} \prod_{n,\mu}(1-q^n)^{-1}(1-\bar{q}^n)^{-1} \\
&= iV_d(q\bar{q})^{-d/24}(4\pi^2\tau_2\alpha')^{-d/2} \prod_n (1-q^n)^{-d}(1-\bar{q}^n)^{-d} \\
&= iV_d \left\{ (4\pi^2\tau_2\alpha')^{-1/2} \left[q^{1/24} \prod_n (1-q^n) \right]^{-1} \left[\bar{q}^{1/24} \prod_n (1-\bar{q}^n) \right]^{-1} \right\}^d \\
&= iV_d \left[(4\pi^2\tau_2\alpha')^{-1/2} |\eta(\tau)|^{-2} \right]^d = iV_d Z_X(\tau)^d
\end{aligned} \qquad [7.51]$$

with

$$Z_X(\tau) = (4\pi^2\tau_2\alpha')^{-1/2}|\eta(\tau)|^{-2} \qquad [7.52]$$

and

$$\eta(\tau) = q^{1/24}\prod_n (1-q^n) \qquad [7.53]$$

Let us check the modular invariance of $Z(\tau)$. Under $\tau \to \tau+1$ we have clearly $4\pi^2\tau_2\alpha' \to 4\pi^2\tau_2\alpha'$ and also

$$\begin{aligned}
|\eta(\tau)|^2 \to & \left[e^{2\pi i(\tau+1)/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n(\tau+1)}\right) \right] \times \left[e^{-2\pi i(\bar{\tau}+1)/24} \prod_{m=1}^{\infty} \left(1 - e^{-2\pi i m(\bar{\tau}+1)}\right) \right] \\
&= e^{2\pi i/24} e^{-2\pi i/24} \left[e^{2\pi i\tau/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) \right] \times \left[e^{-2\pi i\bar{\tau}/24} \prod_{m=1}^{\infty} \left(1 - e^{-2\pi i m\bar{\tau}}\right) \right] \\
&= |\eta(\tau)|^2
\end{aligned} \qquad [7.54]$$

and under $\tau \to -1/\tau$ we find, using (7.2.44), and

$$\tau_1 + i\tau_2 \to -\frac{\tau_1}{\tau_1^2+\tau_2^2} + \frac{i\tau_2}{\tau_1^2+\tau_2^2} \qquad [7.55]$$

that

$$
\begin{aligned}
Z_X(\tau) &\to \left(4\pi\alpha' \frac{i\tau_2}{|\tau|^2}\right)^{-1/2} (-i\tau)^{-1/2}\eta^{-1}(\tau)(i\bar{\tau})^{-1/2}\eta^{-1}(\bar{\tau}) \\
&= (4\pi^2\tau_2\alpha')^{-1/2}|\tau|\frac{1}{|\tau|}\eta^{-1}(\tau)\eta^{-1}(\bar{\tau}) = (4\pi^2\tau_2\alpha')^{-1/2}|\eta(\tau)|^{-2} \\
&= Z_X(\tau)
\end{aligned}
\qquad [7.56]
$$

and so also $Z(-1/\tau) = Z(\tau)$.

7.11 p 210: Eq. (7.2.11) The Change of Metric for the Torus

$$
\begin{aligned}
ds^2 &= (1+\varepsilon^*+\varepsilon)d[w+\epsilon(\bar{w}-w)]d[\bar{w}+\varepsilon^*(w-\bar{w})] \\
&= (1+\varepsilon^*+\varepsilon)[dwd\bar{w}+\varepsilon^* dwd(w-\bar{w})+\varepsilon d(w-\bar{w})d\bar{w}]+o(\varepsilon^2) \\
&= dwd\bar{w}+\varepsilon^* dwd(w-\bar{w})+\varepsilon d(w-\bar{w})d\bar{w}+\varepsilon^* dwd\bar{w}+\varepsilon dwd\bar{w}+o(\varepsilon^2) \\
&= dwd\bar{w}+\varepsilon^* dw^2+\varepsilon d\bar{w}^2
\end{aligned}
\qquad [7.57]
$$

which is exactly how the line element changes under $\delta g_{w\bar{w}} = 0$, $\delta g_{ww} = \varepsilon^*$ and $\delta g_{\bar{w}\bar{w}} = \varepsilon$.

7.12 p 210: Eq. (7.2.12) The Periodicity of w'

Under $w' \to w' + 2\pi$ we have

$$
w' + 2\pi = w + \varepsilon(\bar{w}-w) + 2\pi = w + 2\pi + \varepsilon(\bar{w}-w) = w + \varepsilon(\bar{w}-w) = w' \qquad [7.58]
$$

and under $w' \to w' + 2\pi(\tau - 2i\tau_2\varepsilon)$, using $\tau_2 = (\tau-\bar{\tau})/2i$

$$
\begin{aligned}
w' + 2\pi(\tau - 2i\tau_2\varepsilon) &= w + \varepsilon(\bar{w}-w) + 2\pi(\tau - 2i\tau_2\varepsilon) \\
&= w + 2\pi\tau + \varepsilon(\bar{w}-w) - 4\pi i\frac{\tau-\bar{\tau}}{2i}\varepsilon \\
&= w + \varepsilon(\bar{w}-w-2\pi\tau+2\pi\bar{\tau}) = w + \varepsilon(\bar{w}-w) = w'
\end{aligned}
\qquad [7.59]
$$

7.13 p 210: Eq. (7.2.13) The Change in Modulus form the Change in Metric

The torus described by coordinates $w, \bar{w}$ and moduli τ is thus the same as the torus described by coordinates w' and moduli $\tau' = \tau - 2i\tau_2\varepsilon$. The change in moduli is thus

$$
\delta\tau = \tau' - \tau = -2i\tau_2\varepsilon \qquad [7.60]
$$

7.14 p 210: Eq. (7.2.14) The Change in the Partition Function due to a Change in the Metric

We have

$$\begin{aligned}
\delta Z(\tau) = \frac{\delta Z(\tau)}{\delta g_{ab}}\delta g_{ab} &= \frac{\delta}{\delta g_{ab}}\left(\int_{T^2(\tau)}[dX]\,e^{-S_P}\right)\delta g_{ab} \\
&= -\left(\int_{T^2(\tau)}[dX]\,e^{-S_P}\frac{\delta S_P}{\delta g_{ab}}\right)\delta g_{ab} \\
&= -\left[\int_{T^2(\tau)}[dX]\,e^{-S_P}\frac{1}{4\pi}\int d^2\sigma\, T^{ab}(\sigma)\right]\delta g_{ab} \\
&= -\frac{1}{2\pi}\int d^2w\left(\int_{T^2(\tau)}[dX]\,e^{-S_P}T^{ab}(\sigma)\right)\delta g_{ab} \\
&= -\frac{1}{2\pi}\int d^2w\langle T^{ab}(\sigma)\rangle\delta g_{ab} = -\frac{1}{2\pi}\int d^2w\,\left[\langle T^{ww}(\sigma)\rangle\delta g_{ww} + \langle T^{\bar{w}\bar{w}}(\sigma)\rangle\delta g_{\bar{w}\bar{w}}\right] \\
&= -\frac{2}{\pi}\int d^2w\Big[\langle T_{\bar{w}\bar{w}}(\bar{w})\rangle\delta g_{ww} + \langle T_{ww}(w)\rangle\delta g_{\bar{w}\bar{w}}\Big]
\end{aligned} \qquad [7.61]$$

We have used the definition of the energy momentum tensor $T^{ab} = 4\pi\delta S/\delta g_{ab}$, $d^2\sigma = 2d^2w$ and $4T_{ww} = T^{\bar{w}\bar{w}}$, $4T_{\bar{w}\bar{w}} = T^{ww}$. We now fill in the form of the change in metric and then use the worldsheet translation invariance of the theory to bring the energy momentum tensor to the origin

$$\begin{aligned}
\delta Z(\tau) &= -\frac{2}{\pi}\int d^2w\Big[\langle T_{\bar{w}\bar{w}}(\bar{w})\rangle\varepsilon^* + \langle T_{ww}(w)\rangle\varepsilon\Big] \\
&= -\frac{2}{\pi}\int d^2w\Big[\langle T_{\bar{w}\bar{w}}(0)\rangle\varepsilon^* + \langle T_{ww}(0)\rangle\varepsilon\Big] \\
&= -\frac{1}{\pi}\Big[\langle T_{\bar{w}\bar{w}}(0)\rangle\varepsilon^* + \langle T_{ww}(0)\rangle\varepsilon\Big]\int d^2\sigma
\end{aligned} \qquad [7.62]$$

We know from [7.10] that the surface area of the torus is $4\pi^2\tau_2$ and we have from (7.2.13) that $\varepsilon = i\delta\tau/2\tau_2$

$$\begin{aligned}
\delta Z(\tau) &= -\frac{1}{\pi}\Big[-\frac{i\delta\bar{\tau}}{2\tau_2}\langle T_{\bar{w}\bar{w}}(0)\rangle + \frac{i\delta\tau}{2\tau_2}\langle T_{ww}(0)\rangle\Big]4\pi^2\tau_2 \\
&= -2\pi i\Big[\delta\tau\langle T_{ww}(0)\rangle - \delta\bar{\tau}\langle T_{\bar{w}\bar{w}}(0)\rangle\Big]
\end{aligned} \qquad [7.63]$$

Note that there is an error in the first line of (7.2.14); the coefficient should be $2/\pi$ and not $1/2\pi$. This is not mentioned on Joe's errata page.

7.15 p 211: Eq. (7.2.15) The OPE $\partial X^\mu(w)\partial X_\mu(0)$

From the (2.1.21b) we have

$$X^\mu(w)X^\nu(z) = -\frac{\alpha'\eta^{\mu\nu}}{2}\ln|w-z|^2 + :X^\mu(w)X^\nu(z): \qquad [7.64]$$

and thus

$$\begin{aligned}\partial X^\mu(w)\partial X^\nu(z) &= -\frac{\alpha'\eta^{\mu\nu}}{2(w-z)^2} + :\partial X^\mu(w)\partial X^\nu(z): \\ &= -\frac{\alpha'\eta^{\mu\nu}}{2(w-z)^2} + :\partial X^\mu(z)\partial X^\nu(z): + o(w-z)\end{aligned} \qquad [7.65]$$

Therefore

$$\begin{aligned}\partial X^\mu(w)\partial X_\mu(0) &= -\frac{\alpha'\delta^\mu_\mu}{2w^2} + :\partial X^\mu\partial X^\nu:(0) + o(w) \\ &= -\frac{\alpha' d}{2w^2} - \alpha' T_{ww}(0) + o(w)\end{aligned} \qquad [7.66]$$

where we have used $T(w) = -(1/\alpha'):\partial X\partial X:(w)$.

7.16 p 211: Eq. (7.2.16) The Expectation Value of $\partial X^\mu(w)\partial X_\mu(0)$ on the Torus, I

The Green's function is defined as $\eta^{\mu\nu}G' = \langle\partial^\mu X\partial^\nu X\rangle/\langle\mathbb{1}\rangle_{T^2(\tau)}$. We divide by the vacuum expectation value to describe the connected diagrams only. From this it follows that

$$\frac{1}{Z(\tau)}\partial X^\mu(w)\partial X_\mu(0) = d\lim_{w'\to 0}\partial_w\partial_{w'}G'(w,\bar w;w',\bar w') \qquad [7.67]$$

Let us take the derivatives of the Green's function (7.2.3)

$$G'(w,\bar w;w',\bar w') = -\frac{\alpha'}{2}\ln\left|\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\right|^2 + \alpha'\frac{\big[\mathrm{Im}\,(w-w')\big]^2}{4\pi\tau_2} + k(\tau,\bar\tau) \qquad [7.68]$$

Let us start with the first term

$$\begin{aligned}\partial_{w'}\partial_w\ln\left|\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\right|^2 &= \partial_{w'}\frac{\partial_w\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)} \\ &= -\frac{\partial_w\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\partial_{w'}\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}{\vartheta_1^2\Big(\frac{w-w'}{2\pi},\tau\Big)} + \frac{\partial_{w'}\partial_w\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}\end{aligned} \qquad [7.69]$$

From the definition of the Jacobi theta functions it is clear that ϑ_1 is even in its first argument and so we can replace $\partial_{w'}$ by ∂_w. Thus

$$\partial_{w'}\partial_w \ln\left|\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\right|^2 = +\frac{\Big[\partial_w\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\Big]^2}{\vartheta_1^2\Big(\frac{w-w'}{2\pi},\tau\Big)} - \frac{\partial_w^2\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)}{\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)} \qquad [7.70]$$

and thus

$$\begin{aligned}-\frac{\alpha' d}{2}\lim_{w'\to 0}\partial_{w'}\partial_w \ln\left|\vartheta_1\Big(\frac{w-w'}{2\pi},\tau\Big)\right|^2 &= -\frac{\alpha' d}{2}\times\frac{+\Big[\partial_w\vartheta_1\Big(\frac{w}{2\pi},\tau\Big)\Big]^2 - \vartheta_1\Big(\frac{w}{2\pi},\tau\Big)\partial_w^2\vartheta_1\Big(\frac{w}{2\pi},\tau\Big)}{\vartheta_1^2\Big(\frac{w}{2\pi},\tau\Big)}\\ &= \frac{\alpha' d}{2}\,\frac{\vartheta_1\partial_w^2\vartheta_1 - \partial_w\vartheta_1\partial_w\vartheta_1}{\vartheta_1^2} \qquad [7.71]\end{aligned}$$

with $\vartheta_1 \equiv \vartheta_1(w/2\pi,\tau)$.

Now for the second part. We already worked this out in [7.24]

$$\bar\partial\partial\left\{\alpha'\frac{\left[\mathrm{Im}(w-w')\right]^2}{4\pi\tau_2}\right\} = \frac{\alpha'}{8\pi\tau_2} \qquad [7.72]$$

Combining both expressions we find

$$\frac{1}{Z(\tau)}\partial X^\mu(w)\partial X_\mu(0) = \frac{\alpha' d}{2}\,\frac{\vartheta_1\partial_w^2\vartheta_1 - \partial_w\vartheta_1\partial_w\vartheta_1}{\vartheta_1^2} + \frac{\alpha' d}{8\pi\tau_2} \qquad [7.73]$$

7.17 p 211: Eq. (7.2.17) The Expectation Value of $\partial X^\mu(w)\partial X_\mu(0)$ on the Torus, II

We first show that $\vartheta_1(\nu,\tau)$ is odd in ν. We use the expression (7.2.37d) and find, using $z = e^{2\pi i\nu} \to e^{-2\pi i\nu} = 1/z$,

$$\vartheta_1(-\nu,\tau) = i\sum_{n=-\infty}^{\infty}(-)^n q^{(n-1/2)^2}(1/z)^{n-1/2} = i\sum_{n=-\infty}^{\infty}(-)^n q^{(n-1/2)^2} z^{-n+1/2} \qquad [7.74]$$

Changing summation index $n = -m+1$ gives

$$\begin{aligned}\vartheta_1(-\nu,\tau) &= i\sum_{m=-\infty}^{\infty}(-)^(-m+1) q^{(-m+1/2)^2} z^{m-1/2} = -i\sum_{m=-\infty}^{\infty}(-)^2 q^{(m-1/2)^2} z^{m-1/2}\\ &= -\vartheta_1(\nu,\tau) \qquad [7.75]\end{aligned}$$

Now if a function $f(x)$ is odd, then its $f'(x)$ is even. Indeed

$$f'(x) = \frac{\partial f(x)}{\partial x} \to \frac{\partial f(-x)}{\partial(-x)} = \frac{\partial(-f(x))}{-\partial x} = \frac{\partial f(x)}{\partial x} = f'(x) \tag{7.76}$$

This is easily generalised to that that if f is odd then its odd derivatives are even and its even derivatives are odd. Now an odd function f always has $f(0) = -f(-0) = -f(0)$ and thus all odd functions have $f(0) = 0$, but also all there even derivatives are zero at zero. This is a long story to say that $\vartheta_1(\nu,\tau)$ and all its even derivatives vanish at $\nu = 0$. We can thus expand $\vartheta_1(\nu,\tau)$ as

$$\vartheta_1(\nu,\tau) = \nu\vartheta_1'(0,\tau) + \frac{1}{6}\nu^3\vartheta_1'''(0,\tau) + o(\nu^5) \tag{7.77}$$

Let us now write θ for $\vartheta(0,\tau)$ and likewise for all its derivatives. We thus have

$$\begin{aligned}
\vartheta_1(\nu,\tau) &= \nu\theta' + \frac{1}{6}\nu^3\theta''' + \cdots \\
\partial_\nu\vartheta_1(\nu,\tau) &= \theta' + \frac{1}{2}\nu^2\theta''' + \cdots \\
\partial_\nu^2\vartheta_1(\nu,\tau) &= \nu\theta''' + \cdots
\end{aligned} \tag{7.78}$$

We thus get, ignoring the higher order terms

$$\begin{aligned}
\frac{\vartheta_1\partial_\nu^2\vartheta_1 - \partial_\nu\vartheta_1\partial_\nu\vartheta_1}{\vartheta_1^2} &= \frac{(\nu\theta' + \frac{1}{6}\nu^3\theta''')\nu\theta''' - (\theta' + \frac{1}{2}\nu^2\theta''')^2}{(\nu\theta' + \frac{1}{6}\nu^3\theta''')^2} \\
&= \frac{\nu^2\theta'\theta''' - \theta'\theta' - \nu^2\theta'\theta'''}{\nu^2\theta'\theta'(1 + \frac{1}{6}\nu^2\theta'''/\theta')^2} \\
&= -\frac{1}{\nu^2}(1 - \frac{1}{6}\nu^2\frac{\theta'''}{\theta'})^2 = -\frac{1}{\nu^2} + \frac{1}{3}\frac{\theta'''}{\theta'}
\end{aligned} \tag{7.79}$$

We now need to replace ν by $\nu = w/2\pi$, but both sides have a total of a second derivative, so we also have

$$\frac{\vartheta_1\partial_w^2\vartheta_1 - \partial_w\vartheta_1\partial_w\vartheta_1}{\vartheta_1^2} = -\frac{1}{w^2} + \frac{1}{3}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} \tag{7.80}$$

Using this, we find for (7.2.16)

$$\frac{1}{Z(\tau)}\partial X^\mu(w)\partial X_\mu(0) = -\frac{\alpha' d}{2w^2} + \frac{\alpha' d}{6}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} + \frac{\alpha' d}{8\pi\tau_2} \tag{7.81}$$

We see that the double pole indeed corresponds to the double pole of the expectation value of (7.2.15) and that the order w^0 term is given by (7.2.17).

7.18 p 211: Eq. (7.2.18) The Expectation Value of $\partial X^\mu(w)\partial X_\mu(0)$ on the Torus, III

We take the expectation value of (7.2.15) on the torus

$$\begin{aligned}\langle \partial X^\mu(w)\partial X_\mu(0)\rangle &= -\frac{\alpha' d}{2w^2}\langle \mathbb{1}\rangle_{T^2(\tau)} - \alpha'\langle T_{ww}(0)\rangle + o(w)\\ &= -\frac{\alpha' d}{2w^2}Z(\tau) - \alpha'\langle T_{ww}(0)\rangle + o(w)\end{aligned} \quad [7.82]$$

Comparing with [7.81] we find that

$$\langle T_{ww}(0)\rangle = \left(-\frac{d}{6}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} - \frac{d}{8\pi\tau_2}\right)Z(\tau) \quad [7.83]$$

7.19 p 211: Eq. (7.2.19) The Expectation Value of $\partial X^\mu(w)\partial X_\mu(0)$ on the Torus, IV

Working out the LHS of (7.2.14) we have

$$\frac{\partial Z}{\partial\tau}\delta\tau + \frac{\partial Z}{\partial\bar\tau}\delta\bar\tau = -2\pi i\left[\delta\tau\langle T_{ww}(0)\rangle - \delta\bar\tau\langle T_{\bar w\bar w}(0)\rangle\right] \quad [7.84]$$

Filling in (7.2.18) and its conjugate this gives

$$\begin{aligned}\frac{\partial Z}{\partial\tau}\delta\tau + \frac{\partial Z}{\partial\bar\tau}\delta\bar\tau = -2\pi i\Bigg[&\delta\tau\left(-\frac{d}{6}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} - \frac{d}{8\pi\tau_2}\right)Z(\tau)\\ &-\delta\bar\tau\left(-\frac{d}{6}\frac{\overline{\partial_w^3\vartheta(0,\tau)}}{\partial_w\vartheta(0,\tau)} - \frac{d}{8\pi\tau_2}\right)\overline{Z}(\tau)\Bigg]\end{aligned} \quad [7.85]$$

From the $\delta\tau$ part we thus have Filling in (7.2.18) and its conjugate this gives

$$\frac{\partial Z}{\partial\tau} = \left(\frac{\pi i d}{3}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} + \frac{id}{4\tau_2}\right)Z(\tau) \quad [7.86]$$

Equivalently

$$\partial_\tau \ln Z(\tau) = \frac{\pi i d}{3}\frac{\partial_w^3\vartheta(0,\tau)}{\partial_w\vartheta(0,\tau)} + \frac{id}{4\tau_2} \quad [7.87]$$

7.20 p 211: Eq. (7.2.20) A Jacobi Function Identity

We have

$$\vartheta_1(\nu,\tau) = -\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\nu,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(\nu+1/2)} \tag{7.88}$$

Thus

$$\partial_\nu^2 \vartheta_1(\nu,\tau) = -4\pi^2 \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{2}\right)^2 e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(\nu+1/2)} \tag{7.89}$$

and

$$\partial_\tau \vartheta_1(\nu,\tau) = \pi i \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{2}\right)^2 e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(\nu+1/2)} \tag{7.90}$$

Therefore

$$\partial_\nu^2 \vartheta_1(\nu,\tau) = 4\pi i \partial_\tau \vartheta_1(\nu,\tau) \tag{7.91}$$

and setting $\nu = w/2\pi$

$$\partial_w^2 \vartheta_1 \left(\frac{w}{2\pi},\tau\right) = \frac{i}{\pi} \partial_\tau \vartheta_1 \left(\frac{w}{2\pi},\tau\right) \tag{7.92}$$

7.21 p 211: Eq. (7.2.21) The Differential Equation for the Partition Function

We have

$$\partial_\tau \ln \partial_w \vartheta_1 = \frac{\partial_\tau \partial_w \vartheta_1}{\partial_w \vartheta_1} = \frac{\pi}{i} \frac{\partial_w^3 \vartheta_1}{\partial_w \vartheta_1} \tag{7.93}$$

where we have used (7.2.20). Thus, (7.2.19) becomes

$$\begin{aligned} \partial_\tau \ln Z(\tau) &= \frac{\pi i d}{3} \frac{i}{\pi} \partial_\tau \ln \partial_w \vartheta_1(0,\tau) + \frac{id}{4\tau_2} \\ &= -\frac{d}{3} \partial_\tau \ln \partial_w \vartheta_1(0,\tau) + \frac{id}{4\tau_2} \end{aligned} \tag{7.94}$$

7.22 p 211: Eq. (7.2.22) The Partition Function from the Differential Equation

We check that (7.2.22) satisfies (7.2.21) and its conjugate equation.

$$\ln Z(\tau) \propto -\frac{d}{3}\ln\partial_w\vartheta_1(0,\tau) - \frac{d}{3}\ln\overline{\partial_w\vartheta_1(0,\tau)} - \frac{d}{2}\ln\frac{\tau-\bar{\tau}}{2i} \qquad [7.95]$$

and thus

$$\begin{aligned}\partial_\tau \ln Z(\tau) &= -\frac{d}{3}\partial_\tau\ln\partial_w\vartheta_1(0,\tau) - \frac{d}{2}\frac{1}{\tau-\bar{\tau}} \\ &= -\frac{d}{3}\partial_\tau\ln\partial_w\vartheta_1(0,\tau) - \frac{d}{2}\frac{1}{2i\tau_2} \\ &= -\frac{d}{3}\partial_\tau\ln\partial_w\vartheta_1(0,\tau) + \frac{id}{4\tau_2}\end{aligned} \qquad [7.96]$$

The conjugate equation is satisfied in the same way and so (7.2.22) indeed satisfies (7.2.21).

7.23 p 212: Eq. (7.2.23) The Partition Function for the Ghost System, I

Our starting formula is an expression for the partition function of the matter sector, [7.48] which we repeat here for convenience,

$$\text{Tr}\ e^{-2\pi\tau_2 H + 2\pi i\tau_1 P} = (q\bar{q})^{-d/24}\, V_d \int \frac{d^dk}{(2\pi)^d}(q\bar{q})^{\alpha' k^2/4} \prod_{n=1}^{\infty}\prod_{\mu=0}^{25}\sum_{N_{\mu n},\tilde{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}}\bar{q}^{n\tilde{N}_{\mu n}} \qquad [7.97]$$

What are the changes for the ghost sector? To start with the $-d/24$ gets replaced by $+26/24 = 13/12$. Indeed the d is just the central charge of the matter sector, so for the ghost sector that is -26.

$$\text{Tr}\ e^{-2\pi\tau_2 H + 2\pi i\tau_1 P} = 4(q\bar{q})^{13/12}\,\text{Tr}\ q^{L_0}\bar{q}^{\tilde{L}_0} \qquad [7.98]$$

Next, we use again (4.3.32). We don't need the integration over the momenta for the ghosts, we don't need the spacetime index. Furthermore, the ghost occupation number can be only zero or one so the sum reduces to $1+q^n$. So both the b and the c ghosts contribute a $|1+q^n|^2$ which gives a $|1+q^n|^4$. Moreover we have the normal ordering constant $a^g = 1$ which leads to an extra factor $(q\bar{q})^{-1}$. Finally each holomorphic and anti-holomorphic sectors have two ground states $|\uparrow\rangle$ and $|\downarrow\rangle$ so that the closed string has four ground states $|\uparrow,\uparrow\rangle, |\uparrow,\downarrow\rangle, |\downarrow,\uparrow\rangle$ and $|\downarrow,\downarrow\rangle$. We thus have four times the same contribution. This formula thus reduces for the ghost sector to

$$\text{Tr}\ e^{-2\pi\tau_2 H + 2\pi i\tau_1 P} = 4(q\bar{q})^{13/12}\,(q\bar{q})^{-1}\prod_{n=1}^{\infty}|1+q^2|^4 = 4(q\bar{q})^{1/12}\prod_{n=1}^{\infty}|1+q^2|^4 \qquad [7.99]$$

7.24 p 212: Eq. (7.2.24) The Partition Function for the Ghost System, II

The appearance of the $(-)^F$ is explained in appendix A in (A.2.21) to (A.2.23). Where F is the Fermion number and is even for $|\downarrow\rangle$ and odd for $|\uparrow\rangle$. We thus have

$$\begin{aligned} (-)^F\,|\uparrow\uparrow\rangle &= +\,|\uparrow\uparrow\rangle\,; \qquad (-)^F\,|\downarrow\downarrow\rangle = +\,|\downarrow\downarrow\rangle \\ (-)^F\,|\uparrow\downarrow\rangle &= -\,|\downarrow\uparrow\rangle\,; \qquad (-)^F\,|\uparrow\downarrow\rangle = -\,|\uparrow\downarrow\rangle \end{aligned} \qquad [7.100]$$

and so rather than adding up the four ghost ground states cancel one another.

7.25 p 212: Eq. (7.2.25) The Ghost Insertions

The fact that the trace (7.2.24) vanishes should not be surprising as we know that we need to put ghost insertions in the the expectation value. We need equal number of c insertions as we have moduli and equal number of b insertions as we have CKVs. We saw that the torus has two real moduli (the complex τ) and two real CKVs, hence we need two c and two b insertions. So the vacuum ghost expectation value should indeed be $\langle c(w_1)b(w_2)\tilde{c}(\bar{w}_3)\tilde{b}(\bar{w}_4)\rangle$.

7.26 p 212: Eq. (7.2.26) The Trace with the Ghost Insertions

Let us look at the holomorphic part first. We use the Laurent expansion of the ghosts (2.7.16) and the action of their modes on the ghost ground state (2.7.18).

$$\begin{aligned} \langle c(w_1)b(w_2)\rangle_{T^2(\tau)} &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{w_1^{m-1}w_2^{n+2}}\langle c_m b_n\rangle_{T^2(\tau)} \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{0}\frac{1}{w_1^{m-1}w_2^{n+2}}\langle c_m b_n\rangle_{T^2(\tau)} \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{0}\frac{1}{w_1^{m-1}w_2^{n+2}}\langle -b_n c_m + \delta_{m+n}\rangle_{T^2(\tau)} \\ &= -\frac{w_1}{w_2^2}\langle b_0 c_0\rangle_{T^2(\tau)} + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{0}\frac{1}{w_1^{m-1}w_2^{-m+2}}\langle \mathbb{1}_g\rangle_{T^2(\tau)} \end{aligned} \qquad [7.101]$$

But $\langle \mathbb{1}_g\rangle_{T^2(\tau)} = 0$ by the fact that the trace over the two ground states of the holomorphic sectors cancel one another due to the $(-)^F$ in the trace. We therefore find, adding the anti-holomorphic sector that indeed

$$\langle c(w_1)b(w_2)\tilde{c}(\bar{w}_3)\tilde{b}(\bar{w}_4)\rangle = \mathrm{Tr}\,\left[(-)^F c_0 b_0 \tilde{c}_0 \tilde{b}_0 e^{-2\pi\tau_2 H + 2\pi i\tau_1 P}\right] \qquad [7.102]$$

7.27 p 212: Eq. (7.2.27) The Partition Function for the Ghost System, Final Result

Recall (2.7.18) the action of the ghost zero modes on the ghost ground states: $b_0 |\downarrow\rangle = c_0 |\uparrow\rangle = 0$, $b_0 |\uparrow\rangle = |\downarrow\rangle$ and $b_0 |\downarrow\rangle = |\uparrow\rangle$. This means that

$$c_0 b_0 |\uparrow\rangle = c_0 |\downarrow\rangle = |\uparrow\rangle\,; \quad \text{and} \quad c_0 b_0 |\downarrow\rangle = 0 \qquad [7.103]$$

The ground state $|\downarrow\rangle$ is thus projected out of the trace and both the holomorphic and anti-holomorphic sector only contribute once to the sum. Therefore we lose the factor four in (7.2.23). We also need to evaluate $\mathrm{Tr}\,(-)^F q^{\sum_{n=1}^{\infty} nN_n}$ for the ghost excitations. This looks like an innocuous and straightforward formula, but, at least for me, it is not and needs a detailed derivation. We refer to the explanation of (7.2.6). We need to consider the different level ghost weights and their contributions to the partition function. The issue here is that we can both have c and b excitations at a given weight, so let us work out the first few levels in detail.

- level 0: no ghost excitations so a contribution $q^0 = 1$.
- level 1: two ghost excitations: c_{-1} and b_{-1}. Both have a -1 from the fermion factor $(-)^F$, so a contribution $-2q$.
- level 2: there are three possibilities: $c_{-2}, c_{-1}b_{-1}$ and b_{-2}. Two have fermion number -1 and one has fermion number $+1$. So we have a contribution $-q^2$.
- level 3: the possibilities are $c_{-3}, c_{-2}c_{-1}, c_{-2}b_{-1}, c_{-1}b_{-2}$ and also b_{-3} and $b_{-2}b_{-1}$. There are six possibilities, two of them have fermion number -1 and four have fermion number $+1$. So we have a contribution $+2q^3$.
- level 4: the possibilities are $c_{-4}, c_{-3}c_{-1}, c_{-3}b_{-1}, c_{-2}c_{-1}b_{-1}, c_{-2}b_{-2}, c_{-1}b_{-3}, c_{-1}b_{-2}b_{-1}, b_{-4}$ and $b_{-3}b_{-1}$. There are nine possibilities, four of them have fermion number -1 and five have fermion number $+1$. So we have a contribution $+q^4$.
- etc.

The pattern should now be clear: we are looking for a double distinct partition at every weight. We will leave it to the reader to work out that at level five there are six possibilities with fermion number -1 and eight with fermion number $+1$ giving a $+2q^5$. So we have found so far

$$\mathrm{Tr}\,(-)^F q^{\sum_{n=1}^{\infty} nN_n} = 1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 + \cdots \qquad [7.104]$$

It turns out that we can represent this as $\prod_{n=1}^{\infty}(1-q^n)^2$. Indeed using Mathematica to work this out gives

$$\prod_{n=1}^{\infty}(1-q^n)^2 = 1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 - 2q^6 - 2q^8 - 2q^9 + q^{10} + 2q^{13} + 3q^{14} - 2q^{15} + 2q^{16} - 2q^{19} - 2q^{20} + 2q^{21} - 2q^{22} - 6q^{23} - q^{24} + 2q^{25} + \cdots \qquad [7.105]$$

Adding the anti-holomorphic sector and bringing it all together we see that we can write

$$\langle c(w_1)b(w_2)\tilde{c}(\bar{w}_3)\tilde{b}(\bar{w}_4)\rangle = (q\bar{q})^{1/12}\prod_{n=1}^{\infty}|1-q^n|^4 = |\eta(\tau)|^4 \qquad [7.106]$$

7.28 p 213: Eq. (7.2.29) Modular Invariance Implies Integer Spin

From (7.2.28) we have

$$\begin{aligned} Z(\tau+1) &= \sum_i e^{2\pi i(\tau+1)(h_i-c/24)}e^{-2\pi i(\bar{\tau}+1)(\tilde{h}_i-\tilde{c}/24)} \\ &= e^{2\pi i[h_i-\tilde{h}_i-(c-\tilde{c})/24]}Z(\tau) \end{aligned} \qquad [7.107]$$

Invariance under $\tau \to \tau + 1$ thus requires

$$h_i - \tilde{h}_i - \frac{c-\tilde{c}}{24} \in \mathbb{Z} \qquad [7.108]$$

One operator that is certainly in the theory is the unit operator $\mathbb{1}$, which has $h_{\mathbb{1}} = \tilde{h}_{\mathbb{1}} = 0$. The above equation thus implies that $(c-\tilde{c})/24 \in \mathbb{Z}$ and thus from that, also that $h_i - \tilde{h}_i \in \mathbb{Z}$.

7.29 p 213: Eq. (7.2.30) The Density of States at High Weights

Setting $\tau = i\ell$ the partition function for a general CFT becomes

$$Z(i\ell) = \sum_i e^{-2\pi\ell(h_i-c/24)}e^{-2\pi\ell(\tilde{h}_i-\tilde{c}/24)} = \sum_i e^{-2\pi\ell[h+\tilde{h}_i-(c+\tilde{c})/24]} \qquad [7.109]$$

For $\ell \to 0$ we can expand the partition function

$$Z(i\ell) = \sum_i \left[1 - 2\pi\ell\left(h + \tilde{h}_i - \frac{c+\tilde{c}}{24}\right)\right] + o(\ell^2) \qquad [7.110]$$

and it is the states with the large weights that give the largest contribution to Z.

By modular invariance we must have $Z(\tau) = Z(-1/\tau)$ or with $\tau = i\ell$ this means $Z(i\ell) = Z(i/\ell)$. Now

$$Z(i/\ell) = \sum_i e^{-\frac{2\pi}{\ell}[h+\tilde{h}_i-(c+\tilde{c})/24]} \qquad [7.111]$$

Now as we let ℓ approach zero, it is the smallest weight that contribute, but the state with lowest weight in a unitary compact CFT is the unit state with weight zero. Thus as $\lim_{\ell\to 0} Z(i\ell) = e^{\pi(c+\tilde{c})/12\ell}$ and by invariance under $\tau \to -1/tau$ we have

$$\lim_{\ell\to 0} Z(i\ell) = e^{\pi(c+\tilde{c})/12\ell} \qquad [7.112]$$

7.30 The Jacobi Theta Functions

We will not derive all these equations as it it pure mathematics. For those readers interested in more details see e.g. the seminal book E.T. Whittaker & G.N. Watson, *A Course of Modern Analysis*, pp 462-490.

7.31 p 216: Eq. (7.3.2) The b-Ghost Insertion

By definition

$$B = \frac{1}{4\pi}(b, \partial_\tau g) = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,b^{ab}\partial_\tau g_{ab} \qquad [7.113]$$

We use $2d^2\sigma = d^2w$, $\sqrt{g} = 1$ and. $v^a = g^{ab}v_b = 2v_a$ so that $b^{ww} = 4b_{\bar{w}\bar{w}}$ to get

$$B = \frac{1}{2\pi}\int d^2w\,(b_{\bar{w}\bar{w}}\partial_\tau g_{ww} + b_{ww}\partial_\tau g_{\bar{w}\bar{w}}) \qquad [7.114]$$

Let us now look at the variation of the metric $\delta g_{ww} = \varepsilon^*$ and using (7.2.13) i.e. $\delta\tau = -2i\tau_2\varepsilon$:

$$\begin{aligned}\delta_\tau g_{\bar{w}\bar{w}} &= \partial_\tau g_{\bar{w}\bar{w}}\delta\tau + \partial_{\bar{\tau}} g_{\bar{w}\bar{w}}\delta\bar{\tau} \\ &= \partial_\tau g_{\bar{w}\bar{w}}(-2i\tau_2\varepsilon) + \partial_{\bar{\tau}} g_{\bar{w}\bar{w}}(2i\tau_2\varepsilon^*)\end{aligned} \qquad [7.115]$$

Setting this equal to $\delta g_{\bar{w}\bar{w}} = \varepsilon$ we find that $\partial_{\bar{\tau}} g_{\bar{w}\bar{w}} = 0$ and $\partial_\tau g_{\bar{w}\bar{w}} = i/\tau_2$. We have of course also the conjugate expressions, in particular $\partial_\tau g_{ww} = 0$. Plugging this into [7.114] we find

$$B = \frac{1}{2\pi}\int d^2w\,\frac{i}{2\tau_2}b_{ww}(w) = \frac{i}{4\pi\tau_2}\int d^2w\,b_{ww}(w) \qquad [7.116]$$

We use translation invariant to put the ghost field at the origin and are then left with the surface are of the torus, which we know is $\int d^2w\ = 2\int d^2\sigma = 8\pi^2\tau_2$ and thus

$$B = 2\pi i b_{ww}(0) \qquad [7.117]$$

7.32 p 217: Eq. (7.3.4) The General Amplitude on the Torus

The explanation in Joe's book is quite detailed already, but let us just recall the main point. We have used the CKV to fix the location of one vertex operator in w_1. But we can essentially put this anywhere on the torus. We can then just "average" out its location over the torus. Basically what we do is we replace

$$\mathcal{V}_1(w_1) = \frac{1}{8\pi^2\tau_2}\int d^2w\,\mathcal{V}_1(w_1) = \frac{1}{8\pi^2\tau_2}\int d^2w\,\mathcal{V}_1(w) = \frac{1}{8\pi^2\tau_2}\int d^2w_1\,\mathcal{V}_1(w_1) \qquad [7.118]$$

In the first equation we have kept the vertex operator at the position w_1 and multiplied and divided by (twice) the area of the torus. We have then used translational invariance to put the vertex operator at w and then change integration variables. Applying this, together with (7.3.2), to (7.3.1) gives

$$\begin{aligned} S_{T^2}(1;2;\cdots;n) &= \frac{1}{2}\int_{F_0} d\tau d\bar{\tau} \left\langle 2\pi i b(0)[-2\pi i\tilde{b}(0)]\tilde{c}(0)c(0)\frac{1}{8\pi^2\tau_2}\prod_{i=1}^{n}\mathcal{V}_i(w_i,\bar{w}_i)\right\rangle \\ &= \int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}\left\langle b(0)\tilde{b}(0)\tilde{c}(0)c(0)\prod_{i=1}^{n}\mathcal{V}_i(w_i,\bar{w}_i)\right\rangle \end{aligned} \quad [7.119]$$

7.33 p 217: Eq. (7.3.6) The Vacuum Amplitude on the Torus

We fill in the matter part (7.2.8) and the ghost part (7.2.27) in (7.3.5)

$$\begin{aligned} Z_{T^2} &= iV_{26}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-52}\,|\eta(\tau)|^4 \\ &= iV_{26}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48} \end{aligned} \quad [7.120]$$

7.34 p 217: Eq. (7.3.7) Modular Invariance of the Vacuum Amplitude on the Torus

Under $\tau\to\tau+1$ we have, using (7.2.44a), that $|\eta(\tau)|\to|\eta(\tau+1)| = \left|e^{i\pi/12}\eta(\tau)\right| = |\eta(\tau)|$. All the other factors in the torus vacuum amplitude are obviously invariant as well, so we have invariance under $\tau\to\tau+1$.

To check invariance under $\tau\to-1/\tau$, we first rewrite (7.3.6) as

$$\begin{aligned} Z_{T^2} &= iV_{26}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-13}\tau_2\left[\tau_2|\eta(\tau)|^4\right]^{-12} \\ &= \frac{iV_{26}}{4(4\pi^2\alpha')^{13}}\int_{F_0}\frac{d\tau d\bar{\tau}}{\tau_2^2}\left[\tau_2|\eta(\tau)|^4\right]^{-12} \end{aligned} \quad [7.121]$$

Now under $\tau\to-1/\tau$, using (7.2.44b) and the fact that $\tau_2\to-\tau_2/|\tau|^2$,

$$\tau_2|\eta(\tau)|^4 = -\frac{\tau_2}{|\tau|^2}\left|(-i\tau)^{1/2}\eta(\tau)\right|^4 = \frac{\tau_2}{|\tau|^2}|\tau|^2|\eta(\tau)|^4 = \tau_2|\eta(\tau)|^4 \quad [7.122]$$

To show invariance of the measure $\tau\to\tau'=-1/\tau$, we first note that the Jacobian is simply

$$J = \det\begin{pmatrix}\partial\tau/\partial\tau' & 0 \\ 0 & \partial\bar{\tau}/\partial\bar{\tau}'\end{pmatrix} = \frac{1}{\tau'^4\bar{\tau}'^4} = \frac{1}{|\tau'|^4} \quad [7.123]$$

Thus

$$\frac{d\tau' d\bar{\tau}'}{\tau_2'^2} = \frac{d\tau d\bar{\tau}}{|\tau|^4} \frac{1}{[-\tau_2/|\tau|^2]^2} = \frac{d\tau d\bar{\tau}}{\tau_2^2} \qquad [7.124]$$

and so is invariant as well.

7.35 p 217: Eq. (7.3.8) The Vacuum Amplitude on the Torus for a General Theory

I am not sure about this formula. It seems like Joe is saying that the ghosts cancel the contribution of two of the non-compact dimensions, and that all the other non-compact dimensions don't contribute to the partition function. That seems strange. I would have expected the following. The theory consists of d scalar fields X and a general CFT that has a Hilbert space $\mathcal{H}\perp$ with highest weights $(h_i, \tilde{h}_i)$. The total central charge of the matter sector plus the CFT is $d + c_{\mathrm{CFT}} = 26$. The ghosts cancel the contribution to the transition function of two of the matter oscillators, but we are still left with $d-2$ matter oscillators that contribute a Dedekind function $|\eta(\tau)|^{2(d-2)}$. I would thus expect the partition function for this theory to be

$$Z_{T^2}\left[X^{[d]}; \mathrm{CFT}\right] = iV_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-d/2} |\eta(\tau)|^{2(d-2)} \sum_{i\in\mathcal{H}_\perp} q^{h_i-1}\bar{q}^{\tilde{h}_i-1} \qquad [7.125]$$

7.36 p 217: Eq. (7.3.9) The Partition Function for a Particle on a Circle

This is similar to the derivation of (3.3.22) but limited to the point particle. We will in particular closely follow the discussion in [19] as it solves exercise 5.1 which requires the partition function for a point particle without periodic boundary conditions. We will, of course, extend this to a particle on a circle. The formal expression for the path integral of a particle on a circle is

$$Z_{S_1} = \int_{X(0)=X_0}^{X(1)=X_0} \frac{[dX\, de]}{2V_{\mathrm{diff}} \times V_{\mathrm{trans}}} e^{-S_m[X,e]} \qquad [7.126]$$

with matter action (1.2.5)

$$S_m[X,e] = \frac{1}{2}\int_0^1 d\tau\, e\left(\frac{1}{e^2}\partial X^\mu \partial X_\mu + m^2\right) \qquad [7.127]$$

Here e is the einbein[2], $\tau \in [0,1]$ is a parameter describing the circle and $\partial = d/d\tau$. Because we are on the circle the fields X^μ and e are periodic under $\tau \to \tau + 1$. As discussed in

[2]The reader shall not be confused by the fact that we use the same symbol for the einbein as for the the Euler e.

section 1.2. this action is diffeomorphism invariant. However, as discussed in section 5.1 not all einbeins on a circle are related by a diffeomorphism. Indeed if we define ℓ to be the invariant length of the circle $\ell = \int_0^1 d\tau\, e(\tau)$, then under a diffeomorphism $\tau \to \tau'(\tau)$ we have $\tau'(1) = \ell$, see (5.1.4), and so we cannot simultaneously fix the gauge and keep the coordinate region fixed. The invariant length ℓ is a modulus and describes circles that are not diffeomorphism equivalent. Even after gauge fixing the theory has a residual symmetry of a global translation of τ around the circle, corresponding to the fact that we are free to choose the origin of the circle. Thus we need to divide by the volume of that translation group as well. These translations are given by (5.1.6), i.e. $\tau \to \tau + v \bmod 1$ and the volume of that group is obviously ℓ. For now we will write this as V_{trans} and only use its value later on. Even after that, there still is a residual symmetry of replacing $\tau \to -\tau$ and so we need to divide by factor of two as well.

It will be convenient to introduce an inner product on the space of functions of the unit interval. For two such functions we define

$$(f, g) = \int_0^1 d\tau\, e(\tau) f(\tau) g(\tau) \tag{7.128}$$

This is, of course, similar as the inner product we defined for the string. We can now rewrite the matter action in terms of the inner product

$$S_m[X, e] = \frac{1}{2}(e^{-1}\partial X^\mu, e^{-1}\partial X_\mu) + \frac{1}{2}\ell m^2 \tag{7.129}$$

We chose a fiducial gauge $\hat{e}$ and define the Faddeev-Popov measure

$$1 = \Delta_{\text{FP}}(e) \int d\ell\, [d\varepsilon] \delta(e - \hat{e}^\varepsilon) \tag{7.130}$$

This is the equivalent of [3.27] for the point particle with an integration over all diffeomorphism parameters ε and $\hat{e}^\varepsilon$ being the einbein in the fiducial gauge after a gauge transformation and ℓ is the parameter for the global translations. We plug this into the path integral and perform the usual manipulations, including integrating over all possible moduli ℓ,

$$Z_{S_1} = \int_{X(0)=X_0}^{X(1)=X_0} \frac{[dX\, de]}{2V_{\text{diff}} \times V_{\text{trans}}} \Delta_{\text{FP}}(e) \int d\ell [d\varepsilon] \delta(e - \hat{e}^\varepsilon) e^{-S_m[X,e]} \tag{7.131}$$

We carry out the e integration and rename the dummy variable X

$$Z_{S_1} = \int_0^\infty d\ell \int_{X(0)=X_0}^{X(1)=X_0} \frac{[dX^\varepsilon\, d\varepsilon]}{2V_{\text{diff}} \times V_{\text{trans}}} \Delta_{\text{FP}}(\hat{e}^\varepsilon) e^{-S_m[X^\varepsilon, \hat{e}^\varepsilon]} \tag{7.132}$$

We use the gauge invariance of the Faddeev-Popov measure, the action and the integration measure

$$Z_{S_1} = \int_0^\infty d\ell \int_{X(0)=X_0}^{X(1)=X_0} \frac{[dX d\varepsilon]}{2V_{\text{diff}} \times V_{\text{trans}}} \Delta_{\text{FP}}(\hat{e}) e^{-S_m[X,\hat{e}]} \tag{7.133}$$

We can perform the integration over the gauge parameter ε and this gives V_{diff} which cancels the same factor in the denominator. We now also use the volume of the translations $V_{\text{trans}} = \ell$. All this gives us

$$Z_{S_1} = \int_0^\infty \frac{d\ell}{2\ell} \int_{X(0)=X_0}^{X(1)=X_0} [dX]\, \Delta_{\text{FP}}(\hat{e}) e^{-S_m[X,\hat{e}]} \qquad [7.134]$$

Let us now calculate the Faddeev-Popov determinant. Just as in (3.3.16) we expand the einbein for a small change both in the coordinate $\tau \to \tau^\varepsilon = \tau + e^{-1}\varepsilon$ and the modulus $\ell \to \ell + \delta\ell$. Let us first keep the modulus fixed. Then

$$\begin{aligned} \delta e &= e^\varepsilon(\tau) - e(\tau) = e^\varepsilon(\tau^\varepsilon - e^{-1}\varepsilon) - e(\tau) = e^\varepsilon(\tau^\varepsilon) - e^{-1}\varepsilon\partial_\tau e(\tau) - e(\tau) \\ &= e^\varepsilon(\tau^\varepsilon) - \varepsilon\partial \ln e(\tau) - e(\tau) \end{aligned} \qquad [7.135]$$

Fom $e^\varepsilon(\tau^\varepsilon)d\tau^\varepsilon = e(\tau)d\tau$ we have

$$\begin{aligned} e^\varepsilon(\tau^\varepsilon) &= e(\tau)\frac{d\tau}{d\tau^\varepsilon} = e(\tau)\frac{d}{d\tau}(\tau - e^{-1}\varepsilon) = e(\tau)\left(1 + \frac{1}{e^2}\partial e\varepsilon - e^{-1}\partial\varepsilon\right) \\ &= e(\tau) + \varepsilon\partial \ln e - \partial\varepsilon \end{aligned} \qquad [7.136]$$

Therefore

$$\delta e = e + \varepsilon\partial\ln e - \partial\varepsilon - \varepsilon\partial\ln e - e = -\partial\varepsilon \qquad [7.137]$$

Adding a small change in the modulus as well then give

$$\delta e = -\partial\varepsilon + \partial_\ell e\,\delta\ell \qquad [7.138]$$

Note that as we are working with a fixed coordinate rate and a variable ℓ we need that $\tau^\varepsilon \in [0,1]$ as well. This implies that $\varepsilon(\tau = 0) = \varepsilon(\tau = 1) = 0$. From [7.130] it thus follows that

$$\Delta_{\text{FP}}(e)^{-1} = \int d\ell\,[d\varepsilon]\delta(e - \hat{e}^\varepsilon) = \int d\ell\,[d\varepsilon]\delta(\partial\varepsilon - \partial_\ell e\,\delta\ell) \qquad [7.139]$$

We rewrite the delta function as an exponential

$$\begin{aligned} \Delta_{\text{FP}}(e)^{-1} &= \int d\ell\,[d\varepsilon\,]\delta(e - \hat{e}^\varepsilon) = \int d\ell\,[d\varepsilon\, d\beta] \exp\left[2\pi i \int d\tau\,\beta\,(\partial\varepsilon - \partial_\ell e\,\delta\ell)\right] \\ &= \int d\ell\,[d\varepsilon\, d\beta] \exp\left[2\pi i \int d\tau\, e\beta\left(e^{-1}\partial\varepsilon - e^{-1}\partial_\ell e\,\delta\ell\right)\right] \\ &= \int d\ell\,[d\varepsilon\, d\beta] \exp\left[2\pi i(\beta, e^{-1}\partial\varepsilon - e^{-1}\partial_\ell e\,\delta\ell)\right] \end{aligned} \qquad [7.140]$$

We have chosen to write it in terms of our inner product for our convenience. We invert the integral by introducing Grassmann variables ξ, c and b and evaluate the determinant at $\hat{e}$, our fiducial gauge choice

$$\Delta_{\mathrm{FP}}(\hat{e}) = \int d\xi[dc\,db]\exp\left[\frac{1}{4\pi}\left(b, \hat{e}^{-1}\partial c - \hat{e}^{-1}\partial_\ell \hat{e}\xi\right)\right] \quad [7.141]$$

We can now perform the ξ integration; just keep the linear term as ξ is a Grassmann variable. Ignoring an overall irrelevant sign we get

$$\Delta_{\mathrm{FP}}(\hat{e}) = \int [dc\,db]\frac{1}{4\pi}\left(b, \hat{e}^{-1}\partial_\ell \hat{e}\right)\exp\left[\frac{1}{4\pi}\left(b, \hat{e}^{-1}\partial c\right)\right] \quad [7.142]$$

We can plug this into [7.134] to find our result for the path integral after the Faddeev-Popov procedure

$$Z_{S_1} = \int_0^\infty \frac{d\ell}{2\ell}\int_{X(0)=X_0}^{X(1)=X_0}[dX]\int_{c(0)=0}^{c(1)=0}[dc\,db]\,\frac{1}{4\pi}\left(b, \hat{e}^{-1}\partial_\ell \hat{e}\right)e^{-S_m[X,\hat{e}]-S_g[b,c,\hat{e}]} \quad [7.143]$$

with

$$S_g[b,c,\hat{e}] = -\frac{1}{4\pi}\left(b, \hat{e}^{-1}\partial c\right) \quad [7.144]$$

Note that $\varepsilon(0) = \varepsilon(1) = 0$ translates into the boundary conditions $c(0) = c(1) = 0$ for the c-ghost integration.

Let us now evaluate this partition function. We select a gauge $\hat{e} = \ell$. The inner product then becomes simply

$$(f, g) = \ell\int_0^1 d\tau\, f g \quad [7.145]$$

First look at the ghost part. We expand the ghost fields in Eigenfunctions of the Laplacian

$$\Delta_\ell f = -\ell^{-2}\partial^2 f = \nu^2 f \quad [7.146]$$

on the circle. The coefficient $-\ell^{-2}$ is just a convenient normalisation. A complete set of solutions satisfying these equations are $\sin(\nu\tau/\ell)$ and $\cos(\nu\tau/\ell)$. Requiring periodicity under $\tau \to \tau + 1$ requires $\nu/\ell = 2\pi k$ with $k \in \mathbb{N}$. These Eigenfunctions have Eigenvalues

$$\begin{aligned}\Delta_\ell \sin 2\pi j\tau &= -\ell^{-2}\partial^2 \sin 2\pi j\tau = \frac{4\pi^2 j^2}{\ell^2}\sin 2\pi j\tau \\ \Delta_\ell \cos 2\pi j\tau &= -\ell^{-2}\partial^2 \cos 2\pi j\tau = \frac{4\pi^2 j^2}{\ell^2}\cos 2\pi j\tau\end{aligned} \quad [7.147]$$

i.e.

$$\nu_j^2 = \frac{4\pi^2 j^2}{\ell^2} \qquad [7.148]$$

We thus expand the b and c ghost fields

$$b = \sqrt{\frac{1}{\ell}} b_0 + \sqrt{\frac{2}{\ell}} \sum_{j=1}^{\infty} b_j \cos 2\pi j\tau; \qquad c = \sqrt{\frac{2}{\ell}} \sum_{j=1}^{\infty} c_j \sin 2\pi j\tau \qquad [7.149]$$

Note that the c-ghost has no zero mode because we need $c(0) = c(1) = 0$. The coefficients are just convenient normalisations.

We now work out the ghost part of the partition function. We start with the ghost action [7.144]

$$\begin{aligned} S_g[b, c, \hat{e}] &= -\frac{1}{4\pi} \left(b, \hat{e}^{-1}\partial c\right) \\ &= -\frac{1}{4\pi} \int_0^1 d\tau \left(\sqrt{\frac{1}{\ell}} b_0 + \sqrt{\frac{2}{\ell}} \sum_{j=1}^{\infty} b_j \cos 2\pi j\tau \right) \times \partial \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} c_k \sin 2\pi k\tau \\ &= -\frac{1}{\sqrt{2}\ell} \int_0^1 d\tau \left(b_0 + \sqrt{2} \sum_{j=1}^{\infty} b_j \cos 2\pi j\tau \right) \sum_{k=1}^{\infty} k c_k \cos 2\pi k\tau \end{aligned} \qquad [7.150]$$

To work this out we need the following integrals

$$\int_0^1 d\tau \, \cos 2\pi j\tau = 0; \qquad \int_0^1 d\tau \, \cos 2\pi j\tau \cos 2\pi k\tau = \frac{1}{2}\delta_{jk} \qquad [7.151]$$

This gives

$$S_g[b, c, \hat{e}] = -\frac{1}{\sqrt{2}\ell} \sqrt{2} \sum_{j,k=1}^{\infty} k b_j c_k \frac{1}{2} \delta_{jk} = \frac{1}{2\ell} \sum_{j=1}^{\infty} j b_j c_j \qquad [7.152]$$

The b-ghost insertion becomes

$$\begin{aligned} \frac{1}{4\pi}(b, \hat{e}^{-1}\partial_\ell \hat{e}) &= \frac{\ell}{4\pi} \int_0^1 d\tau \, b \, \ell^{-1} \partial_\ell \ell = \frac{1}{4\pi} \int_0^1 d\tau \, b \\ &= \frac{1}{4\pi} \int_0^1 d\tau \left(\sqrt{\frac{1}{\ell}} b_0 + \sqrt{\frac{2}{\ell}} \sum_{j=1}^{\infty} b_j \cos 2\pi j\tau \right) = \frac{1}{4\pi\sqrt{\ell}} b_0 \end{aligned} \qquad [7.153]$$

The ghost contribution to the partition function thus becomes

$$Z_g = \int \prod_{j=1}^{\infty} [dc_j \, db_0 \, db_j] \, \frac{1}{4\pi\sqrt{\ell}} b_0 \exp\left[\frac{1}{2\ell}\sum_{j=1}^{\infty} j b_j c_j\right] = \frac{1}{4\pi\sqrt{\ell}} \prod_{j=1}^{\infty} \frac{j}{2\ell} \qquad [7.154]$$

Using [7.148] we can write this as

$$Z_g = \frac{1}{4\pi\sqrt{\ell}} \prod_{j=1}^{\infty} \frac{\nu_j}{4\pi^2} = \frac{1}{4\pi\sqrt{\ell}} \det{}' \left(\frac{\Delta_\ell}{16\pi^2}\right)^{1/2} \qquad [7.155]$$

where $\det{}'$ denotes the determinant excluding the zero mode.

Let us now turn to the matter part. We can either use the approach of section 6.2 and expand the matter fields in generic Eigenfunctions, or we can take the approach of [19] and use the same Eigenfunctions we have for the ghost system. The latter approach requires that we first split the matter field in a classical and in a quantum piece, and that is the approach that we will follow. So we first look at the classical equation of motion for the matter field, $\ddot{X}^\mu = 0$. It has general solution

$$X^\mu_{\text{cl}} = x^\mu_0 + (x^\mu_1 - x^\mu_0)\tau \qquad [7.156]$$

where $x_0 = X^\mu(0)$ and x_1 are some constants determined by the boundary conditions. We then define the quantum fields X^μ as

$$X^\mu = X^\mu_{\text{cl}} + \mathsf{X}^\mu \qquad [7.157]$$

We note that X^μ must also be periodic under $\tau \to \tau + 1$ and that is satisfies $\mathsf{X}^\mu(0) = X^\mu(0) - X^\mu_{\text{cl}}(0) = X^\mu(0) - x_0 = 0$ and likewise for $\mathsf{X}^\mu(1)$. We can thus expand the quantum fields in terms of the $\sin 2\pi j\tau$ Eigenmodes, just as the c-ghost and the matter field becomes

$$X^\mu = X^\mu_{\text{cl}} + \sqrt{\frac{2}{\ell}} \sum_{j=1}^{\infty} x^\mu_j \sin 2\pi j\tau \qquad [7.158]$$

The matter action [7.127] then becomes in the gauge $\hat{e} = \ell$

$$\begin{aligned} S_m &= \frac{1}{2}\int_0^1 d\tau \, \ell \left[\frac{1}{\ell^2}\partial\left(X^\mu_{\text{cl}} + \sqrt{\frac{2}{\ell}}\sum_{j=1}^{\infty} x^\mu_j \sin 2\pi j\tau\right)^2 + m^2\right] \\ &= \frac{1}{2\ell}\int_0^1 d\tau \left[\left(x^\mu_1 - x^\mu_0 + 2\pi\sqrt{\frac{2}{\ell}}\sum_{j=1}^{\infty} j x^\mu_j \cos 2\pi j\tau\right)^2 + \ell^2 m^2\right] \end{aligned} \qquad [7.159]$$

To work this out we use [7.151] and find

$$S_m = \frac{1}{2\ell}\left[(x_1^\mu - x_0^\mu)^2 + \ell^2 m^2 + 4\pi^2 \frac{2}{\ell}\sum_{j=1}^{\infty} j^2 x_j^2 \frac{1}{2}\right]$$
$$= \frac{(x_1 - x_0)^2}{2\ell} + \frac{\ell m^2}{2} + \frac{\pi^2}{\ell^2}\sum_{j=1}^{\infty} j^2 x_j^2 \qquad [7.160]$$

The matter part of the partition function thus becomes

$$Z_X = \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} + \frac{\ell m^2}{2}\right] \int \prod_{\mu=1}^{D}\prod_{j=1}^{\infty} dx_j^\mu \, \exp\left(-\frac{\pi^2}{\ell^2}\sum_{j=1}^{\infty} j^2 x_j^2\right) \qquad [7.161]$$

The integration is now Gaussian and can be performed and we find for Z_X

$$\exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\left(\prod_{j=1}^{\infty}\sqrt{\frac{\pi}{\pi^2 j^2/\ell^2}}\right)^d = \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\left(\prod_{j=1}^{\infty}\frac{\pi j^2}{\ell^2}\right)^{-d/2}$$
$$= \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\left(\prod_{j=1}^{\infty}\frac{\nu_j^2}{4\pi}\right)^{-d/2} = \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\det{}'(\Delta_\ell)^{-d/2}$$
$$[7.162]$$

It is time to bring everything together. The partition function of the particle on a circle becomes

$$Z_{S_1} = \int_0^\infty \frac{d\ell}{2\ell} Z_X Z_g$$
$$= \int_0^\infty \frac{d\ell}{2\ell}\exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\det{}'\left(\frac{\Delta_\ell}{4\pi}\right)^{-d/2}\frac{1}{4\pi\sqrt{\ell}}\det{}'\left(\frac{\Delta_\ell}{16\pi^2}\right)^{1/2}$$
$$= C\int_0^\infty \frac{d\ell}{\ell^{3/2}}\exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right]\det{}'(\Delta_\ell)^{(1-d)/2} \qquad [7.163]$$

where C is some constant coefficient that encapsulates all the previous coefficients, but whose exact form is not important to us.

In the bosonic string example we ignored the functional determinant at this stage, but here we cannot do this as it depends on the modulus ℓ and so we need to extract that behaviour. The functional determinant is divergent[3] and needs to regularised. Appendix

[3]This is obvious as $\det{}'\Delta_\ell \propto \prod_{j=1}^{\infty} j^2/\ell^2$.

A.1 of Joe's book explains how this can be done via Pauli-Villars regularisation. This adds a contribution of a field with a very large frequency Ω and the determinant then becomes

$$\det{}'\Delta_\ell \to \frac{\det{}'\Delta_\ell}{\det{}'(\Delta_\ell + \Omega^2)} = \prod_{j=1}^{\infty} \frac{\pi^2 j^2}{\pi^2 j^2 + \Omega^2 \ell^2} = \frac{\Omega\ell}{\sinh \Omega\ell} \qquad [7.164]$$

The ℓ^2 in the infinite product is easily seen to come from the fact that $\Delta = -\ell^{-1}\partial^2$. For $\Omega \to +\infty$ we have $\sinh \Omega\ell \to e^{\Omega\ell}/2$ and so

$$\frac{\det{}'\Delta_\ell}{\det{}'(\Delta_\ell + \omega^2)} \sim 2\Omega\ell e^{-\Omega\ell} \qquad [7.165]$$

The partition function function thus becomes

$$\begin{aligned} Z_{S_1} &= C \int_0^\infty \frac{d\ell}{\ell^{3/2}} \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right] \left(2\Omega\ell e^{-\Omega\ell}\right)^{(1-d)/2} \\ &= \frac{C}{(2\Omega)^{(d-1)/2}} \int_0^\infty d\ell \ell^{-d/2-1} \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell[m^2 - (d-1)\Omega]}{2}\right] \end{aligned} \qquad [7.166]$$

Let us quickly analyse the divergences. The pre-factor $\Omega^{(1-d)/2}$ is a multiplicative factor that can be removed via field strength renormalisation, i.e. by redefining redefining $X^\mu \to Z^{1/2} X^\mu$ for some coefficient Z that will be infinite as well. The divergence in the exponential is different, but it can be removed by adding a counterterm to the Lagrangian of the form ℓA^2 for some constant A. This counterterm is of the same form as the mass term, so we immediately know that it will change the exponential into

$$\exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell[m^2 - (d-1)\Omega + 2A^2]}{2}\right] \qquad [7.167]$$

We can then chose a renormalisation condition so that m is the physical mass, which implies choosing $A^2 = (d-1)\Omega$. The counterterm is then divergent as well, but it cancels the divergence of the original integral, which is the whole point of counterterms.[4] The upshot is that the partition function becomes

$$Z_{S_1} \propto \int_0^\infty d\ell \ell^{-d/2-1} \exp\left[-\frac{(x_1 - x_0)^2}{2\ell} - \frac{\ell m^2}{2}\right] \qquad [7.168]$$

This is the analogous of formula (34) of chapter 5 in [19] for a particle moving from one point to another. The difference is that we have an extra factor ℓ^{-1} here, which we can

[4]To readers who find it hard to follow these points, I would suggest to consult the sections on perturbative renormalisation in any decent text book on QFT. Alternatively they can consult my Notes on QFT, available on hepnotes.com.

trace back to the fact that we needed to divide in the path integral by the volume of the group of global translations on the circle.

In order to make sense of this we write it as a function of $x = x_1 - x_0$ and go to the Fourier transform

$$\begin{aligned}\tilde{Z}(k) &\propto \int d^d x e^{ik\cdot x} \int_0^\infty d\ell \ell^{-d/2-1} \exp\left[-\frac{x^2}{2\ell} - \frac{\ell m^2}{2}\right] \\ &= \int_0^\infty d\ell \ell^{-d/2-1} \exp\left[-\frac{\ell m^2}{2}\right] \int d^d x \exp\left[ik\cdot x - \frac{x^2}{2\ell}\right]\end{aligned} \quad [7.169]$$

The x integration is Gaussian and can be performed after completing the square

$$\begin{aligned}\tilde{Z}(k) &\propto \int_0^\infty d\ell \ell^{-d/2-1} \exp\left(-\frac{\ell m^2}{2}\right) \int d^d x \, \exp\left[-\frac{(x - i\ell k)^2 + \ell^2 k^2}{2\ell}\right] \\ &= \int_0^\infty d\ell \ell^{-d/2-1} \exp\left(-\frac{\ell m^2}{2}\right) \exp\left(-\frac{lk^2}{2}\right) (2\pi\ell)^{d/2} \\ &= (2\pi)^{d/2} \int_0^\infty \frac{d\ell}{\ell} \exp\left(-\frac{\ell(k^2 + m^2)}{2}\right)\end{aligned} \quad [7.170]$$

This is the partition function for a particular momentum. To get all the particle states propagating over the circle we need to sum over all possible momenta. This gives the integration with with volume factor of (7.3.9).

7.37 p 218: Eq. (7.3.10) The Point Particle and the String Spectrum

I must admit that I don't fully understand what is happening here. We use the partition function of a point particle on a circle and then sum over the spectrum of the string, as if every string state has the partition function of a point particle. Joe wants to illustrate the difference, and in particular the difference in divergent behaviour, between what this approach would yield versus what a full string approach gives.

In order to achieve this we need to relate the partition function of the point particle on a circle that that of a particular state of a string. This can be done by relating the mass m^2 in (7.3.9) to the mass-shell formula for a string state (4.3.32):

$$\frac{\alpha'}{4} m^2 = L_0 - 1 = h - 1; \qquad \frac{\alpha'}{4} \tilde{m}^2 = \tilde{L}_0 - 1 = \tilde{h} - 1 \quad [7.171]$$

Adding both together we do indeed find, assuming $m = \tilde{m}$ that

$$\frac{\alpha'}{2} m^2 = h + \tilde{h} - 2 \quad [7.172]$$

This is the generic relation between the mass and the weight of a state in the closed string.

7.38 p 218: Eq. (7.3.11) The Constraints on the Weights

Let us work out the RHS of (7.3.11) and call $g = h - \tilde{h} \in \mathbb{N}$

$$RHS(g) = \frac{1}{2\pi i g} e^{ig\theta}\Big|_{-\pi}^{+\pi} = \frac{1}{2\pi i g}\left[e^{ig\pi} - e^{-ig\pi}\right] = \frac{\sin \pi g}{\pi g} \qquad [7.173]$$

Now if $g \neq 0$ then $\sin \pi g = 0$ and the result is zero. If $g = 0$ then we need to take the limit witt the l'Hôpital's rule

$$\lim_{g\to 0} RHS(g) = \lim_{g\to 0} \frac{\sin \pi g}{\pi g} = \lim_{g\to 0} \frac{\pi \cos \pi g}{\pi} = 1 \qquad [7.174]$$

and indeed $RHS(g) = \delta_g = \delta_{h,\tilde{h}}$.

7.39 p 218: Eq. (7.3.12) The Partition Function for a Stringy Particle

So we now assume that the partition function for a string state with mass given by (7.3.10) is given by the particle particle on a circle partition function and see what that means if we sum this over all possible states in the transverse Hilbert space $\mathcal{H}^\perp$, taking into account the condition (7.3.11):

$$\begin{aligned}
\mathsf{Z}_{S_1} &= \sum_{|i\rangle\in\mathcal{H}^\perp} Z_{S_1}(m_i) = iV_d \int_0^\infty \frac{d\ell}{2\ell}(2\pi\ell)^{-d/2} \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \sum_{|i\rangle\in\mathcal{H}^\perp} e^{i(h_i-\tilde{h}_i)\theta} e^{-m_i^2\ell/2} \\
&= iV_d \int_0^\infty \frac{d\ell}{2\ell} \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} (2\pi\ell)^{-d/2} \sum_{|i\rangle\in\mathcal{H}^\perp} e^{i(h_i-\tilde{h}_i)\theta-(h_i+\tilde{h}_i-2)\ell/\alpha'} \\
&= \frac{1}{2} iV_d (2\pi)^{-d/2-1} \int_0^\infty d\ell \int_{-\pi}^{+\pi} d\theta\, \ell^{-d/2-1} \sum_{|i\rangle\in\mathcal{H}^\perp} e^{ih_i(\theta+i\ell/\alpha')-i\tilde{h}(\theta-i\ell/\alpha')+2\ell/\alpha'}
\end{aligned} \qquad [7.175]$$

We set $\theta + i\ell/\alpha' = 2\pi tau = 2\pi(\tau_1 + i\tau_2)$. We then find that $d\ell = \alpha' 2\pi d\tau_2$ and the τ_2 integration runs from zero to infinity. Similarly $d\theta = 2\pi d\tau_1$ and the integration over τ_1 is from $-1/2$ to $+1/2$. Let us call this integration region $\mathcal{R}$. We can thus write

$$\begin{aligned}
\mathsf{Z}_{S_1} &= \frac{1}{2} iV_d (2\pi)^{-d/2-1} \int_{\mathcal{R}} 2\pi\alpha' d\tau_2 2\pi d\tau_1 (2\pi\alpha'\tau_2)^{-d/2-1} \sum_{|i\rangle\in\mathcal{H}^\perp} e^{ih_i 2\pi\tau - i\tilde{h}_i 2\pi\bar{\tau} + 4\pi\tau_2} \\
&= \frac{1}{2} iV_d (2\pi)^{-d/2-1+1+1-d/2-1} \alpha'^{1-d/2-1} \int_{\mathcal{R}} d\tau_1 d\tau_2 \tau_2^{-d/2-1} \\
&\qquad \times \sum_{|i\rangle\in\mathcal{H}^\perp} e^{2\pi\tau h_i} e^{-2\pi\bar{\tau}\tilde{h}_i} e^{+4\pi(\tau-\bar{\tau})/2i}
\end{aligned} \qquad [7.176]$$

We regroup and use $2d\tau_1\, d\tau_2 = d\tau\, d\bar{\tau}$ and write $q = e^{2\pi i\tau}$

$$\mathsf{Z}_{S_1} = iV_d \int_{\mathcal{R}} \frac{d\tau\, d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-d/2} \sum_{|i\rangle\in\mathcal{H}^\perp} q^{h_i} \bar{q}^{\tilde{h}_i} (q\bar{q})^{-1} \qquad [7.177]$$

which is (7.3.12).

7.40 p 218: Eq. (7.3.13)-(7.3.14) The Integration Region for the Particle on a Circle and the String on a Torus

The integration region for the particle on a circle summed over all string states is the region $\mathcal{R}$ in (7.3.12). The integration region for the closed string on a torus is the fundamental domain F_0 of the torus, see fig.5.2. Both these regions are illustrated in the figure below.

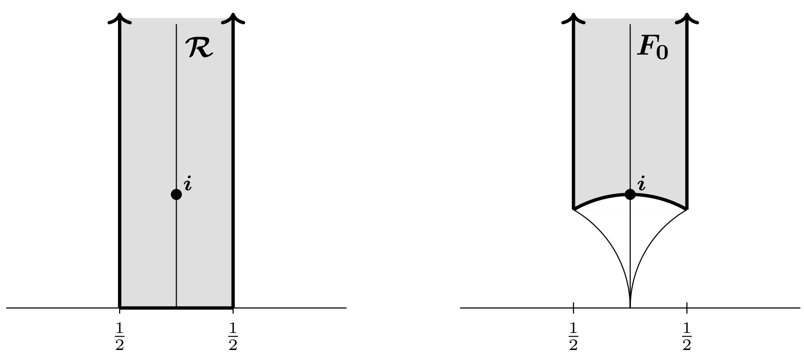


Figure 7.7: The integration region for a particle on a circle summed over all string states $\mathcal{R}$ and for a torus F_0

To see the divergence in (7.3.9) note that integral is basically the Gamma function

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \qquad [7.178]$$

So we find a expression for the integral proportional to $\Gamma(-d/2)$, which is divergent for even dimensions, in particular for $d = 26$ and $d = 4$.

7.41 p 219: Eq. (7.3.15) The Torus Vacuum Energy for $\tau \to \infty$ in Flat Spacetime

Let us first write (7.3.8b) in terms of τ_1 and τ_2 and take $d = 26$. With a serious abuse of notation, we write

$$\begin{aligned} Z_{T^2}(\tau \to \infty) &= iV_{26} \int \frac{2d\tau_1 d\tau_2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \sum_{|i\rangle \in \mathcal{H}^\perp} e^{2\pi i\tau(h_i-1)} e^{-2\pi i\bar{\tau}(\tilde{h}_i-1)} \\ &= iV_{26} \int \frac{2d\tau_1 d\tau_2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \sum_{|i\rangle \in \mathcal{H}^\perp} e^{2\pi(i\tau_1-\tau_2)(h_i-1)} e^{-2\pi(i\tau_1+\tau_2)(\tilde{h}_i-1)} \\ &= iV_{26} \int \frac{2d\tau_1 d\tau_2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \sum_{|i\rangle \in \mathcal{H}^\perp} e^{-2\pi\tau_2(h_i+\tilde{h}_i-2)} e^{2\pi i\tau_1(h_i-\tilde{h}_i)} \end{aligned} \quad [7.179]$$

If we consider the theory of 26 flat spacetime dimensions, then what does the the Hilbert Space $\mathcal{H}^\perp$ consists of? We have the unit operator $\mathbb{1}$ of weight $(0,0)$, potential higher weight states formed from derivatives the form $\partial^m X^i \partial^n X^j$ of weight (m,n) for $m,n = 1,\cdots$. Note that if $m-n = p \neq 0$ is still integer and then the τ_1 integration becomes zero. If $m = n$ then the τ_1 integration gives one. Now, as we take $\tau_2 \to \infty$ it is the states with lowest weight that give the largest contribution to $Z_{T^2}(\tau \to \infty)$. So let us consider the unit state of weight zero and the states of weight $(1,1)$ obtained by acting with the operators $\partial X^i \partial X^j$.

$$Z_{T^2}(\tau \to \infty) = iV_{26} \int \frac{d\tau_2}{2\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \left(e^{4\pi\tau_2} + \sum_{(m,n)=(1,1)}^{24} 1 + \cdots \right) \quad [7.180]$$

The states of weight $(1,1)$ are obtained by acting with the operators $\partial X^i \partial X^j$, so there are 24×24 of them and we obtain (7.3.15):

$$Z_{T^2}(\tau \to \infty) = iV_{26} \int \frac{d\tau_2}{2\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \left(e^{4\pi\tau_2} + 24^2 + \cdots \right) \quad [7.181]$$

Let us now look at this expression the contribution ($e^{4\pi\tau_2}$ comes from unit operator which comes from a propagating tachyon $e^{ik\cdot X}$.Recall that we are integrating over all momenta, so the k-dependence disappears. This gives a divergence in the $\tau_2 = \infty$ integration boundary. The second term in the sum comes from the "graviton" contribution $\epsilon_{ij}\partial X^i \bar{\partial} X^j e^{ik\cdot X}$. The τ_2 integration is straightforward and the contribution at $\tau_2 = \infty$ vanishes. The next contribution would be from a state with weight $(2,2)$ and give rise to a dampening factor $e^{-4\pi\tau_2}$ and vanishes as $\tau_2 \to \infty$.

7.42 p 219: Eq. (7.3.16) The Torus Vacuum Energy for $\tau \to \infty$ for a General CFT

Our starting point is [7.179]:

$$Z_{T^2}(\tau \to \infty) = = iV_{26} \int \frac{2d\tau_1 d\tau_2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \sum_i e^{-2\pi\tau_2(h_i+\tilde{h}_i-2)} e^{2\pi i\tau_1(h_i-\tilde{h}_i)} \qquad [7.182]$$

We saw earlier that unitarity requires $h_i = \tilde{h}_i$ so the τ_1 integration decouples in the limit $\tau_2 \to \infty$. Now use (7.3.10)

$$Z_{T^2}(\tau \to \infty) = = iV_{26} \int \frac{d\tau_2}{2\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \sum_i e^{-\pi\alpha' m_i^2 \tau_2} \qquad [7.183]$$

showing the divergence for a theory with a state of negative mass.

7.43 p 220: A BRST Null State is Proportional to a Total Derivative on Moduli Space

This statement refers to (5.4.6) where we saw that the BRST variation of the b-ghost insertion was

$$\delta_B(b, \partial_k \hat{g}) = i\varepsilon(T, \partial_k \hat{g}) \qquad [7.184]$$

where ∂_k is a derivative w.r.t. the modulus t^k.

7.44 p 219: Eq. (7.3.16) The Torus Vacuum Energy for $\tau \to \infty$ for a General CFT

The one loop partition function for the particle on a circle $Z_{S^1}(m^2)$ in (7.3.9) can easily be extended to all loops. Indeed, this is a free theory so there are no interactions and the higher loops are just disconnected circles, as in fig.7.8

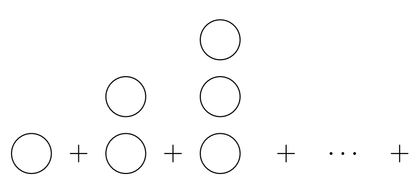


Figure 7.8: Multiloop partition function for a particle on a circle

A contribution with n loops can be formed in $n!$ different ways, so the total vacuum energy is indeed given by

$$\boldsymbol{Z}_{\text{vac}}(m^2) = \sum_{n=0}^{\infty} \frac{1}{n!} Z_{S_1}(m^2) = e^{Z_{S_1}(m^2)} \tag{7.185}$$

7.45 p 220: Eq. (7.3.20) The $\ell \to 0$ Limit of the Particle Partition Function, I

We use a different way from Joe. We define

$$\mathcal{I}(\varepsilon) = \int_0^{\infty} \frac{d\ell}{2\ell^{1-\varepsilon}} \exp\left(-\Delta\ell\right) \tag{7.186}$$

where we defined $\Delta = (k^2 + m^2)/2$ and will then take $\lim_{\varepsilon\to 0} \mathcal{I}(\varepsilon)$. First we change integration variables $\ell = x/\Delta$:

$$\mathcal{I}(\varepsilon) = \frac{1}{2} \int_0^{\infty} \frac{dx}{\Delta} \left(\frac{x}{\Delta}\right)^{\varepsilon-1} e^{-x} = \frac{1}{2}\frac{1}{\Delta^{\varepsilon}} \int_0^{\infty} x^{\varepsilon-1} e^{-x} dx = \frac{1}{2\Delta^{\varepsilon}}\Gamma(\varepsilon) \tag{7.187}$$

where we have used the definition of the Gamma function. Now, if you have had any exposure to dimensional regularisation you will recognise the expression $\Gamma(\varepsilon)/\Delta^{\varepsilon}$. If you haven't had any exposure to dimensional regularisation, then this is an appropriate time to remedy this. In any case, any text book on QFT will probably have – most likely in an appendix – the expansion

$$\lim_{\varepsilon\to 0} \frac{\Gamma(\varepsilon)}{\Delta^{\varepsilon}} = \frac{1}{\varepsilon} - \ln\Delta - \gamma + o(\varepsilon) \tag{7.188}$$

where γ is the Euler constant. Using this we find

$$\lim_{\varepsilon\to 0} \mathcal{I}(\varepsilon) = \frac{1}{2}\left[\frac{1}{\varepsilon} - \ln\Delta\gamma + o(\varepsilon)\right] = \frac{1}{2\varepsilon} - \frac{1}{2}\ln\bigl(k^2 + m^2\bigr) + \frac{1}{2}\ln 2 - \frac{1}{2}\gamma + o(\varepsilon) \tag{7.189}$$

The regularisation scheme is not just to ignore the divergent term $frac12\varepsilon$ but also to ignore the constant $(\ln 2 - \gamma)/2$. This is known in QFT as the Modified Minimal Subtraction or $\overline{\text{MS}}$ to the aficionados.[5] After all, if we are neglecting an infinite term, what is the harm of including in that a few small constants? The upshot is that after regularisation, we indeed find

$$\int_0^{\infty} \frac{d\ell}{2\ell^{1-\varepsilon}} \exp\left(-\frac{k^2 + m^2}{2}\ell\right) \to -\frac{1}{2}\ln\bigl(k^2 + m^2\bigr) \tag{7.190}$$

[5]Once more, if this is gibberish to you, consult your favourite QFT text book, or even better consult my QFT Notes on hepnotes.com.

7.46 p 220: Eq. (7.3.21) The $\ell \to 0$ Limit of the Particle Partition Function, II

In Euclidean space we have

$$\begin{aligned}LHS &= i\int_0^\infty \frac{d\ell}{2\ell}\int_{-\infty}^\infty \frac{idk^0}{2\pi}\exp\left[-\frac{k_0^2+\boldsymbol{k}^2+m^2}{2}\,\ell\right]\\&= -\frac{1}{2\pi}\int_0^\infty \frac{d\ell}{2\ell}\left(\int_{-\infty}^\infty dk^0 e^{-k_0^2\ell/2}\right)\exp\left[-\frac{\boldsymbol{k}^2+m^2}{2}\,\ell\right]\\&= -\frac{1}{2\pi}\int_0^\infty \frac{d\ell}{2\ell}\sqrt{\frac{2\pi}{\ell}}\exp\left[-\frac{\boldsymbol{k}^2+m^2}{2}\,\ell\right]\\&= -\frac{1}{2\sqrt{2\pi}}\int_0^\infty d\ell\,\ell^{-3/2}\left[-\frac{\boldsymbol{k}^2+m^2}{2}\,\ell\right]\end{aligned} \quad [7.191]$$

We introduce $\ell = 2/(\boldsymbol{k}^2+m^2)$

$$\begin{aligned}LHS &= -\frac{1}{2\sqrt{2\pi}}\int_0^\infty \frac{2}{\boldsymbol{k}^2+m^2}dx\left(\frac{2}{\boldsymbol{k}^2+m^2}x\right)^{-3/2}e^{-x}\\&= -\frac{1}{4\sqrt{\pi}}\sqrt{\boldsymbol{k}^2+m^2}\int_0^\infty x^{-1/2-1}e^{-x} = -\frac{1}{4\pi}\sqrt{\boldsymbol{k}^2+m^2}\Gamma(-1/2)\\&= -\frac{1}{4\pi}\sqrt{\boldsymbol{k}^2+m^2}\int_0^\infty x^{-1/2-1}e^{-x} = -\frac{1}{4\sqrt{\pi}}\sqrt{\boldsymbol{k}^2+m^2}(-2\sqrt{\pi})\\&= \frac{1}{2}\omega_{\boldsymbol{k}}\end{aligned} \quad [7.192]$$

7.47 p 221: Eq. (7.3.23) The Vacuum Energy of a Scalar Field

All the necessary formulas are available in Appendix A. For an action (A.1.38)

$$S = \frac{1}{2}\int d^dx\,\phi\Delta\phi \quad [7.193]$$

we can write the vacuum energy, see (A.1.39) and (A.1.48) as

$$Z[J=0] = \left(\det\frac{\Delta}{2\pi}\right)^{-1/2} \to (\det\Delta)^{-1/2} \quad [7.194]$$

where we have dropped the 2π as it can be removed by a rescaling of the field ϕ. In our case we have for a free scalar field of mass m the Klein-Gordon Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 = \phi\frac{\partial^2-m^2}{2}\phi \quad [7.195]$$

and so we take $\Delta = -\partial^2 + m^2$ where we have absorbed the $1/2$ in a redefinition of ϕ. We now use $\det A = \exp \mathrm{Tr}\, \ln A$ to find

$$\begin{aligned} \ln Z[0] &= \ln \left[\det\left(-\partial^2 + m^2\right)\right]^{-1/2} = -\frac{1}{2} \ln \exp \mathrm{Tr} \ln\left(-\partial^2 + m^2\right) \\ &= -\frac{1}{2} \mathrm{Tr} \ln\left(-\partial^2 + m^2\right) \end{aligned} \tag{7.196}$$

The trace can be taken by using a complete set of solutions to the Klein-Gordon equation $e^{ik\cdot x}$

$$\begin{aligned} \mathrm{Tr} \ln\left(-\partial^2 + m^2\right) &= V_d \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \ln\left(-\partial^2 + m^2\right) e^{ik\cdot x} \\ &= V_d \int \frac{d^d k}{(2\pi)^d} \ln\left(k^2 + m^2\right) \end{aligned} \tag{7.197}$$

and thus

$$\ln Z[0] = -\frac{1}{2} V_d \int \frac{d^d k}{(2\pi)^d} \ln\left(k^2 + m^2\right) \tag{7.198}$$

7.48 p 223: Eq. (7.4.1) The Vacuum Energy of a Cylinder

The cylinder represents an open string that evolves over worldsheet time and comes back where it started after $2\pi t$. The vacuum energy is thus calculated by summing over the time evolution of all states for that period and overlapping with the original state:

$$Z_{C_2}(t) = \sum_{\text{all states } |\psi\rangle} \langle\psi| \, e^{-2\pi t H} \, |\psi\rangle \tag{7.199}$$

with $H = L_o - c/24$ the open string Hamiltonian. Introducing $q = e^{-2\pi t}$ this becomes

$$Z_{C_2}(t) = q^{-c/24} \mathrm{Tr}\, q^{L_0} \tag{7.200}$$

Let us evaluate the matter sector first. From (4.3.21) and (4.3.22) we have for the matter sector of the open string

$$L_0 = \alpha'(p^2 + m^2) \tag{7.201}$$

and

$$\alpha' m^2 = \sum_{n=1}^{\infty} n \sum_{\mu=0}^{\infty} N_{\mu n} \tag{7.202}$$

The calculation is now similar as for the torus. For the matter sector we find

$$\begin{aligned} Z_X(t) &= q^{-d/24} \mathrm{Tr}\, q^{\alpha' p^2 + \sum_{n=1}^{\infty} n \sum_{\mu=0}^{\infty} N_{\mu n}} \\ &= q^{-d/24} V_d \int \frac{d^d k}{(2\pi)^d} q^{\alpha' k^2} \prod_{n=1}^{\infty} \prod_{\mu=0}^{25} \sum_{N_{\mu n}} q^{n N_{\mu n}} \end{aligned} \quad [7.203]$$

we refer to the derivation of (7.2.6) for this last step. Performing the Gaussian momentum integral gives

$$\begin{aligned} Z_X(t) &= i V_d q^{-d/24} (8\pi^2 t\alpha')^{-d/2} \prod_{n,\mu} \sum_{N_{\mu n}} q^{n N_{\mu n}} \\ &= i V_d q^{-d/24} (8\pi^2 t\alpha')^{-d/2} \prod_{n,\mu} (1-q^n)^{-1} \\ &= i V_d q^{-d/24} (8\pi^2 t\alpha')^{-d/2} \prod_{n} (1-q^n)^{-d} \\ &= i V_d (8\pi^2 t\alpha')^{-d/2} \left[q^{1/24} \prod_n (1-q) \right]^{-d} \\ &= i V_d (8\pi^2 t\alpha')^{-d/2} \eta(it)^{-d} \end{aligned} \quad [7.204]$$

with $\eta(it)$ defined in [7.53] and $\tau = it$ in our case.

For the ghost sector we can use the exactly the same argument as for the holomorphic side of the of the ghost sector on the torus. The result is the analogue of (7.2.27)

$$Z_g = \eta(it)^2 \quad [7.205]$$

The vacuum energy, without Chan-Paton factors, on the cylinder is then equivalent to the (7.3.5) on the torus, i.e.

$$Z_{C_2} = \int_0^{\infty} \frac{dt}{2t} \langle b(0) c(0) \rangle_{C_2} \quad [7.206]$$

The differences are that we integrate over all the values of the modulus t from 0 to ∞ as there is no modular invariance on the cylinder that limits the integration region to a fundamental domain and that we loose the factor $1/2/$ that was due to the symmetry $w \to -w$ on the torus. We now fill in the matter and ghost parts, and limit ourselves to the critical dimension $d = 26$

$$\begin{aligned} Z_{C_2} &= \int_0^{\infty} \frac{dt}{2t} Z_X Z_g = \int_0^{\infty} \frac{dt}{2t} i V_d (8\pi^2 t\alpha')^{-d/2} \eta(it)^{-d} \eta(it)^2 \\ &= i V_{26} \int_0^{\infty} \frac{dt}{2t} (8\pi^2 t\alpha')^{-13} \eta(it)^{-24} \end{aligned} \quad [7.207]$$

Let us now add Chan-Paton factors to this. It might be useful to review section 6.5 and in particular (6.5.4) to remind oneself about Chan-Paton factors. The equivalent of (6.5.4) or our fig.6.10 for the cylinder is the figure below

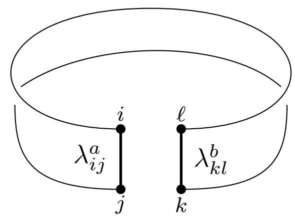


Figure 7.9: Chan-Paton factors for the cylinder. Note that in the λ matrix the first index refers to the $\sigma = 0$ point and the second index to the $\sigma = \pi$ point.

Joining the two ends mean that the Chan-Paton factors give a contribution

$$\lambda^a_{ij}\delta^{ab}\delta_{i\ell}\delta_{jk}\lambda^b_{k\ell} = \lambda^a_{ij}\lambda^a_{ji} = \mathrm{Tr}\,\lambda^a\lambda^a = \delta^{aa} = n^2 \qquad [7.208]$$

where we have used (6.5.2), i.e. $\mathrm{Tr}\,\lambda^a\lambda^b = \delta^{ab}$ and we recall that a runs from one to n^2. Note that it is important to connect the endpoints of the string correctly and to make sure in the λ matrix the first index refers to the $\sigma = 0$ point and the second index to the $\sigma = \pi$ point. Using this coefficient from the Chan-Paton factors we find

$$Z_{C_2} = iV_{26}n^2 \int_0^\infty \frac{dt}{2t}(8\pi^2 t\alpha')^{-13}\eta(it)^{-24} \qquad [7.209]$$

which is (7.4.1).

7.49 p 224: Eq. (7.4.2) Modular Transformation of the Dedekind Function for $\tau = it$

This follows immediately from (7.2.44) by setting $\tau = it$:

$$\eta(-1/it) = \sqrt{-i(it)}\eta(it) \quad \Rightarrow \eta(it) = t^{-1/2}\eta(i/t) \qquad [7.210]$$

7.50 p 224: Eq. (7.4.3) The Vacuum Energy for a Very Long Cylinder

We first use (7.4.2) in (7.4.1) and then change variables $t = \pi/s$, so that $ds = -dt/t^2$

$$\begin{aligned} Z_{C_2} &= iV_{26}n^2 \int_0^\infty \frac{dt}{2t}(8\pi^2 t\alpha')^{-13}t^{12}\eta(i/t)^{-24} = \frac{iV_{26}n^2}{2(8\pi^2\alpha')^13}\int_0^\infty \frac{dt}{t^2}\,\eta(i/t)^{-24} \\ &= \frac{iV_{26}n^2}{2(8\pi^2\alpha')^{13}}\int_\infty^0 (-ds)\,\eta(is/\pi)^{-24} = \frac{iV_{26}n^2}{2(8\pi^2\alpha')^13}\int_0^\infty ds\,\eta(is/\pi)^{-24} \end{aligned} \qquad [7.211]$$

7.51 p 224: Eq. (7.4.4) Expanding the Dedekind Function for a Very Long Cylinder

With $q = e^{2\pi i\tau}$ and $\tau = is/\pi$ we have $q = e^{-2s}$. We use the expansion of the Dedekind function

$$\eta(q) = q^{1/24}(1 - q - q^2 + q^5 + q^7 + \cdots) \qquad [7.212]$$

We are slightly abusing notation here because η should really be $\eta(\tau)$ with $q = e^{2\pi i\tau}$. From this we find

$$\begin{aligned}\eta(is/\pi)^{-24} &= q^{-1}(1 - q - q^2 + q^5 + q^7 + \cdots)^{-24} \\ &= e^{2s}\left(1 - e^{-2s} - e^{-4s} + e^{-10s} + e^{-14s} + \cdots\right)^{-24}\end{aligned} \qquad [7.213]$$

We now expand to the leading contributions in s:

$$\eta(is/\pi)^{-24} = e^{2s}(1 + 24e^{-2s} + \cdots) = e^{2s} + 24 + o\left(e^{-2s}\right) \qquad [7.214]$$

7.52 p 224: The Long Cylinder is a Closed String

The partition function for the cylinder can now be written as

$$Z_{C_2}(s \to \infty) = \frac{iV_{26}n^2}{2(8\pi^2\alpha')^{13}} \int_0^\infty ds\ \left[e^{2s} + 24 + o\left(e^{-2s}\right)\right] \qquad [7.215]$$

We can compare this with the vacuum energy of a torus for large τ_2 in (7.3.15)

$$Z_{T^2}(\tau \to \infty) = iV_{26} \int \frac{d\tau_2}{2\tau_2}(4\pi^2\alpha'\tau_2)^{-13}\left(e^{4\pi\tau_2} + 24^2 + \cdots\right) \qquad [7.216]$$

It should of course not be identical because a torus is a cylinder closed on itself. But we see that there is a similarity in the behaviour for large s and large τ_2. In both cases the partition function is divergent due to the contribution from the tachyonic states. The limit $s \to \infty$ corresponds to $t \to 0$, which is a very long cylinder with very small diameter. In that sense we can say that it looks like a close string propagating from one point to another, when we have the Euclidean worldsheet coordinates interchanged. We tend to think as σ^2 as being worldsheet time and σ^1 as worldsheet space. But there is no one living on that worldsheet that can tell us what is time and what is space. As far as the worldsheet is concerned they are just two coordinates, and even more so, they have the same signature. So who is to say which coordinate represents what?

7.53 p 224: Eq. (7.4.5) The Analytic Continuation of the Tachyon Divergence

If we consider the integral

$$I(\Lambda) = \int_0^\Lambda ds\, e^{\beta s} = \frac{1}{\beta}\left(e^{\beta\Lambda} - 1\right) \qquad [7.217]$$

then the analytic continuation is obtained by just ignoring the Λ dependence. It just means you ignore the divergence. I don't understand why this is an analytic continuation. I also don't understand why the second term gives a divergence $1/0$ as the corresponding integral is $\int_0^\infty ds$. The argument that this looks like a zero-momentum closed string propagator between two disks (i.e. open strings) makes sense and thus gives a $1/k^2$ divergence, but that is a purely heuristic argument.

7.54 p 225: UV and IR Divergences

Let us digress a little bit on UV and IR divergences. If what follows is complete gibberish, then it would be fair to ask the question if the reader should not spend her or his time more usefully than by trying to read Joe's book. If it sounds vaguely familiar or reminiscent of things known a long time ago, the reader could consult her or his favourite QFT text book or, even better, my QFT notes on hepnotes.com.

UV divergences are divergences at high momentum/energy. This means that as we increase the energy of the process our theory is not well-defined and breaks down. It is usually a strong suggestion that the theory is just some kind of an effective field theory that lacks the details and granularity to describe the process at high energy.

A typical example is the Fermi four point interaction that is quite successful at describing the weak interaction at low energies. As an illustration, the interaction Lagrangian for this is of the general form

$$\mathcal{L}_{\text{in}} = 2\sqrt{2}G_F \left[\bar{\ell}\gamma^\mu \frac{1-\gamma^5}{2}\nu\right] \left[\bar{u}\gamma_\mu \frac{1-\gamma^5}{2}d\right] + \text{h.c.} \qquad [7.218]$$

This describes a four-point interaction between a lepton, neutrino, up quark and down quark, represented by the fields ℓ, ν, u and d respectively. The $(1-\gamma^5)/2$ is a projection operator ensuring all fermion fields are left-handed as the weak interaction is chiral. h.c. stands for Hermitian conjugate and G_F is the Fermi constant. This theory describes the weal interaction well at low energies but not at high energies; in fact it is not renormalisable. It is a low-energy effective theory for the electroweak theory where the four-point interaction is replaced by the propagation of an intermediate W boson. This then reduces the four-point vertex to two three-point vertices with coupling constant g and reduces the dimensionality of the coupling constant and makes the theory renormalisable and a better, let us just say correct, description at higher energy; see the figure below for an illustration.

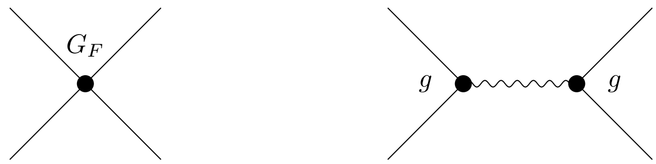


Figure 7.10: The non-renormalisable Fermi interaction vs the weak interaction. The propagating W particle ensures that there are only three–point vertices in this process. As a result the divergence can be renormalised.

The relation between the Fermi constant G_F and the weak coupling constant g is, see e.g, Peskin & Schröder (17.32).

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} \qquad [7.219]$$

where m_W is the mass of the intermediate W boson. Let us do some dimensional analysis. In the real world, i.e. four dimensions, fermion fields have (mass-)dimension $[\psi] = 3/2$. For the action to be dimensionless we need

$$0 = [G_F] + 4[x] + 4[\psi] = [G_F] + 4(-1) + 4(3/2) \quad \Rightarrow [G_F] = -2 \qquad [7.220]$$

so the Fermi constant has dimension minus two. We can then find the dimension of the weak coupling constant g from [7.219]

$$[g^2] = [G_F] + [m_W]^2 = -2 + 2 = 0 \qquad [7.221]$$

so g is dimensionless. Now it is, or at least should be well known, that interactions with negative dimension coupling constants are non-renormalisable, whilst interactions with dimensionless and positive dimension coupling constants are renormalisable, super-renormalisable respectively. The UV divergence of the four-point function is not renormalisable, but the the introduction of the propagating W-boson still gives a UV divergence but one that is renormalisable.

IR divergences, on the other hand, occur for very small momenta, or, equivalently, very large distances. These divergences are generally an artefact of how we solve the theory. Let us give the standard example from QED for the radiation of soft photons during Bremsstrahlung. This is the process where a low momentum photon is created and emitted during the scattering of an electron, see fig.7.11.

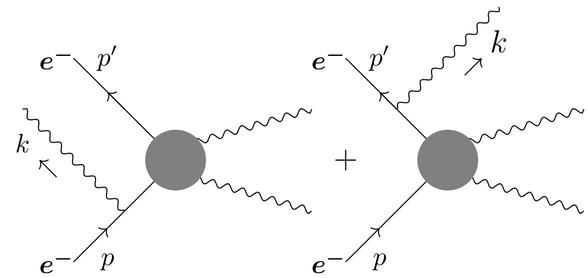


Figure 7.11: Bremsstrahlung in electron scattering. A very low momentum photon is emitted fro the incoming or outgoing electron, before and after the scattering process respectively.

The emission of such a soft photon creates a divergence. This is rather surprising because it is actually very hard, if not impossible, to measure a very soft photon. So something that is too small to measure would lead to a divergence?

The solution to this is to look at the so-called electron vertex correction. This is the one loop correction to the electron-electron-photon interaction vertex, see fig.7.12.

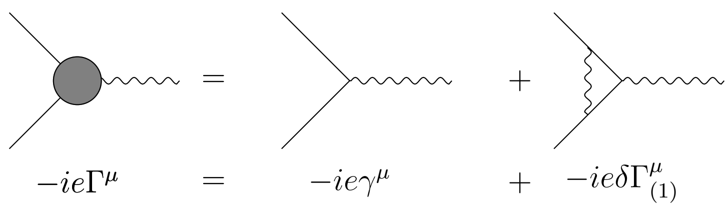


Figure 7.12: Electron vertex one loop radiative correction

The one-loop correction to this vertex also has an IR divergence, but it is exactly of the same form, but of opposite sign to the soft photon radiation above, so taken together these Feynman diagrams are perfectly well behaved, as shown in fig,7.13, where we also indicate the type of divergence . Note, en passant, that this type of divergence has basically a $(\log q^2)^2$ form and is known as a Sudakov double logarithm.

There is, of course, much more to IR divergences than just that. But the upshot of this is that, in general, whilst individual Feynman diagrams may have IR divergences, when all diagrams of the same order are combined, there is no IR divergence at all. Nature, as far as we know, does not care about the fact that we calculate results in QFT via diagrams, let alone on how we split them. So the appearance of IR divergencies, is, at least in this case, an artefact of how Feynman has told us we can easily calculate things.

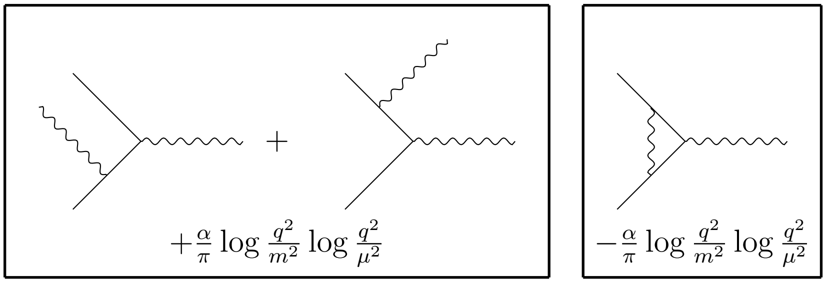


Figure 7.13: Cancellation of the first order QED IR divergence. Here q^2 is the momentum squared of the emitted photon and μ is a very small we have given to the photon.

7.55 p 226: Eq. (7.4.11)–(7.4.13) The Partition Function for the Cylinder from the Closed String

I must admit I am completely lost by this part. I understand the idea that you have a closed string propagating from a time $\sigma^1 = 0$ to a time $\sigma^1 = s$ and that the partition function is then

$$Z_{C_2} = \langle B| \, c_0 b_0 e^{-s(L_0+\tilde{L}_o)} \, |B\rangle \qquad [7.222]$$

with $|B\rangle$ denoting the closed string state at the boundary. The b_0c_0 are the ghost insertions for the cylinder which has one modulus and one CKV. The central charge term$(c + \tilde{c})/24$ vanishes because in the critical dimension the total central charge of matter plus ghost sector is zero.

But, I don't understand that you can determine the boundary state $|B\rangle$ by requiring it to vanish under $\partial_1 X^\mu, c^1$ and b b_{12}. Why these components? Why not $\partial_2 X^\mu$, c^2 or b_{11}? And what happens with the anti-holomorphic side?

Let us assume that these are the right conditions. In the traditional cylinder, see (2.6.4) we have $z = e^{-iw} = e^{-i\sigma^1+\sigma^2}$. Here we have interchanged the role of σ^1 and σ^2 and so we have

$$z = e^{-i\sigma^2+\sigma^1} \qquad [7.223]$$

and consequently

$$\partial_1 = \partial_1 z \partial + \partial_1 \bar{z}\bar{\partial} = z\partial + \bar{z}\bar{\partial} \qquad [7.224]$$

We then expand $\partial_1 X^\mu |B\rangle$, using the mode expansion (2.7.1)

$$\begin{aligned}\partial_1 X^\mu |B\rangle &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\left(z\frac{\alpha_m^\mu}{z^{m+1}} + \bar{z}\frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}\right)|B\rangle \\ &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\left(\frac{\alpha_m^\mu}{z^{m}} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m}}\right)|B\rangle \qquad [7.225]\end{aligned}$$

We now evaluate this at $\sigma^1 = 0$,

$$\begin{aligned}\partial_1 X^\mu |B\rangle \Big|_{\sigma^1=0} &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\left(\frac{\alpha_m^\mu}{e^{-im\sigma^2}} + \frac{\tilde{\alpha}_m^\mu}{e^{+im\sigma^2}}\right)|B\rangle \\ &= -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty}\left(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\right)e^{im\sigma^2}|B\rangle \qquad [7.226]\end{aligned}$$

and so requiring $\partial_1 X^\mu |B\rangle = 0$ is indeed equivalent to $\left(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\right)|B\rangle = 0$ for all n.

Consider now the condition that $c^1(w)$ vanishes at the boundary. Going to complex indices, we use (2.1.5). i.e. $v^w = v^2 + iv^1$ and $v^{\bar{w}} = v^2 - iv^1$, where we have interchanged the indices one and two, as we should. Thus $v^1(w) = (v^w(w) - w^{\bar{w}}(w))/2i$ and thus

$$c^1(w) = \frac{1}{2i}[c(w) - \tilde{c}(w)] \qquad [7.227]$$

We now still need to go to the ghost field on the complex z plane with $z = e^{-iw}$. For that we need the the rule for the transformation of a weight h primary field under $w \to z$. That is

$$(\partial_w z)^h \mathcal{O}(z) = \mathcal{O}(w) \qquad [7.228]$$

For the c-ghost with weight -1 this becomes

$$(-iz)^{-1}c^a(z) = c^a(w) \quad \Rightarrow c^a(w) = iz^{-1}c(z) \qquad [7.229]$$

and similarly $\tilde{c}^a(w) = -i\bar{z}^{-1}\tilde{c}(\bar{z})$. Therefore

$$\begin{aligned}c^1(w)|B\rangle &= \frac{1}{2i}[c(w) - \tilde{c}(w)]|B\rangle = \frac{1}{2i}\left[iz^{-1}c(z) - \left(-i\bar{z}^{-1}\tilde{c}(\bar{z})\right)\right]|B\rangle \\ &= \frac{1}{2}\left[z^{-1}c(z) + \bar{z}^{-1}\tilde{c}(\bar{z})\right]|B\rangle \qquad [7.230]\end{aligned}$$

We know fill in the Laurent expansion

$$\begin{aligned}c^1(w)|B\rangle &= \frac{1}{2}\left[z^{-1}\sum_{m=-\infty}^{\infty}\frac{c_m}{z_{m-1}} + \bar{z}^{-1}\sum_{m=-\infty}^{\infty}\frac{\tilde{c}_m}{\bar{z}_{m-1}}\right]|B\rangle \\ &= \frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\frac{c_m}{z_m} + \frac{\tilde{c}_m}{\bar{z}_m}\right)|B\rangle \qquad [7.231]\end{aligned}$$

We now use [7.223] and set $\sigma^1 = 0$

$$c^1(w)\,|B\rangle = \frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\frac{c_m}{e^{-im\sigma^2}} + \frac{\tilde{c}_m}{e^{+im\sigma^2}}\right)|B\rangle = \frac{1}{2}\sum_{m=-\infty}^{\infty}(c_m + \tilde{c}_{-m})e^{im\sigma^2}\,|B\rangle \qquad [7.232]$$

Requiring $c^1(w)$ to vanish on the boundary this implies that $(c_m + \tilde{c}_{-m})\,|B\rangle = 0$ for all m.

The reader should not be confused that we use a different method for the condition on from $\partial_1 X^\mu$ and from c^1. They are actually consistent. In the $\partial_1 X^\mu$ we such used the fact that∂_1 increases the weight by one, so that X^μ has $h = 0$. One more comment that may have passed unnoticed. The indices on the c and b ghosts have been chosen so that they transform nicely under conformal transformations.

Finally, let us consider the condition that b_{12} vanishes on the boundary. We follow the same procedure as for the c^1 condition. We first go to complex w coordinates using (2.1.5) and $v_w = (v_2 - iv_1)/2$ and $v_{\bar{w}} = (v_2 + iv_1)$ where we have, dutifully interchanged the indices. From this we have

$$v_1 = v_w + v_{\bar{w}} \qquad \text{and} \qquad v_2 = i(v_w - v_{\bar{w}}) \qquad [7.233]$$

And thus

$$b_{12} = b_{w2} + b_{\bar{w}2} = i(b_{ww} - b_{w\bar{w}} + b_{\bar{w}w} - b_{\bar{w}\bar{w}}) = i(b_{ww} - b_{\bar{w}\bar{w}}) \qquad [7.234]$$

where we used the fact that b is traceless by the Faddeev-Popov construction, i.e. $b_{w\bar{w}} = b_{\bar{w}w} = 0$. We now perform the conformal transformation to z and as b has weight two we find

$$(-iz)^2 b^{ab}(z) = b^{ab}(w) \quad \Rightarrow b^{ab}(w) = -z^2 b^{ab}(z) \qquad [7.235]$$

and similarly $b^{ab}(\bar{w}) = -\bar{z}^2 b^{ab}(\bar{z})$. Therefore

$$\begin{aligned} b_{12}(w)\,|B\rangle &= i[b_{ww} - \tilde{b}_{\bar{w}\bar{w}}]\,|B\rangle = i[-z^2 b_{zz} - (-\bar{z}^2\tilde{b}_{\bar{z}\bar{z}}]\,|B\rangle \\ &= -\,i[z^2 b_{zz} - \bar{z}^2\tilde{b}_{\bar{z}\bar{z}}]\,|B\rangle \end{aligned} \qquad [7.236]$$

Using the Laurent expansion we therefore find

$$\begin{aligned} b_{12}(w)\,|B\rangle &= i\left[z^2\sum_{m=-\infty}^{\infty}\frac{b_m}{z_{m+2}} - \bar{z}^2\sum_{m=-\infty}^{\infty}\frac{\tilde{b}_m}{\bar{z}_{m+2}}\right]|B\rangle \\ &= i\sum_{m=-\infty}^{\infty}\left(\frac{b_m}{z_m} - \frac{\tilde{b}_m}{\bar{z}_m}\right)|B\rangle \end{aligned} \qquad [7.237]$$

We now use [7.223] and set $\sigma^1 = 0$

$$b_{12}(w)\,|B\rangle = i\sum_{m=-\infty}^{\infty}\left(\frac{b_m}{e^{-im\sigma^2}} - \frac{\tilde{b}_m}{e^{+im\sigma^2}}\right)|B\rangle = i\sum_{m=-\infty}^{\infty}(b_m - \tilde{b}_{-m})e^{im\sigma^2}\,|B\rangle \qquad [7.238]$$

Requiring $b_{12}(w)$ to vanish on the boundary this implies that $(b_m - \tilde{b}_{-m})\,|B\rangle = 0$ for all m.

7.56 p 226: Eq. (7.4.13) The Explicit Form of the Boundary State

Taking into account the correction on Joe's errata page, we need to show that the state

$$|B\rangle = e^{-\sum_{n=1}^{\infty}\left(\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}+b_{-n}\tilde{c}_{-n}+\tilde{b}_{-n}c_{-n}\right)}(c_0+\tilde{c}_0)\,|0;0\rangle \quad [7.239]$$

satisfies the conditions (7.4.12). Let us for simplicity define

$$\mathfrak{X} = -\sum_{n=1}^{\infty}\left(\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}+b_{-n}\tilde{c}_{-n}+\tilde{b}_{-n}c_{-n}\right) \quad [7.240]$$

We start by considering $(c_m+\tilde{c}_{-m})e^{\mathfrak{X}}(c_0+\tilde{c}_0)\,|0;0\rangle$. We fist work out $c_m e^{\mathfrak{X}}$ and $\tilde{c}_m e^{\mathfrak{X}}$. We need to look at three separate cases depending on whether $m<0, m=0$ or $m>0$.

- $\underline{m<0}$: In this case c_m and $\tilde{c}_m$ both commute with $\mathfrak{X}$ and we simply have

$$c_m\,|B\rangle = c_m e^{\mathfrak{X}}(c_0+\tilde{c}_0)\,|0;0\rangle = -e^{\mathfrak{X}}(c_0+\tilde{c}_0)c_m\,|0;0\rangle \quad [7.241]$$
$$\tilde{c}_m\,|B\rangle = \tilde{c}_m e^{\mathfrak{X}}(c_0+\tilde{c}_0)\,|0;0\rangle = -e^{\mathfrak{X}}(c_0+\tilde{c}_0)\tilde{c}_m\,|0;0\rangle \quad [7.242]$$

- $\underline{m=0}$: Here c_m and $\mathfrak{X}$ commute as well and we have

$$\begin{aligned} c_m\,|B\rangle &= c_0 e^{\mathfrak{X}}(c_0+\tilde{c}_0)\,|0;0\rangle = e^{\mathfrak{X}}c_0(c_0+\tilde{c}_0)\,|0;0\rangle = e^{\mathfrak{X}}c_0\tilde{c}_0\,|0;0\rangle \\ \tilde{c}_m\,|B\rangle &= \tilde{c}_0 e^{\mathfrak{X}}(c_0+\tilde{c}_0)\,|0;0\rangle = e^{\mathfrak{X}}\tilde{c}_0(c_0+\tilde{c}_0)\,|0;0\rangle = -e^{\mathfrak{X}}c_0\tilde{c}_0\,|0;0\rangle \end{aligned} \quad [7.243]$$

- $\underline{m>0}$: Here we need to be more careful as c_m and $\mathfrak{X}$ don't commute, but we have

$$\begin{aligned} [c_m,\mathfrak{X}] &= \left[c_m, -\sum_{n=1}^{\infty}\left(\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}+b_{-n}\tilde{c}_{-n}+\tilde{b}_{-n}c_{-n}\right)\right] \\ &= -\sum_{n=1}^{\infty}[c_m, b_{-n}\tilde{c}_{-n}] = -\tilde{c}_{-m} \end{aligned} \quad [7.244]$$

  Thus

$$c_m\mathfrak{X} = \mathfrak{X}c_m - \tilde{c}_{-m} \quad \text{and} \quad \tilde{c}_m\mathfrak{X} = \mathfrak{X}\tilde{c}_m - c_{-m} \quad [7.245]$$

  Next

$$\begin{aligned} c_m\mathfrak{X}^2 &= (c_m\mathfrak{X})\mathfrak{X} = (\mathfrak{X}c_m - \tilde{c}_{-m})\mathfrak{X} = \mathfrak{X}c_m\mathfrak{X} - \tilde{c}_{-m}\mathfrak{X} \\ &= \mathfrak{X}(\mathfrak{X}c_m - \tilde{c}_{-m}) - \mathfrak{X}\tilde{c}_{-m} = \mathfrak{X}^2 c_m - 2\mathfrak{X}\tilde{c}_{-m} \end{aligned} \quad [7.246]$$

  because if $m>0$ then $\tilde{c}_{-m}$ and $\mathfrak{X}$ commute. Similarly, of course,

$$\tilde{c}_m\mathfrak{X}^2 = \mathfrak{X}^2\tilde{c}_m - 2\mathfrak{X}c_{-m} \quad [7.247]$$

Next

$$c_m\mathfrak{X}^3 = (c_m\mathfrak{X}^2)\mathfrak{X} = (\mathfrak{X}^2 c_m - 2\mathfrak{X}\tilde{c}_{-m})\mathfrak{X} \\ = \mathfrak{X}^2(\mathfrak{X}c_m - \tilde{c}_{-m}) - 2\mathfrak{X}^2\tilde{c}_{-m} = \mathfrak{X}^3 - 3\mathfrak{X}^2\tilde{c}_{-m} \quad [7.248]$$

We see the general pattern: for $m > 0$ and $k > 0$ we have

$$c_m\mathfrak{X}^k = \mathfrak{X}^k c_m - k\mathfrak{X}^{k-1}\tilde{c}_{-m} \quad [7.249]$$

Therefore

$$c_m e^{\mathfrak{X}} = \sum_{k=0}^{\infty}\frac{1}{k!}c_m\mathfrak{X}^k = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}c_m\mathfrak{X}^k = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\mathfrak{X}^k c_m - k\mathfrak{X}^{k-1}\tilde{c}_{-m}\right) \\ = \sum_{k=0}^{\infty}\frac{1}{k!}\mathfrak{X}^k c_m - \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\mathfrak{X}^{k-1}\tilde{c}_{-m} = e^{\mathfrak{X}}c_m - e^{\mathfrak{X}}\tilde{c}_{-m} \quad [7.250]$$

From this

$$c_m\left|B\right\rangle = c_m e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\left|0;0\right\rangle = \left(e^{\mathfrak{X}}c_m - e^{\mathfrak{X}}\tilde{c}_{-m}\right)(c_0 + \tilde{c}_0)\left|0;0\right\rangle \\ = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\tilde{c}_{-m}\left|0;0\right\rangle \quad [7.251]$$

and similarly

$$\tilde{c}_m\left|B\right\rangle = \tilde{c}_m e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\left|0;0\right\rangle = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)c_{-m}\left|0;0\right\rangle \quad [7.252]$$

We can now work out the impact of $c_m + \tilde{c}_{-m}$ on $\left|B\right\rangle$. If $m = 0$ the we simply have

$$(c_0 + \tilde{c}_0)\left|B\right\rangle = e^{\mathfrak{X}}c_0\tilde{c}_0\left|0;0\right\rangle - e^{\mathfrak{X}}c_0\tilde{c}_0\left|0;0\right\rangle = 0 \quad [7.253]$$

Now take $m > 0$. We then use [7.251] and [7.242] to find

$$(c_m + \tilde{c}_{-m})\left|B\right\rangle = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\tilde{c}_{-m}\left|0;0\right\rangle - e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\tilde{c}_{-m}\left|0;0\right\rangle = 0 \quad [7.254]$$

Of course, a similar relation holds if $m < 0$. We have thus shown that $(c_m + \tilde{c}_{-m})\left|B\right\rangle = 0$, which is what we set out to do.

Next, let us turn to $(\alpha^m u_n + \tilde{\alpha}^\mu{-n})\left|B\right\rangle$. Here we consider two cases

- $\underline{m \leq 0}$: α^μ_m commutes through $\mathfrak{X}$ and we have

$$\alpha^\mu_m\left|B\right\rangle = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\alpha^\mu_m\left|0;0\right\rangle \\ \tilde{\alpha}^\mu_m\left|B\right\rangle = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\tilde{\alpha}^\mu_m\left|0;0\right\rangle \quad [7.255]$$

- $\underline{m > 0}$: We have

$$\alpha^\mu_m\mathfrak{X} = -\sum_{n=1}^{\infty}\frac{1}{n}\alpha^\mu_m\alpha_{-n}\cdot\tilde{\alpha}_{-n} = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}\cdot\tilde{\alpha}_{-n}\alpha^\mu_m + m\delta_{m-n}\delta^\mu_\nu\tilde{\alpha}^\nu_{-m}\right) \\ = \mathfrak{X}\alpha^\mu_m - \tilde{\alpha}^\mu_{-m} \quad [7.256]$$

This is a similar relation as for the c-ghost and so we can immediately write down

$$\alpha^\mu_m e^{\mathfrak{X}} = e^{\mathfrak{X}}\alpha^\mu_m - e^{\mathfrak{X}}\tilde{\alpha}^\mu_{-m} \quad [7.257]$$

Thus, for $m = 0$ we have

$$(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\,|B\rangle = (\alpha_0^\mu + \tilde{\alpha}_0^\mu)e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\,|0;0\rangle = e^{\mathfrak{X}}(c_0 + \tilde{c}_0)(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\,|0;0\rangle = 0 \qquad [7.258]$$

by the fact that $\alpha_m^\mu = \tilde{\alpha}_m^\mu\,|0;0\rangle = 0$ for $m \geq 0$, the definition of the matter ground state. For $m > 0$ we have

$$\begin{aligned}(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu)\,|B\rangle &= (\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu)e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\,|0;0\rangle \\ &= \left(e^{\mathfrak{X}}\alpha_m^\mu - e^{\mathfrak{X}}\tilde{\alpha}_{-m}^\mu\right)(c_0 + \tilde{c}_0)\,|0;0\rangle + e^{\mathfrak{X}}\tilde{\alpha}_{-m}^\mu(c_0 + \tilde{c}_0)\,|0;0\rangle = 0 \qquad [7.259]\end{aligned}$$

as two of the terms cancel and the other one annihilates the ground state. The same, of course, holds for $m < 0$ and we have thus shown that $(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu)\,|B\rangle = 0$ as well for all m.

Let us finally tackle the case of $(b_n - \tilde{b}_{-n})\,|B\rangle$ and see how this sign comes in handy here. We split in two cases

- $\underline{m \leq 0}$: The b-ghost commutes through the $\mathfrak{X}$ and we simply have

$$\begin{aligned}b_m\,|B\rangle &= e^{\mathfrak{X}}b_m(c_0 + \tilde{c}_0)\,|0;0\rangle \\ \tilde{b}_m\,|B\rangle &= e^{\mathfrak{X}}\tilde{b}_m(c_0 + \tilde{c}_0)\,|0;0\rangle \qquad [7.260]\end{aligned}$$

- $\underline{m > 0}$: We have

$$\begin{aligned}b_m\mathfrak{X} &= -\sum_{n=1}^{\infty} b_m\tilde{b}_{-n}c_{-n} = +\sum_{n=1}^{\infty}\tilde{b}_{-n}b_m c_{-n} = -\sum_{n=1}^{\infty}\tilde{b}_{-nc_{-n}}b_m + \tilde{b}_{-m} \\ &= \mathfrak{X}b_m + \tilde{b}_{-m} \qquad [7.261]\end{aligned}$$

  This generalises to

$$b_m\mathfrak{X}^k = \mathfrak{X}^k b_m + k\mathfrak{X}^{k-1}\tilde{b}_{-m} \qquad [7.262]$$

  and

$$b_m e^{\mathfrak{X}} = e^{\mathfrak{X}}b_m + e^{\mathfrak{X}}\tilde{b}_{-m} \qquad [7.263]$$

Thus we have

$$\begin{aligned}(b_0 - \tilde{b}_0)\,|B\rangle &= (b_0 - \tilde{b}_0)e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\,|0;0\rangle = e^{\mathfrak{X}}(b_0 - \tilde{b}_0)(c_0 + \tilde{c}_0)\,|0;0\rangle \\ &= e^{\mathfrak{X}}(b_0c_0 + b_0\tilde{c}_0 - \tilde{b}_0c_0 - \tilde{b}_0\tilde{c}_0)\,|0;0\rangle \qquad [7.264]\end{aligned}$$

The second and the third term vanish because the b_0 and $\tilde{b}_0$ can be moved to the right and gives zero by the definition of the ghost ground state. For the first and last term we can mode the b-ghost to the right but pick up a factor one for each of them due to the anti-commutator. These two factor cancel and we thus find that $(b_0 - \tilde{b}_0)\,|B\rangle = 0$. Now, for $m > 0$ we have

$$\begin{aligned}(b_m - \tilde{b}_{-m})\,|B\rangle &= (b_m - \tilde{b}_{-m})e^{\mathfrak{X}}(c_0 + \tilde{c}_0)\,|0;0\rangle \\ &= \left(e^{\mathfrak{X}}b_m + e^{\mathfrak{X}}\tilde{b}_{-m}\right)(c_0 + \tilde{c}_0)\,|0;0\rangle - e^{\mathfrak{X}}\tilde{b}_{-m}(c_0 + \tilde{c}_0)\,|0;0\rangle \\ &= 0 \end{aligned} \qquad [7.265]$$

because in the first term we can move the b_m to the right and it gives zero, whilst in the second term cancels the third term. The same, of course, holds for $m < 0$. We have thus shown that $(b_m - \tilde{b}_{-m})\,|B\rangle = 0$ for all m.

We will not show that plugging (7.4.13) into (7.4.11) gives, up to a normalisation, (7.4.3). That seems like a real nightmare calculation. If anyone has a simple way to demonstrate this, please send me an e-mail via hep.notes@hotmail.com.

7.57 p 226: Eq. (7.4.14) The Vacuum Amplitude for the Klein Bottle, I

Recall that the Klein bottle is a cylinder with a parity transformation Ω on one of the boundaries and the boundaries then sewn together. The Klein bottle has one modulus and one CKV. We thus have one b and one c insertion. We thus have the same integration over the moduli as for the cylinder. The momentum integration contribution of $8\pi^2\alpha' t$ that came from the open string of the cylinder now gets replaced by $4\pi^2\alpha' t$ from the closed string contribution. The extra factor $1/2$ comes, I believe, from the same symmetry as for the torus $w \to -w$. All of this gives (7.4.14).

7.58 p 226: Eq. (7.4.14) The Vacuum Amplitude for the Klein Bottle, II

Consider a typical state of the form

$$|\psi\rangle = (\alpha^{\mu}_{-k})^K(\alpha^{\nu}_{-\ell})^L(\tilde{\alpha}^{\sigma}_{-m})^M\,|0;0\rangle \qquad [7.266]$$

The contribution of that state to the trace is

$$\begin{aligned}\langle\psi|\,\Omega\,|\psi\rangle &= \left\langle (\alpha^{\mu}_{-k})^K(\alpha^{\nu}_{-\ell})^L(\tilde{\alpha}^{\sigma}_{-m})^M \middle|\Omega\middle| (\alpha^{\mu}_{-k})^K(\alpha^{\nu}_{-\ell})^L(\tilde{\alpha}^{\sigma}_{-m})^M \right\rangle \\ &= \left\langle (\alpha^{\mu}_{-k})^K(\alpha^{\nu}_{-\ell})^L(\tilde{\alpha}^{\sigma}_{-m})^M \middle| (\tilde{\alpha}^{\mu}_{-k})^K(\tilde{\alpha}^{\nu}_{-\ell})^L(\alpha^{\sigma}_{-m})^M \right\rangle \end{aligned} \qquad [7.267]$$

where we have use (1.4.19) i.e. $\Omega\alpha^{\mu}_{n}\Omega^{-1} = \tilde{\alpha}^{\mu}_{n}$, its conjugate and the fact that the ground state is invariant under Ω. We see that this contribution is zero and that a non-zero contribution to the trace can only occur if every left-handed oscillator is matched a right-handed oscillator with the same index, and vice-versa. Only then do we have just the right amount of left- and right-handed oscillators to commute through one another and leave a non-zero result. If we have a general state of left-moving oscillators that we denote by $\boldsymbol{L}$ and right-moving oscillators that we denote by $\boldsymbol{R}$ then that state has a non-vanishing contribution to the trace $\langle \boldsymbol{LR}|\,\Omega\,|\boldsymbol{LR}\rangle$ iff $\boldsymbol{L} = \Omega\boldsymbol{R}\Omega^{-1}$, and, of course, it follows that conversely $\boldsymbol{R} = \Omega\boldsymbol{L}\Omega^{-1}$. So, if we denote by $\widetilde{\boldsymbol{A}}$ the operation of adding a tilde where $\boldsymbol{A}$ has none and removing one when $\boldsymbol{A}$ already has one, then the trace is s sum over all states A of the form $\langle \widetilde{\boldsymbol{A}}\boldsymbol{A}|\widetilde{\boldsymbol{A}}\boldsymbol{A}\rangle$. So if we sum over all left-moving states and call that $\boldsymbol{A}$, then all the right-moving states are included. Hence this is the same as the sum over open string states. Looking at (7.4.1) this would give us a contribution $\eta(it)^{-24}$. But recall that the t comes from the sum in the exponent of $q^{L_0} = q^{\alpha' p^2 + \sum_{n=1}^{\infty} n \sum_{\mu=0}^{\infty} N_{\mu n}}$ in [7.203]. Now we have $q^{L_0+\tilde{L}_0}$ with only non-vanishing contributions to the trace if $L_0 = \tilde{L}_0$ so we double that amount and need to replace t by $2t$.[6]

Taking all this into account, we conclude that we can write the vacuum amplitude for the Klein bottle from looking at the vacuum amplitude for the cylinder (7.4.1) as

$$Z_{K_2} = iV_{26} \int_0^\infty \frac{dt}{t} (4\pi^2\alpha' t)^{-13} \eta(2it)^{-24} \qquad [7.268]$$

which is (7.4.15).

7.59 p 227: Eq. (7.4.17) An Identification for the Klein Bottle, I

With $w = \sigma_1 + i\sigma_2$, we have form

$$w \cong w + 2\pi \quad \Leftrightarrow \quad \sigma^1 + i\sigma^2 \cong \sigma^1 + 2\pi + i\sigma 2 \quad \Rightarrow \quad \sigma^1 \cong \sigma^1 + 2\pi \qquad [7.269]$$

and also

$$w \cong \bar{w} + 2\pi i t \quad \Leftrightarrow \quad \sigma^1 + i\sigma^2 \cong -\sigma^1 + i\sigma^2 + 2\pi i t \cong (2\pi - \sigma^1) + i(\sigma^2 + 2\pi t) \qquad [7.270]$$

where we have used the periodicity of σ^1 to bring it back in the range $[0, \pi]$. This is exactly the description of the Klein bottle, see fig.7.5.

7.60 p 227: Eq. (7.4.17) An Identification for the Klein Bottle, II

We show that (7.4.18) follows from (7.4.17).

$$w \cong w + 4\pi i t \quad \Leftrightarrow \quad \sigma^1 + i\sigma^2 \cong \sigma^1 + i(\sigma^2 + 4\pi t) \quad \Rightarrow \quad \sigma^2 \cong \sigma^2 + 4\pi \qquad [7.271]$$

[6]This is also consistent with replacing the opens string factor $8\pi^2\alpha' t$ by $4\pi^2\alpha' t$. Indeed a factor two comes from replacing t by $2t$ and a factor $1/4$ comes because for the closed string we replace α' by $\alpha'/4$, so overall we have to divide the open string contribution by two.

We can obtain this by applying the second part of (7.4.17) twice. Indeed, that part says that

$$\sigma^1 \cong -\sigma^1 \quad \text{with} \quad \sigma^2 \cong \sigma^2 + 2\pi t \tag{7.272}$$

Let us apply it once more and we have

$$-\sigma^1 \cong \sigma^1 \quad \text{with} \quad \sigma^2 + 2\pi t \cong \sigma^2 + 2\pi t + 2\pi t = \sigma^2 + 4\pi t \tag{7.273}$$

and so we have indeed $\sigma^2 \cong \sigma^2 + 4\pi t$ or $w \cong w + 4\pi i t$.

The second part of (7.4.18) is obtained by first taking the second part of (7.4.17) and then applying the first part

$$w \cong -\bar{w} + 2\pi i t \cong -(\bar{w} + 2\pi) + 2\pi i t \tag{7.274}$$

from which it follows that, by adding π to both sides,

$$w + \pi \cong -(\bar{w} + \pi) + 2\pi i t \tag{7.275}$$

If we work this out for the components we find

$$\sigma^1 + \pi + i\sigma^2 \cong -(\sigma^1 + \pi) + i(\sigma^2 + 2\pi t) \tag{7.276}$$

or, using $\sigma_1 \cong \sigma_1 + 2\pi$,

$$\sigma^1 + \pi \cong \pi - \sigma^1 \quad \text{and} \quad \sigma^2 \cong \sigma^2 + 2\pi t \tag{7.277}$$

which fits the description in the text of a cross-cap.

7.61 p 227: Eq. (7.4.19) The Klein Bottle Amplitude as a Cylinder with Two Cross-Caps

We perform the transformation $t = 1/2s$ in (7.4.15).

$$\begin{aligned} Z_{K^2} &= iV_{26} \int_\infty^0 \left(-\frac{ds}{2s^2}\right) \frac{1}{4/2s} \left(\frac{4\pi^2\alpha'}{2s}\right)^{-13} \eta\left(\frac{2i}{2s}\right)^{-24} \\ &= \frac{iV_{26}}{4} \left(\frac{8\pi^2\alpha'}{4}\right)^{-13} \int_0^\infty ds\, s^{12} \eta(i/s)^{-24} \\ &= i\frac{2^{26} V_{26}}{4(8\pi^2\alpha')^{13}} \int_0^\infty ds\, s^{12} \eta(-1/is)^{-24} \end{aligned} \tag{7.278}$$

Use (7.2.44b) $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$ to find

$$\begin{aligned} Z_{K^2} &= i\frac{2^{26}V_{26}}{4(8\pi^2\alpha')^{13}} \int_0^\infty ds\, s^{12} \left[(-iis)^{1/2}\eta(is)\right]^{-24} \\ &= i\frac{2^{26}V_{26}}{4(8\pi^2\alpha')^{13}} \int_0^\infty ds\, s^{12} s^{-12}\eta(is)^{24} \\ &= i\frac{2^{26}V_{26}}{4\pi(8\pi^2\alpha')^{13}} \int_0^\infty ds\, \eta(is/\pi)^{24} \end{aligned} \quad [7.279]$$

In the last line we have just rescaled $s \to s/\pi$.

7.62 p 227: Eq. (7.4.21) The Oscillator Trace for the Möbius Strip

The trace over the matter sector is similar as for the left-handed side of the torus, see table 7.1, but all odd level acquire a minus sign. The trace is then, for one matter field,

$$\mathrm{Tr}\,\Omega q^{\sum_{n=1}^\infty nN_n - 1} = q^{-1}\sum_{k=0}^\infty (-)^k p(k) q^k \quad [7.280]$$

$q = e^{2\pi i\tau} = e^{-2\pi t}$ and $p(k)$ the number of partitions of n. One can rewrite this as

$$\mathrm{Tr}\,\Omega q^{\sum_{n=1}^\infty nN_n - 1} = q^{-1}\prod_{\ell=1}^\infty \sum_{m=0}^\infty (-q)^{\ell m} \quad [7.281]$$

as is easily checked by working out the first few factors of the infinite product

$$\begin{aligned} &\left(1 - q + q^2 - q^3 + q^4 - \cdots\right) \times \left(1 - q^2 + q^4 - q^6 + \cdots\right) \times \left(1 - q^3 + q^6 - q^9 + \cdots\right) \\ &\times \left(1 - q^4 + q^8 + \cdots\right) \times \cdots \end{aligned} \quad [7.282]$$

The geometric sums can be rewritten as

$$\mathrm{Tr}\,\Omega q^{\sum_{n=1}^\infty nN_n - 1} = q^{-1}\prod_{\ell=1}^\infty \frac{1}{1-(-q)^\ell} \quad [7.283]$$

Taking into account 24 transverse oscillations we this get for the trace

$$q^{-1}\prod_{n=1}^\infty \left[1 - (-q)^n\right]^{-24} \quad [7.284]$$

We now use (7.2.38a) the product representation for $\vartheta_{00}(\nu, \tau)$ and (7.2.43) the definition of the Dedekind function, with $\tau = 2it$

$$\begin{aligned}\vartheta_{00}(0, 2it)\eta(2it) &= \left[\prod_{m=1}^{\infty}(1 - e^{-4\pi tm})(1 - e^{-4\pi t(m-1/2)})^2\right] e^{-4\pi t/24} \prod_{m=1}^{\infty}(1 - e^{-4\pi tm}) \\ &= e^{-4\pi t/24} \prod_{m=1}^{\infty}(1 - e^{-4\pi tm})^2(1 - e^{-4\pi t(m-1/2)})^2 \end{aligned} \qquad [7.285]$$

so that

$$\begin{aligned}\vartheta_{00}(0, 2it)^{-12}\eta(2it)^{-12} &= e^{2\pi t} \prod_{m=1}^{\infty}\left[(1 - e^{-4\pi tm})(1 - e^{-4\pi t(m-1/2)})\right]^{-24} \\ &= q^{-1} \prod_{m=1}^{\infty}\left[(1 - q^{2m})(1 - q^{2(m-1/2)})\right]^{-24} \\ &= q^{-1} \prod_{m=1}^{\infty}(1 - q^{2m} + q^{2m-1} - q^{4m-1})^{-24} \end{aligned} \qquad [7.286]$$

It so happens that

$$\prod_{m=1}^{\infty}(1 - (-q)^m) = \prod_{m=1}^{\infty}(1 - q^{2m} + q^{2m-1} - q^{4m-1}) \qquad [7.287]$$

as can easily be checked by e.g. working out the first few powers of q. Adding the momentum integral and the ghost insertions we thus find for the vacuum energy of the Möbius strip

$$Z_{M_2} = iV_{26} \int_0^{\infty} \frac{dt}{4t}(8\pi^2\alpha' t)^{-13}\vartheta_{00}(0, 2it)^{-12}\eta(2it)^{-12} \qquad [7.288]$$

7.63 p 227: Eq. (7.4.23) The Möbius Strip as Cylinder with a Boundary and a Cross-Cap

Setting $t = \pi/4s$ we find

$$\begin{aligned}Z_{M_2} &= \pm\, inV_{26} \int_{\infty}^{0} \left(-\frac{ds}{4s}\right)\left(8\pi^2\alpha'\frac{\pi}{4s}\right)^{-13}\left[\vartheta_{00}\left(0, 2i\frac{\pi}{4s}\right)\eta\left(2i\frac{\pi}{4s}\right)\right]^{-12} \\ &= \pm \frac{inV_{26}}{4(8\pi^2\alpha')^{13}}\left(\frac{4}{\pi}\right)^{13}\int_0^{\infty} ds\, s^{-12}\left[\vartheta_{00}\left(0, -\frac{\pi}{2is}\right)\eta\left(-\frac{\pi}{2is}\right)\right]^{-12}\end{aligned} \qquad [7.289]$$

Using the transformation of ϑ_{00} in (7.2.40) and of η in (7.2.44)

$$\begin{aligned} Z_{M_2} &= \pm \frac{inV_{26}}{4(8\pi^2\alpha')^{13}} \left(\frac{4}{\pi}\right)^{13} \int_0^\infty ds\, s^{-12} \left[\left(-i\frac{2is}{\pi}\right)^{1/2} \vartheta_{00}\left(0, \frac{2is}{\pi}\right) \left(-i\frac{2is}{\pi}\right)^{1/2} \left(\frac{2is}{\pi}\right)\right]^{-12} \\ &= \pm \frac{inV_{26}}{4(8\pi^2\alpha')^{13}} \left(\frac{4}{\pi}\right)^{13} \int_0^\infty ds\, s^{-12} \left(\frac{2s}{\pi}\right)^{-12} \left[\vartheta_{00}\left(0, \frac{2is}{\pi}\right) \left(\frac{2is}{\pi}\right)\right]^{-12} \\ &= \pm \frac{2in2^{13}V_{26}}{4\pi(8\pi^2\alpha')^{13}} \int_0^\infty ds \left[\vartheta_{00}\left(0, \frac{2is}{\pi}\right) \left(\frac{2is}{\pi}\right)\right]^{-12} \end{aligned} \quad [7.290]$$

Chapter 8

Toroïdal Compactification and T-Duality

Open Questions

I have a number of unanswered points for this chapter. They are briefly mentioned here and more detail is given under the respective headings. Any help in resolving them can be sent to hepnotes@hotmail.com and is more than welcome.

- ♣ (8.1.6) In deriving the infinite tower of fields due to the Kaluza-Klein compactification, one needs to assume $G^{\mu d} = 0$. I believe this derivation of the equation and the corresponding mass of these states is meant to be in flat spacetime and merely serves as an illustration of the more general case.
- ♣ p240 When considering the action of a DDF operator on a state of given momentum q, i.e. corresponding to the OPE $\partial X^i e^{ink_0 X^+}(z) f(\partial^i \bar{\partial}^j X^k) e^{iq\cdot X}(0)$ Joe finds a term with a factor $z^{-\alpha' n k_0 q^-/2}$. I believe this should rather be a term with a factor $z^{-\alpha' n k_0 q^+/2}$. Indeed this term comes from the $X^+(z)X^-(0)$ OPE and the $X^-(0)$ comes with q^+. It is a bit strange that this is not on Joe's errata page.
- ♣ (8.3.3) In the compactified bosonic string the massless scalar $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\,|0;k\rangle$ is said to be "the modulus for the radius of the compactified direction". It is clear that this state is linked to the compactified dimension, but what does this statement exactly mean?
- ♣ (8.4.25) Show that if Γ is an even self-dual lattice then so is $\Lambda\Gamma$ with Λ and $O(k,k;\mathbb{R})$ transformation.
- ♣ (8.5.19) In discussing general orbifold theory with a non-Abelian subgroup H and projection operator $P_H = [\mathrm{order}(h)]^{-1}\sum_{h_2\in H}\hat{h}_2$ Joe argues that the diagonal matrix elements of $\hat{h}_2$ are zero. I don's see that.
- ♣ (8.6.6) I have a sign difference with the Hamiltonian for a point particle with charge q in a compactified dimension. It must be linked to some Minkowski-Euclidean transformation I did incorrectly, but I can't figure it out.
- ♣ (8.6.9) It isn't clear to me why $|ij\rangle$ has charge $+1$ under $U(1)_i$ and change -1 under $U(1)_j$. The charge is linked to the phase the open string picks up in Wilson loop. The open string has an λ^i Chan-Paton factor on one end and a λ^j Chan Paton factor at the other end. Why do they pick up a different sign?
- ♣ (8.7.5) In his Little Book of Strings [16] Joe refers to Pythagoras. Yes, it looks like an application of Pythagoras, but where does it come from? The original infinitesimal two-dimensional element $dX^1\,dX^2$ gets collapsed into a one-dimensional one. How does good-old Pythagoras come into play? This is a Greek mystery to me.

♣ (8.7.23) I cannot reproduce the graviton propagator from the spacetimeaction.

8.1 p 231: Eq. (8.1.2) The Metric in $D = d + 1$ Dimensions

Write out (8.1.2)

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\mu + G_{dd}(dx^d dx^d + 2A_\mu dx^\mu dx^d + A_\mu A_\nu dx^\mu dx^\nu) \\ &= (G_{\mu\nu} + G_{dd} A_\mu A_\nu) dx^\mu dx^\nu + 2G_{dd} A_\mu dx^\mu dx^d + G_{dd} dx^d dx^d \end{aligned} \quad [8.1]$$

Comparing this with $ds^2 = G^D_{MN}$ we see that

$$\begin{aligned} G^D_{\mu\nu} &= G_{\mu\nu} + G_{dd} A_\mu A_\nu \\ G^D_{\mu d} &= G_{dd} A_\mu \\ G^D_{dd} &= G_{dd} \end{aligned} \quad [8.2]$$

8.2 p 232: Eq. (8.1.4) The Gauge Transformation of the Kaluza-Klein Vector

Let us do this carefully. In the primed reference frame we have

$$\begin{aligned} ds'^2 = &[G'_{\mu\nu}(x'^\mu) + G'_{dd} A'_\mu(x'^\mu) A'_\nu(x'^\mu)] dx'^\mu dx'^\nu + 2G'_{dd}(x'^\mu) A'_\mu(x'^\mu) dx'^\mu dx'^d \\ &+ G'_{dd}(x'^\mu) dx'^d dx'^d \end{aligned} \quad [8.3]$$

We are only looking at a transformation of x^d so $x'^\mu = x^\mu$:

$$\begin{aligned} ds'^2 = &[G'_{\mu\nu}(x^\mu) + G'_{dd}(x^\mu) A'_\mu(x^\mu) A'_\nu(x^\mu)] dx^\mu dx^\nu + 2G'_{dd}(x^\mu) A'_\mu(x^\mu) dx^\mu dx'^d \\ &+ G'_{dd}(x^\mu) dx'^d dx'^d \end{aligned} \quad [8.4]$$

We don't write the x^μ dependence anymore and use $dx'^d = dx^d + \partial_\mu \lambda dx^\mu$

$$\begin{aligned} ds'^2 &= (G'_{\mu\nu} + G'_{dd} A'_\mu A'_\nu) dx^\mu dx^\nu + 2G'_{dd} A'_\mu dx^\mu (dx^d + \partial_\nu \lambda dx^\nu) \\ &\quad + G'_{dd}(dx^d + \partial_\mu \lambda dx^\mu)(dx^d + \partial_\nu \lambda dx^\nu) \\ &= [G'_{\mu\nu} + G'_{dd} A'_\mu A'_\nu + G'_{dd}(A'_\mu \partial_\nu \lambda + A'_\nu \partial_\mu \lambda) + G'_{dd} \partial_\mu \lambda \partial_\nu \lambda)] dx^\mu dx^\nu \\ &\quad + (2G'_{dd} A'^\mu + 2G'_{dd} \partial_\mu \lambda) dx^\mu dx^d + G'_{dd} dx^d dx^d \end{aligned} \quad [8.5]$$

Requiring $ds'^2 = ds^2$ we find from the $dx^d dx^d$ term

$$G'_{dd} = G_{dd} \quad [8.6]$$

so G_{dd} transforms as a scalar under this transformation. From the $dx^\mu dx^d$ term we see that we have invariance if the vector field transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda \tag{8.7}$$

Indeed, using this we find that

$$\begin{aligned} ds'^2 = &[G'_{\mu\nu} + G_{dd}\big[(A_\mu - \partial_\mu\lambda)(A_\nu - \partial_\nu\lambda) + (A_\mu - \partial_\mu\lambda)\partial_\nu\lambda \\ &+ (A_\nu - \partial_\nu\lambda)\partial_\mu\lambda + \partial_\mu\lambda\partial_\nu\lambda)\big]dx^\mu dx^\nu + A^\mu dx^\mu dx^d + G_{dd}dx^d dx^d \\ = &(G'_{\mu\nu} + G_{dd}A_\mu A_\nu)dx^\mu dx^\nu + 2A^\mu dx^\mu dx^d + G_{dd}dx^d dx^d \\ = &G'_{\mu\nu}dx^\mu dx^\nu + G_{dd}(A^\mu dx^\mu + dx^d)^2 \end{aligned} \tag{8.8}$$

and so we have invariance if $G'_{\mu\nu} = G_{\mu\nu}$ as well.

We conclude that we have invariance under $x^d \to x'^d = x^d + \lambda(x^\mu)$ and $x^\mu \to x'^\mu = x^\mu$ with $G_{\mu\nu}$ and G_{dd} unchanged and $A^\mu \to A'_\mu = A_\mu - \partial_\mu\lambda$. This is a $U(1)$ gauge transformation.

8.3 p 232: Eq. (8.1.5) Expanding the Compact Coordinate in a Complete Set

A complete set of functions for any coordinate x is e^{ipx}; it just provides the Fourier transform. For the compact coordinate we need to impose the boundary condition

$$e^{ip_d x^d} \cong e^{ip_d(x^d+2\pi R)} = e^{ip_d x^d}e^{2\pi i p_d R} \tag{8.9}$$

so this implies $p_d R = n$ for $n \in \mathbb{Z}$, i.e. quantisation of the "momentum" $p_d = n/R$. We can thus expand the dependence on the compact dimension of any scalar field in this complete set

$$\phi(x) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{ip_d x^d} = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^d/R} \tag{8.10}$$

8.4 p 232: Eq. (8.1.6) The Wave Equation for the Kaluza-Klein Theory

The location of the indices is crucial here. As $\partial_d\phi = (in/R)\phi$ we have

$$\begin{aligned} \partial_M\partial^M\phi = &(\partial_\mu\partial^\mu + \partial_d\partial^d)\sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^d/R} \\ = &\partial_\mu\partial^\mu \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^d/R} + \partial^d \sum_{n=-\infty}^{\infty} \frac{in}{R}\phi_n(x^\mu)e^{inx^d/R} \end{aligned} \tag{8.11}$$

Now $\partial^d = G^{d\mu}\partial_\mu + G^{dd}\partial_d$ and thus

$$\begin{aligned}\partial_M\partial^M\phi &= \partial_\mu\partial^\mu \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^d/R} + G^{\mu d}\partial_\mu \sum_{n=-\infty}^{\infty} \frac{in}{R}\phi_n(x^\mu)e^{inx^d/R} \\ &\quad + G^{dd}\partial_d \sum_{n=-\infty}^{\infty} \frac{in}{R}\phi_n(x^\mu)e^{inx^d/R} \\ &= \sum_{n=-\infty}^{\infty}\left[\partial_\mu\partial^\mu + G^{\mu d}\frac{in}{R}\partial_\mu - \frac{n^2}{R^2}\right]\phi_n(x^\mu)e^{inx^d/R} \qquad [8.12]\end{aligned}$$

and so the wave equation $\partial_M\partial^M\phi = 0$ becomes

$$\left(\partial_\mu\partial^\mu + \frac{in}{R}G^{\mu d}\partial^\mu\right)\phi_n = \frac{n^2}{R^2}\phi_n \qquad [8.13]$$

We have an additional term compared to Joe. One could argue that $G^{\mu d}\partial_\mu = \partial^d$ and that we have assumed there is no dependence on the compactified dimension. But that is no dependence on x^d. And $\partial^d = \partial\ /\partial x_d$ with $x_d = G_{dM}x^M = G_{\mu d}x^\mu + G_{dd}x^d$. So one needs to assume as well that $G_{\mu d} = 0$ but that would imply that there is no vector field A^μ. I actually believe that Joe meant this to be an example in flat spacetime, so that indeed $G^{\mu d} = 0$ and that it just serves as a simple illustration. This certainly seems to be suggested in the Notes of Chris Pope on Kaluza-Klein theory on page 3 of http://people.physics.tamu.edu/pope/ihplec.pdf.

Assuming this to be the case, we deduce that

$$\partial_\mu\partial^\mu\phi_n(x^\mu) = \frac{n^2}{R^2}\phi_n(x^\mu) \qquad [8.14]$$

where we have re-instated the x^μ dependence.

8.5 p 232: Eq. (8.1.7) The Infinite Tower of Kaluza Klein Fields

Eq. (8.1.6) is nothing but the Klein-Gordon equation for a field ϕ_n with mass squared n^2/R^2. The compactification of one of the dimensions of a scalar field ϕ thus leads to an infinite set of fields ϕ_n that behave like scalar fields of mass n^2/R^2 in the d spacetime dimensions. The mass shell condition for that field is $-p^2 = n^2/R^2$.

8.6 p 232: Eq. (8.1.18) The Ricci Scalar in the Kaluza-Klein Theory

This is an entirely straightforward albeit exceedingly tedious calculation. Even if one considers the fact that we should really only consider the terms that involve a G_{dd} and/or a

A_μ as everything without any of these automatically leads to $\boldsymbol{R}_d$. There is an argument based on dimensional analysis and symmetries that bypasses some of the calculation, even though there is still a lot left to do.

Let us denote the D dimensional quantities with a $\widetilde{}$ and the d-dimensional quantities without one. We start by looking at the mass-dimensions of the various quantities. These are

quantity	**dimension**	**explanation**
x^μ, x^d, r	-1	standard dimension of length
$\widetilde{G}_{\mu\nu}, G_{\mu\nu}, \sigma$	0	$[ds^2] = -2$ and $ds^2 = G_{MN}dx^M dx^N$
$\widetilde{R}, R, F_{\mu\nu}$	$+2$	$R \propto \partial\Gamma \propto \partial^2 G$ and $[\partial] = +1, [F] = [\partial A]$ and $[A] = +1$

Table 8.1: Mass-dimensions of Kaluza-Klein Fields. x^d is the compactified dimension and r is the radius of compactification.

Let us now write down all combinations of mass dimension two that we could use to expand $\widetilde{R}$. But let us keep in mind that we have a d-dimensional diffeomorphism so that we should have scalars under that and also that a gauge symmetry of the metric, i.e. under $x^d \longrightarrow x^d + \lambda$ we have $A_\mu \longrightarrow A_\mu - \partial\lambda$, so that A_μ can only appear in gauge invariant combinations. We are then lead to a combination

$$\widetilde{R} = aR + bF_{\mu\nu}F^{\mu\nu} + c(\nabla\sigma)^2 + d\sigma\nabla^2\sigma + e\nabla^2\sigma + f \qquad [8.15]$$

Here a, b, c, d, e and f are to be determined. They can depend on σ as that is a scalar in d-dimensional space time, but they cannot depend on $G_{\mu\nu}$ or on A_μ, as any such dependence is already in R and $F_{\mu\nu}F^{\mu\nu}$ respectively. We could combine $d\sigma\nabla^2\sigma + e\nabla^2\sigma$ into a $g\nabla^2\sigma$ with g a function of σ, but it is convenient to split it in this way.

What else can we say? Under a scaling

$$\begin{aligned} x^d &\longrightarrow \lambda x^d \\ A_\mu &\longrightarrow \lambda A_\mu \\ e^{2\sigma} &\longrightarrow \lambda^{-2} e^{2\sigma} \end{aligned} \qquad [8.16]$$

and keeping x^μ and $G_{\mu\nu}$ fixed, with λ a constant, the line element $ds^2 = G_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dx^d + A_\mu dx^\mu)^2$ is manifestly invariant. Under this scaling we obviously have

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &\longrightarrow \lambda^2 F_{\mu\nu}F^{\mu\nu} \\ \nabla_\mu\sigma &\longrightarrow \nabla_\mu\sigma \end{aligned} \qquad [8.17]$$

The latter relation follows from $\sigma \longrightarrow \sigma - \ln\lambda$. How does the curvature scale under this? It turns out that $\widetilde{R}$ remains unchanged under this scaling, and so does R as well.

This is not too hard to see if we investigate the scaling of the metric components. Let us for arguments sake momentarily take $D = 5$. We will just use this for illustration and will see that the arguments are general. The metric $\widetilde{G}_{MN}$ is now

$$\widetilde{G}_{MN} = \begin{pmatrix} G_{00} + e^{2\sigma} A_0 A_0 & G_{01} + e^{2\sigma} A_0 A_1 & G_{02} + e^{2\sigma} A_0 A_2 & G_{03} + e^{2\sigma} A_0 A_3 & e^{2\sigma} A_0 \\ G_{10} + e^{2\sigma} A_1 A_0 & G_{01} + e^{2\sigma} A_1 A_1 & G_{02} + e^{2\sigma} A_1 A_2 & G_{03} + e^{2\sigma} A_1 A_3 & e^{2\sigma} A_1 \\ G_{20} + e^{2\sigma} A_2 A_0 & G_{01} + e^{2\sigma} A_2 A_1 & G_{02} + e^{2\sigma} A_2 A_2 & G_{03} + e^{2\sigma} A_2 A_3 & e^{2\sigma} A_2 \\ G_{30} + e^{2\sigma} A_3 A_0 & G_{01} + e^{2\sigma} A_3 A_1 & G_{02} + e^{2\sigma} A_3 A_2 & G_{03} + e^{2\sigma} A_3 A_3 & e^{2\sigma} A_3 \\ e^{2\sigma} A_0 & e^{2\sigma} A_1 & e^{2\sigma} A_2 & e^{2\sigma} A_3 & e^{2\sigma} \end{pmatrix} \quad [8.18]$$

The first thing we notice is that the determinant of the D dimensional metric is very simply related to the determinant of the d dimensional metric. This can either be done by direct calculation, for those with sufficient stamina, or by noting that if one adds a column (row) of a matrix to another column (row) then the determinant of original and new matrices are the same. We see that the first three columns are just the G metric plus a coefficient times the fourth column.[1] This means that

$$\det \widetilde{G} = \det \begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} & e^{2\sigma} A_0 \\ G_{10} & G_{11} & G_{12} & G_{13} & e^{2\sigma} A_1 \\ G_{20} & G_{21} & G_{22} & G_{23} & e^{2\sigma} A_2 \\ G_{30} & G_{31} & G_{32} & G_{33} & e^{2\sigma} A_3 \\ 0 & 0 & 0 & 0 & e^{2\sigma} \end{pmatrix} = e^{2\sigma} \det G \quad [8.19]$$

The second observation is that the inverse of $\widetilde{G}_{MN}$ is very simple

$$\widetilde{G}^{MN} = \begin{pmatrix} G^{00} & G^{01} & G^{02} & G^{03} & -A^0 \\ G^{10} & G^{11} & G^{12} & G^{13} & -A^1 \\ G^{20} & G^{21} & G^{22} & G^{23} & -A^2 \\ G^{30} & G^{31} & G^{32} & G^{33} & -A^3 \\ -A_0 & -A_1 & -A_2 & -A_3 & e^{-2\sigma} + A_\mu A^\mu \end{pmatrix} \quad [8.20]$$

In other words

$$\begin{aligned} \widetilde{G}^{\mu\nu} &= G^{\mu\nu} \\ \widetilde{G}^{\mu d} &= -\, A^\mu \\ \widetilde{G}^{dd} &= e^{-2\sigma} + A_\mu A^\mu \end{aligned} \quad [8.21]$$

and let us just remind ourselves that

$$\begin{aligned} \widetilde{G}_{\mu\nu} &= G_{\mu\nu} + e^{2\sigma} A_\mu A_\nu \\ \widetilde{G}_{\mu d} &= e^{2\sigma} A_\mu \\ \widetilde{G}_{dd} &= e^{2\sigma} \end{aligned} \quad [8.22]$$

[1] It is this property of factorisation of the determinant, which, as we will see later, remains valid if more than one dimension is compactified, that warrants the choice of metric (8.1.2).

Here $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$. It is easily checked that $\widetilde{G}_{MN}\widetilde{G}^{NK} = \delta_M^K$. Indeed

$$\begin{aligned}
\widetilde{G}_{\mu N}\widetilde{G}^{N\rho} &= \widetilde{G}_{\mu\nu}\widetilde{G}^{\nu\rho} + \widetilde{G}_{\mu d}\widetilde{G}^{d\rho} = (G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu)G^{\nu\rho} + e^{2\sigma}A_\mu\,(-A^\rho) \\
&= G_{\mu\nu}G^{\nu\rho} = \delta_\nu^\rho \\
\widetilde{G}_{\mu N}\widetilde{G}^{Nd} &= \widetilde{G}_{\mu\nu}\widetilde{G}^{\nu d} + \widetilde{G}_{\mu d}\widetilde{G}^{dd} = (G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu)(-A^\nu) + e^{2\sigma}A_\mu(e^{-2\sigma} + A_\nu A^\nu) \\
&= -\,G_{\mu\nu}A^\nu + A^\mu = 0 \\
\widetilde{G}_{dN}\widetilde{G}^{Nd} &= \widetilde{G}_{d\nu}\widetilde{G}^{\nu d} + \widetilde{G}_{dd}\widetilde{G}^{dd} = e^{2\sigma}A_\nu(-A^\nu) + e^{2\sigma}(e^{-2\sigma} + A_\mu A^\mu) = 1
\end{aligned} \qquad [8.23]$$

From these expressions of the metric we find the scaling

$$\begin{aligned}
\widetilde{G}^{\mu\nu} &\longrightarrow \widetilde{G}_{\mu\nu} \\
\widetilde{G}^{\mu d} &\longrightarrow \lambda\widetilde{G}_{\mu d} \\
\widetilde{G}^{dd} &\longrightarrow \lambda^2\widetilde{G}_{dd}
\end{aligned} \qquad [8.24]$$

and for the inverse metric

$$\begin{aligned}
\widetilde{G}_{\mu\nu} &\longrightarrow \widetilde{G}^{\mu\nu} \\
\widetilde{G}_{\mu d} &\longrightarrow \lambda^{-1}\widetilde{G}^{\mu d} \\
\widetilde{G}_{dd} &\longrightarrow \lambda^{-2}\widetilde{G}^{dd}
\end{aligned} \qquad [8.25]$$

We see that any upper index ^{d} gives a factor λ and any lower index $_d$ gives a factor λ^{-1}. As the Ricci scalar is formed from the curvature tensor with all indices contracted and any ^{d} can only contract with a $_d$ we conclude that $\widetilde{R}$ indeed remains unchanged. R is also unchanged as both $\widetilde{G}_{\mu\nu}$ and $\widetilde{G}^{\mu\nu}$ are invariant under our scaling.

Let us now go back to [8.15] which we repeat for convenience

$$\widetilde{R} = aR + be^{2\sigma}F_{\mu\nu}F^{\mu\nu} + c(\nabla\sigma)^2 + d\sigma\nabla^2\sigma + e\nabla^2\sigma + f \qquad [8.26]$$

The LHS is invariant under our scaling. So the RHS must be invariant as well. Recall that a, b, c, d, e and f are still allowed to be functions of σ, but as R is invariant and $a(\sigma)R$ must also be invariant we need to have that a is independent of σ. Similarly as $F_{\mu\nu}F^{\mu\nu}$ scales as λ^2, we must have that b scales as λ^{-2} and so $b \propto e^{2\sigma}$. Finally $\nabla\sigma$ is invariant, so c, d and e must be independent of σ. We have thus established that

$$\widetilde{R} = \alpha R + \beta e^{2\sigma}F_{\mu\nu}F^{\mu\nu} + \gamma(\nabla\sigma)^2 + \delta\sigma\nabla^2\sigma + \varepsilon\nabla^2\sigma + f \qquad [8.27]$$

for some constants $\alpha, \beta, \gamma, \delta$ and ε. We now see the reason why we kept the split into d and e. We can still have f to be a function of σ, but with $f(\sigma = 0) = 0$, obviously.

To fix these constants we can look at special cases of the metric. But in order to do so we unfortunately need the expression for the Ricci scalar and this starts with an expression for the connections. Before we tackle this calculation, let me give you some pep-talk.

It turns out that Kaluza had first developed his theory with $\widetilde{G}_{\mu\nu} = G_{\mu\nu}$, $\widetilde{G}_{\mu d} = \kappa A_\mu$ and $\widetilde{G}_{dd} = \kappa$. This, however, made it prohibitively difficult to "split" the four dimensional theory from the fifth dimension, as we can see from the fact that the determinant $\widetilde{G}$ is then not easily expressible in terms of G, and from the fact that the inverse metric $\widetilde{G}^{MN}$ is already a mess. Imagine how the connections and curvature tensor must look like, let alone the Ricci tensor. It was Klein's insight, about five years later, to write the metric as we are now doing. And this led to dramatic simplifications in the calculations. Dramatic being a relative term here.

With this anecdote off our chest, we are ready to tackle the calculation of the connections. We start with

$$\begin{aligned}
\widetilde{\Gamma}^\lambda_{\mu\nu} =& \frac{1}{2}\widetilde{G}^{M\lambda}(\partial_\mu \widetilde{G}_{M\nu} + \partial_\nu \widetilde{G}_{M\mu} - \partial_M \widetilde{G}_{\mu\nu}) \\
=& \frac{1}{2}\widetilde{G}^{\rho\lambda}(\partial_\mu \widetilde{G}_{\rho\nu} + \partial_\nu \widetilde{G}_{\rho\mu} - \partial_\rho \widetilde{G}_{\mu\nu}) + \frac{1}{2}\widetilde{G}^{d\lambda}(\partial_\mu \widetilde{G}_{d\nu} + \partial_\nu \widetilde{G}_{d\mu} - \partial_d \widetilde{G}_{\mu\nu}) \\
=& \frac{1}{2}\Big\{ G^{\rho\lambda}\left[\partial_\mu(G_{\rho\nu} + e^{2\sigma}A_\rho A_\nu) + \partial_\nu(G_{\rho\mu} + e^{2\sigma}A_\rho A_\mu) - \partial_\rho(G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu)\right] \\
& - A^\lambda\left[\partial_\mu(e^{2\sigma}A_\nu) + \partial_\nu(e^{2\sigma}A_\mu) - \partial_d(G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu)\right]\Big\} \\
=& \Gamma^\lambda_{\mu\nu} + \frac{1}{2}e^{2\sigma}\Big\{G^{\rho\lambda}\big[2\partial_\mu\sigma A_\rho A_\nu + \partial_\mu A_\rho A_\nu + A_\rho \partial_\mu A_\nu \\
& + 2\partial_\nu\sigma A_\rho A_\mu + \partial_\nu A_\rho A_\mu + A_\rho\partial_\nu A_\mu - 2\partial_\rho\sigma A_\mu A_\nu - \partial_\rho A_\mu A_\nu - A_\mu\partial_\rho A_\nu\big] \\
& - A^\lambda\big[2\partial_\mu\sigma A_\nu + \partial_\mu A_\nu + 2\partial_\nu\sigma A_\mu + \partial_\nu A_\mu\big]\Big\}
\end{aligned} \qquad [8.28]$$

We have used the fact that there is no x^d dependence so that $\partial_d(G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu) = 0$. Let us first consider the terms with a $\partial\sigma$:

$$\begin{aligned}
& e^{2\sigma}\Big[G^{\rho\lambda}\big(\partial_\mu\sigma A_\rho A_\nu + \partial_\nu\sigma A_\rho A_\mu - \partial_\rho\sigma A_\mu A_\nu\big) - A^\lambda\partial_\mu\sigma A_\nu - A^\lambda\partial_\nu\sigma A_\mu\Big] \\
=& e^{2\sigma}\big(\partial_\mu\sigma A^\lambda A_\nu + \partial_\nu\sigma A^\lambda A_\mu - \partial^\lambda\sigma A_\mu A_\nu - \partial_\mu\sigma A^\lambda A_\nu - \partial_\nu\sigma A^\lambda A_\mu\big) \\
=& -e^{2\sigma}\partial^\lambda\sigma A_\mu A_\nu
\end{aligned} \qquad [8.29]$$

The terms without a $\partial\sigma$ give

$$\begin{aligned}
&\frac{1}{2}e^{2\sigma}\ \Big[G^{\rho\lambda}\big(\partial_\mu A_\rho A_\nu + A_\rho\partial_\mu A_\nu + \partial_\nu A_\rho A_\mu + A_\rho\partial_\nu A_\mu - \partial_\rho A_\mu A_\nu - A_\mu\partial_\rho A_\nu\big)\\
&- A^\lambda\partial_\mu A_\nu - A^\lambda\partial_\nu A_\mu\Big]\\
=&\frac{1}{2}e^{2\sigma}\big(G^{\rho\lambda}\partial_\mu A_\rho A_\nu + A^\lambda\partial_\mu A_\nu + G^{\rho\lambda}\partial_\nu A_\rho A_\mu + A^\lambda\partial_\nu A_\mu - \partial^\lambda A_\mu A_\nu - A_\mu\partial^\lambda A_\nu\\
&- A^\lambda\partial_\mu A_\nu - A^\lambda\partial_\nu A_\mu\big)\\
=&\frac{1}{2}e^{2\sigma}\Big[A_\nu\big(G^{\rho\lambda}\partial_\mu A_\rho - \partial^\lambda A_\mu\big) + A_\mu\big(G^{\rho\lambda}\partial_\nu A_\rho - \partial^\lambda A_\nu\big)\Big]\\
=&\frac{1}{2}e^{2\sigma}\Big[A_\nu G^{\rho\lambda}\big(\partial_\mu A_\rho - \partial_\rho A_\mu\big) + A_\mu G^{\rho\lambda}\big(\partial_\nu A_\rho - \partial_\rho A_\nu\big)\Big]\\
=&\frac{1}{2}e^{2\sigma}\big(A_\nu G^{\rho\lambda}F_{\mu\rho} + A_\mu G^{\rho\lambda}F_{\nu\rho}\big) = \frac{1}{2}e^{2\sigma}\big(A_\nu F_\mu{}^\lambda + A_\mu F_\nu{}^\lambda\big)
\end{aligned} \tag{8.30}$$

We thus find

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - e^{2\sigma}\partial^\lambda\sigma A_\mu A_\nu + \frac{1}{2}e^{2\sigma}\big(A_\nu F_\mu{}^\lambda + A_\mu F_\nu{}^\lambda\big) \tag{8.31}$$

Next, we have

$$\begin{aligned}
\widetilde{\Gamma}^\lambda_{\mu d} &= \frac{1}{2}\widetilde{G}^{M\lambda}(\partial_\mu\widetilde{G}_{Md} + \partial_d\widetilde{G}_{M\mu} - \partial_M\widetilde{G}_{\mu d})\\
&= \frac{1}{2}\widetilde{G}^{\rho\lambda}(\partial_\mu\widetilde{G}_{\rho d} + \partial_d\widetilde{G}_{\rho\mu} - \partial_\rho\widetilde{G}_{\mu d}) + \frac{1}{2}\widetilde{G}^{d\lambda}(\partial_\mu\widetilde{G}_{dd} + \partial_d\widetilde{G}_{d\mu} - \partial_d\widetilde{G}_{\mu d})
\end{aligned} \tag{8.32}$$

We use the fact that all ∂_d's are zero

$$\begin{aligned}
\widetilde{\Gamma}^\lambda_{\mu d} &= \frac{1}{2}G^{\rho\lambda}\Big[\partial_\mu\big(e^{2\sigma}A_\rho\big) - \partial_\rho\big(e^{2\sigma}A_\mu\big)\Big] - \frac{1}{2}A^\lambda\partial_\mu\big(e^{2\sigma}\big)\\
&= \frac{1}{2}e^{2\sigma}\Big(2\partial_\mu\sigma A^\lambda + G^{\rho\lambda}\partial_\mu A_\rho - 2\partial^\lambda\sigma A_\mu - G^{\rho\lambda}\partial_\rho A_\mu - 2\partial_\mu\sigma A^\lambda\Big)\\
&= -e^{2\sigma}\partial^\lambda\sigma A_\mu + \frac{1}{2}e^{2\sigma}G^{\rho\lambda}F_{\mu\rho} = -e^{2\sigma}\partial^\lambda\sigma A_\mu + \frac{1}{2}e^{2\sigma}F_\mu{}^\lambda
\end{aligned} \tag{8.33}$$

And on we go

$$\begin{aligned}
\widetilde{\Gamma}^\lambda_{dd} &= \frac{1}{2}\widetilde{G}^{M\lambda}(\partial_d\widetilde{G}_{Md} + \partial_d\widetilde{G}_{Md} - \partial_M\widetilde{G}_{dd})\\
&= \frac{1}{2}\widetilde{G}^{\rho\lambda}(\partial_d\widetilde{G}_{\rho d} + \partial_d\widetilde{G}_{\rho d} - \partial_\rho\widetilde{G}_{dd}) + \frac{1}{2}\widetilde{G}^{d\lambda}(\partial_d\widetilde{G}_{dd} + \partial_d\widetilde{G}_{dd} - \partial_d\widetilde{G}_{dd})\\
&= -\frac{1}{2}\widetilde{G}^{\rho\lambda}\partial_\rho\widetilde{G}_{dd} = -\frac{1}{2}G^{\rho\lambda}\partial_\rho e^{2\sigma} = -e^{2\sigma}\partial^\lambda\sigma
\end{aligned} \tag{8.34}$$

Next

$$\begin{aligned}\widetilde{\Gamma}^{d}_{\mu\nu} &= \frac{1}{2}\widetilde{G}^{Md}(\partial_\mu \widetilde{G}_{M\nu} + \partial_\nu \widetilde{G}_{M\mu} - \partial_M \widetilde{G}_{\mu\nu}) \\ &= \frac{1}{2}\widetilde{G}^{\rho d}(\partial_\mu \widetilde{G}_{\rho\nu} + \partial_\nu \widetilde{G}_{\rho\mu} - \partial_\rho \widetilde{G}_{\mu\nu}) + \frac{1}{2}\widetilde{G}^{dd}(\partial_\mu \widetilde{G}_{d\nu} + \partial_\nu \widetilde{G}_{d\mu} - \partial_d \widetilde{G}_{\mu\nu}) \\ &= -\frac{1}{2}A^\rho\Big[\partial_\mu\big(G_{\rho\nu} + e^{2\sigma}A_\rho A_\nu\big) + \partial_\nu\big(G_{\rho\mu} + e^{2\sigma}A_\rho A_\mu\big) - \partial_\rho\big(G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu\big)\Big] \\ &\quad + \frac{1}{2}\Big(e^{-2\sigma} + A_\rho A^\rho\Big)\Big[\partial_\mu\big(e^{2\sigma}A_\nu\big) + \partial_\nu\big(e^{2\sigma}A_\mu\big)\Big] \end{aligned} \tag{8.35}$$

Let us first consider the terms without a $\partial\sigma$: ignoring a factor $1/2$ these are

$$\begin{aligned}& - A^\rho\partial_\mu G_{\rho\nu} - A^\rho\partial_\nu G_{\rho\mu} + A^\rho\partial_\rho G_{\mu\nu} \\ & + e^{2\sigma}\Big(- A^\rho\partial_\mu A_\rho A_\nu - A^\rho A_\rho\partial_\mu A_\nu - A^\rho\partial_\nu A_\rho A_\mu - A^\rho A_\rho\partial_\nu A_\mu + A^\rho\partial_\rho A_\mu A_\nu + A^\rho A_\mu\partial_\rho A_\nu\Big) \\ & + \partial_\mu A_\nu + \partial_\nu A_\mu + e^{2\sigma}\Big(A_\rho A^\rho\partial_\mu A_\nu + A_\rho A^\rho\partial_\nu A_\mu\Big)\end{aligned} \tag{8.36}$$

Now look at the terms without $e^{2\sigma}$ and consider

$$\begin{aligned}\nabla_\mu A_\nu + \nabla_\nu A_\mu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu}A_\lambda + \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu}A_\lambda \\ &= \partial_\mu A_\nu + \partial_\nu A_\mu - G^{\lambda\rho}\big(\partial_\mu G_{\rho\nu} + \partial_\nu G_{\rho\mu} - \partial_\rho G_{\mu\nu}\big)A_\lambda \\ &= \partial_\mu A_\nu + \partial_\nu A_\mu - A^\rho\big(\partial_\mu G_{\rho\nu} + \partial_\nu G_{\rho\mu} - \partial_\rho G_{\mu\nu}\big)\end{aligned} \tag{8.37}$$

which we see gives exactly these terms. Let us now focus on the terms with $e^{2\sigma}$. This simplifies to

$$A^\rho A_\nu(-\partial_\mu A_\rho + \partial_\rho A_\mu) + A^\rho A_\mu(-\partial_\nu A_\rho + \partial_\rho A_\nu) = A^\rho A_\nu F_{\rho\mu} + A^\rho A_\mu F_{\rho\nu} \tag{8.38}$$

We can thus write

$$\widetilde{\Gamma}^{d}_{\mu\nu} = \frac{1}{2}\Big(\nabla_\mu A_\nu + \nabla_\nu A_\mu\Big) + \frac{1}{2}e^{2\sigma}\Big(A^\rho A_\nu F_{\rho\mu} + A^\rho A_\mu F_{\rho\nu}\Big) + \partial\sigma \text{ terms} \tag{8.39}$$

Let us now consider those terms with a $\partial\sigma$ in [8.35]:

$$\begin{aligned}& e^{2\sigma}\Big(- A^\rho\partial_\mu\sigma A_\rho A_\nu - A^\rho\partial_\nu\sigma A_\rho A_\mu + A^\rho\partial_\rho\sigma A_\mu A_\nu\Big) \\ &\quad + \partial_\mu\sigma A_\nu + \partial_\nu\sigma A_\mu + e^{2\sigma}\Big(A_\rho A^\rho\partial_\mu\sigma A_\nu + A_\rho A^\rho\partial_\nu\sigma A_\mu\Big) \\ & = \partial_\mu\sigma A_\nu + \partial_\nu\sigma A_\mu + e^{2\sigma}A_\mu A_\nu A^\rho\partial_\rho\sigma\end{aligned} \tag{8.40}$$

We thus have

$$\begin{aligned}\widetilde{\Gamma}^{d}_{\mu\nu} &= \frac{1}{2}\Big(\nabla_\mu A_\nu + \nabla_\nu A_\mu\Big) + \frac{1}{2}e^{2\sigma}\Big(A^\rho A_\nu F_{\rho\mu} + A^\rho A_\mu F_{\rho\nu}\Big) \\ &\quad + \partial_\mu\sigma A_\nu + \partial_\nu\sigma A_\mu + e^{2\sigma}A_\mu A_\nu A^\rho\partial_\rho\sigma\end{aligned} \tag{8.41}$$

Two more to go

$$
\begin{aligned}
\widetilde{\Gamma}^d_{\mu d} &= \frac{1}{2}\widetilde{G}^{Md}(\partial_\mu \widetilde{G}_{Md} + \partial_d \widetilde{G}_{M\mu} - \partial_M \widetilde{G}_{\mu d}) \\
&= \frac{1}{2}\widetilde{G}^{\rho d}(\partial_\mu \widetilde{G}_{\rho d} - \partial_\rho \widetilde{G}_{\mu d}) + \frac{1}{2}\widetilde{G}^{dd}\partial_\mu \widetilde{G}_{dd} \\
&= -\frac{1}{2}A^\rho\Big[\partial_\mu\big(e^{2\sigma}A_\rho\big) - \partial_\rho\big(e^{2\sigma}A_\mu\big)\Big] + \frac{1}{2}\Big(e^{-2\sigma} + A_\rho A^\rho\Big)\partial_\mu e^{2\sigma} \\
&= \frac{1}{2}e^{2\sigma}\Big(- A^\rho\partial_\mu A_\rho + A^\rho\partial_\rho A_\mu\Big) + e^{2\sigma}\Big(- A^\rho A_\rho \partial_\mu\sigma + A^\rho A_\mu \partial_\rho\sigma + A_\rho A^\rho\partial_\mu\sigma\Big) + \partial_\mu\sigma \\
&= \frac{1}{2}e^{2\sigma}A^\rho F_{\rho\mu} + e^{2\sigma}A_\mu A^\rho\partial_\rho\sigma + \partial_\mu\sigma
\end{aligned} \qquad [8.42]
$$

and finally

$$
\begin{aligned}
\widetilde{\Gamma}^d_{dd} &= \frac{1}{2}\widetilde{G}^{Md}(\partial_d \widetilde{G}_{Md} + \partial_d \widetilde{G}_{Md} - \partial_M \widetilde{G}_{dd}) = -\frac{1}{2}\widetilde{G}^{\mu d}\partial_\mu \widetilde{G}_{dd} \\
&= \frac{1}{2}A^\mu \partial_\mu e^{2\sigma} = e^{2\sigma}A^\mu\partial_\mu\sigma
\end{aligned} \qquad [8.43]
$$

Let us for convenience write all the connections out once more:

$$
\begin{aligned}
\widetilde{\Gamma}^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} - e^{2\sigma}\partial^\lambda\sigma A_\mu A_\nu + \frac{1}{2}e^{2\sigma}\big(A_\nu F_\mu{}^\lambda + A_\mu F_\nu{}^\lambda\big) \\
\widetilde{\Gamma}^\lambda_{\mu d} &= -e^{2\sigma}\partial^\lambda\sigma A_\mu + \frac{1}{2}e^{2\sigma}F_\mu{}^\lambda \\
\widetilde{\Gamma}^\lambda_{dd} &= -e^{2\sigma}\partial^\lambda\sigma \\
\widetilde{\Gamma}^d_{\mu\nu} &= \frac{1}{2}\Big(\nabla_\mu A_\nu + \nabla_\nu A_\mu\Big) + \frac{1}{2}e^{2\sigma}\Big(A^\rho A_\nu F_{\rho\mu} + A^\rho A_\mu F_{\rho\nu}\Big) \\
&\quad + \partial_\mu\sigma A_\nu + \partial_\nu\sigma A_\mu + e^{2\sigma}A_\mu A_\nu A^\rho\partial_\rho\sigma \\
\widetilde{\Gamma}^d_{\mu d} &= \frac{1}{2}e^{2\sigma}A^\rho F_{\rho\mu} + e^{2\sigma}A_\mu A^\rho\partial_\rho\sigma + \partial_\mu\sigma \\
\widetilde{\Gamma}^d_{dd} &= e^{2\sigma}A^\mu\partial_\mu\sigma
\end{aligned} \qquad [8.44]
$$

As an aside we note that, evidently, the connections scale with a factor λ for each ^{d} and a factor λ^{-1} for each $_d$.

We now return to [8.27] which we repeat for convenience.

$$
\widetilde{R} = \alpha R + \beta e^{2\sigma}F_{\mu\nu}F^{\mu\nu} + \gamma(\nabla\sigma)^2 + \delta\sigma\nabla\sigma + \varepsilon\nabla^2\sigma + f \qquad [8.45]
$$

To fix these constants we can look at special cases of the metric. Let us set $A^\mu = \sigma = 0$. In that case we simply have $\widetilde{R} = R$ and so we find that $\alpha = 1$.

Next, let us take $G_{\mu\nu} = \delta_{\mu\nu}$ and $A_\mu = 0$, leaving only σ free.[2] The only non-vanishing metric component with upper indices and connections are then

$$\begin{aligned}\widetilde{G}^{\mu\nu} &= \delta^{\mu\nu}\\ \widetilde{G}^{dd} &= e^{-2\sigma}\\ \widetilde{\Gamma}^{\mu}_{dd} &= -\, e^{2\sigma}\partial^\mu\sigma\\ \widetilde{\Gamma}^{d}_{\mu d} &= \partial_\mu\sigma\end{aligned} \qquad [8.46]$$

The Ricci scalar thus reduces to

$$\begin{aligned}\widetilde{R} &= \widetilde{G}^{LM}\left(\partial_N\widetilde{\Gamma}^N_{ML} - \partial_M\widetilde{\Gamma}^N_{NL} + \widetilde{\Gamma}^N_{NK}\widetilde{\Gamma}^K_{ML} - \widetilde{\Gamma}^N_{MK}\widetilde{\Gamma}^K_{NL}\right)\\ &= \widetilde{G}^{\lambda\mu}\left(\partial_N\widetilde{\Gamma}^N_{\mu\lambda} - \partial_\mu\widetilde{\Gamma}^N_{N\lambda} + \widetilde{\Gamma}^N_{NK}\widetilde{\Gamma}^K_{\mu\lambda} - \widetilde{\Gamma}^N_{\mu K}\widetilde{\Gamma}^K_{N\lambda}\right)\\ &\quad + \widetilde{G}^{dd}\left(\partial_N\widetilde{\Gamma}^N_{dd} - \partial_d\widetilde{\Gamma}^N_{Nd} + \widetilde{\Gamma}^N_{NK}\widetilde{\Gamma}^K_{dd} - \widetilde{\Gamma}^N_{dK}\widetilde{\Gamma}^K_{Nd}\right)\\ &= \widetilde{G}^{\lambda\mu}\left(-\partial_\mu\widetilde{\Gamma}^d_{d\lambda} - \widetilde{\Gamma}^d_{\mu d}\widetilde{\Gamma}^d_{d\lambda}\right) + \widetilde{G}^{dd}\left(\partial_\nu\widetilde{\Gamma}^\nu_{dd} + \widetilde{\Gamma}^d_{d\kappa}\widetilde{\Gamma}^\kappa_{dd} - \widetilde{\Gamma}^\nu_{dd}\widetilde{\Gamma}^d_{\nu d} - \widetilde{\Gamma}^d_{d\kappa}\widetilde{\Gamma}^\kappa_{dd}\right)\\ &= -\,\partial_\mu\partial^\mu\sigma - \partial_\mu\sigma\partial^\mu\sigma + e^{-2\sigma}\left[\partial_\mu\left(-e^{2\sigma}\partial^\mu\sigma\right) - \partial_\mu\sigma\left(-e^{2\sigma}\partial^\mu\sigma\right)\right]\\ &= -\,\partial_\mu\partial^\mu\sigma - \partial_\mu\sigma\partial^\mu\sigma - 2\partial_\mu\sigma\partial^\mu\sigma - \partial_\mu\partial^\mu\sigma + \partial_\mu\sigma\partial^\mu\sigma\\ &= -\,2(\partial_\mu\partial^\mu\sigma + \partial_\mu\sigma\partial^\mu\sigma)\end{aligned} \qquad [8.47]$$

From [8.27] we have for this choice of metric

$$\widetilde{R} = \gamma(\partial\sigma)^2 + \delta\sigma\partial^2\sigma + \varepsilon\partial^2\sigma + f \qquad [8.48]$$

and we thus find that $\gamma = \varepsilon = -2$ and $\delta = f = 0$. To link this to the expression in (8.1.8) note that

$$e^{-\sigma}\partial^2 e^\sigma = e^{-\sigma}\partial_\mu\left(\partial_\mu\sigma e^\sigma\right) = \partial_\mu\partial^\mu\sigma + \partial_\mu\sigma\partial^\mu\sigma \qquad [8.49]$$

and $e^{-\sigma}\nabla^2 e^\sigma$ is just the covariant expression of this. So we have a contribution $-2e^{-\sigma}\nabla^2 e^\sigma$.

Finally, let us take $G_{\mu\nu} = \delta_{\mu\nu}$ and $\sigma = 0$, leaving only the A_μ free. Here we don't have to do any calculations as the theory we have is just a Euclidean d dimensional flat spacetime with an Abelian gauge field A_μ and we know that the action reduces to $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ so that $\beta = -\frac{1}{4}$. For the assiduous reader who is not yet tired of these calculations we will do them in detail. Less assiduous readers can immediately skip to [8.57]. The only non-vanishing metric component with upper indices and connections are in this case

$$\begin{aligned}\widetilde{G}^{\mu\nu} &= \delta^{\mu\nu}\\ \widetilde{G}^{\mu d} &= -\, A^\mu\\ \widetilde{G}^{dd} &= 1 + A_\mu A^\mu\end{aligned} \qquad [8.50]$$

[2] Note that taking $G_{\mu\nu} = 0$ is not an allowed choice as the metric would then not be invertible.

and

$$
\begin{aligned}
\widetilde{\Gamma}^{\lambda}_{\mu\nu} &= \frac{1}{2}\big(A_\nu F_\mu{}^\lambda + A_\mu F_\nu{}^\lambda\big)\\
\widetilde{\Gamma}^{\lambda}_{\mu d} &= \frac{1}{2}F_\mu{}^\lambda\\
\widetilde{\Gamma}^{d}_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu A_\nu + \partial_\nu A_\mu\Big) + \frac{1}{2}\Big(A^\rho A_\nu F_{\rho\mu} + A^\rho A_\mu F_{\rho\nu}\Big)\\
\widetilde{\Gamma}^{d}_{\mu d} &= \frac{1}{2}A^\rho F_{\rho\mu}
\end{aligned}
\qquad [8.51]
$$

The Ricci scalar is given by

$$
\widetilde{R} = \widetilde{G}^{LM}\left(\partial_N \widetilde{\Gamma}^N_{ML} - \partial_M \widetilde{\Gamma}^N_{NL} + \widetilde{\Gamma}^N_{NK}\widetilde{\Gamma}^K_{ML} - \widetilde{\Gamma}^N_{MK}\widetilde{\Gamma}^K_{NL}\right) \qquad [8.52]
$$

Let us do the four terms separately. Because $G_{\mu\nu} = \delta_{\mu\nu}$ we do not have to make a distinction between upper and lower indices in d-spacetime and will move them all downstairs, but we do need to be careful with the order. We start with

$$
\begin{aligned}
\widetilde{G}^{LM}\partial_N\widetilde{\Gamma}^N_{ML} &= \widetilde{G}^{LM}\partial_\nu\widetilde{\Gamma}^\nu_{ML} = \widetilde{G}^{\lambda M}\partial_\nu\widetilde{\Gamma}^\nu_{M\lambda} + \widetilde{G}^{dM}\partial_\nu\widetilde{\Gamma}^\nu_{Md}\\
&= \widetilde{G}^{\lambda\mu}\partial_\nu\widetilde{\Gamma}^\nu_{\mu\lambda} + \widetilde{G}^{\lambda d}\partial_\nu\widetilde{\Gamma}^\nu_{d\lambda} + \widetilde{G}^{d\mu}\partial_\nu\widetilde{\Gamma}^\nu_{\mu d} + \widetilde{G}^{dd}\partial_\nu\widetilde{\Gamma}^\nu_{dd}\\
&= \frac{1}{2}\delta_{\lambda\mu}\partial_\nu\big(A_\mu F_{\lambda\nu} + A_\lambda F_{\mu\nu}\big) - \frac{1}{2}A_\lambda\partial_\nu F_{\lambda\nu} - \frac{1}{2}A_\mu\partial_\nu F_{\mu\nu}\\
&= \frac{1}{2}\Big(\partial_\nu A_\mu F_{\mu\nu} + A_\mu\partial_\nu F_{\mu\nu} + \partial_\nu A_\mu F_{\mu\nu} + A_\mu\partial_\nu F_{\mu\nu} - A_\mu\partial_\nu F_{\mu\nu} - A_\mu\partial_\nu F_{\mu\nu}\Big)\\
&= \partial_\nu A_\mu F_{\mu\nu} = \frac{1}{2}F_{\mu\nu}(\partial_\nu A_\mu - \partial_\mu A_\nu) = -\frac{1}{2}F_{\mu\nu}F_{\mu\nu}
\end{aligned}
\qquad [8.53]
$$

where in the last line we have antisymmetrised the result. Next we have

$$
\begin{aligned}
-\widetilde{G}^{LM}\partial_M\widetilde{\Gamma}^N_{NL} &= -\widetilde{G}^{L\mu}\partial_\mu\widetilde{\Gamma}^N_{NL} = -\widetilde{G}^{L\mu}\partial_\mu\widetilde{\Gamma}^\nu_{\nu L} - \widetilde{G}^{L\mu}\partial_\mu\widetilde{\Gamma}^d_{dL}\\
&= -\widetilde{G}^{\lambda\mu}\partial_\mu\widetilde{\Gamma}^\nu_{\nu\lambda} - \widetilde{G}^{d\mu}\partial_\mu\widetilde{\Gamma}^\nu_{\nu d} - \widetilde{G}^{\lambda\mu}\partial_\mu\widetilde{\Gamma}^d_{d\lambda} - \widetilde{G}^{d\mu}\partial_\mu\widetilde{\Gamma}^d_{dd}\\
&= -\frac{1}{2}\delta_{\lambda\mu}\partial_\mu\big(A_\nu F_{\lambda\nu} + A\lambda F_{\nu\nu}\big) + \frac{1}{2}A_\mu\partial_\mu F_{\nu\nu} - \frac{1}{2}\delta_{\lambda\mu}\partial_\mu\big(A^\rho F_{\rho\lambda}\big)\\
&= \frac{1}{2}\Big(-\partial_\mu A_\nu F_{\mu\nu} - A_\nu\partial_\mu F_{\mu\nu} - \partial_\mu A_\rho F_{\rho\mu} - A_\rho\partial_\mu F_{\rho\mu}\Big) = 0
\end{aligned}
\qquad [8.54]
$$

The third and fourth terms with the products of the connections are a lot more tedious to work out, and is most easily done with a software package such as Mathematica. The result of this is

$$
\widetilde{G}^{LM}\widetilde{\Gamma}^N_{NK}\widetilde{\Gamma}^K_{ML} = 0 \quad \text{and} \quad -\widetilde{G}^{LM}\widetilde{\Gamma}^N_{MK}\widetilde{\Gamma}^K_{NL} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \qquad [8.55]
$$

The Mathematica calculation is shown in fig.8.1.

Bringing the four terms together we conclude that

$$\widetilde{R} = -\frac{1}{2}F_{\mu\nu}F_{\mu\nu} + \frac{1}{4}F_{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \qquad [8.56]$$

and hence $\beta = -1/4$.

```
In[167]:= dim = 5;
Do[g[a, b] = 0, {a, dim}, {b, dim}]
Do[g[a, a] = 1, {a, dim - 1}]
Do[g[a, dim] = -A[a], {a, dim - 1}]
Do[g[dim , a] = -A[a], {a, dim - 1}]
g[dim , dim] = 1 + Sum[A[m] * A[m], {m, dim - 1}];

Do[G[a, b, c] = 0, {a, dim}, {b, dim}, {c, dim}]
F[a_ , b_] := dA[a, b] - dA[b, a];
Do[G[a, b, c] = (1 / 2) * ( A[b] * F[c, a] +  A[c] * F[b, a]), {a, dim - 1}, {b, dim - 1}, {c, dim - 1}]
Do[G[a, b, dim] = (1 / 2) * F[b, a], {a, dim - 1}, {b, dim - 1}]
Do[G[a, dim , b] = (1 / 2) * F[b, a], {a, dim - 1}, {b, dim - 1}]
Do[G[dim , m, n] = (1 / 2) *
   ( dA[m, n] + dA[n, m] + Sum[A[r] * A[n] * F[r, m] + A[r] * A[m] * F[r, n], {r, dim - 1}]), {a, dim - 1}]
Do[G[dim , m, dim] = (1 / 2) * Sum[A[r] * F[r, m], {r, dim - 1}], {m, dim - 1}]
Do[G[dim , dim , m] = (1 / 2) * Sum[A[r] * F[r, m], {r, dim - 1}], {m, dim - 1}]
F2 = Sum[F[a, b] * F[a, b], {a, dim - 1}, {b, dim - 1}];

In[164]:= GG1 =
 Simplify [Expand [Sum[g[l, m] * G[n, n, k] * G[k, m, l], {l, dim}, {k, dim}, {m, dim}, {n, dim}]]]
GG2 = Simplify [
  Expand [- Sum[g[l, m] * G[n, m, k] * G[k, n, l], {l, dim}, {k, dim}, {m, dim}, {n, dim}]]]
Expand [GG2 - 1 / 4 * F2]

Out[164]= 0

Out[165]= 1/2 (dA[1, 2]^2 + dA[1, 3]^2 + dA[1, 4]^2 + dA[1, 5]^2 - 2 dA[1, 2] dA[2, 1] + dA[2, 1]^2 +
   dA[2, 3]^2 + dA[2, 4]^2 + dA[2, 5]^2 - 2 dA[1, 3] dA[3, 1] + dA[3, 1]^2 - 2 dA[2, 3] dA[3, 2] +
   dA[3, 2]^2 + dA[3, 4]^2 + dA[3, 5]^2 - 2 dA[1, 4] dA[4, 1] + dA[4, 1]^2 - 2 dA[2, 4] dA[4, 2] +
   dA[4, 2]^2 - 2 dA[3, 4] dA[4, 3] + dA[4, 3]^2 + dA[4, 5]^2 - 2 dA[1, 5] dA[5, 1] + dA[5, 1]^2 -
   2 dA[2, 5] dA[5, 2] + dA[5, 2]^2 - 2 dA[3, 5] dA[5, 3] + dA[5, 3]^2 - 2 dA[4, 5] dA[5, 4] + dA[5, 4]^2)

Out[166]= 0
```

Figure 8.1: Mathematica code for Ricci scalar in Kaluza-Klein theory. We are illsutarting this with $D = 5$, but it is obviously a general result.

Taking everything together, we conclude thus that $\alpha = 1, \beta = -1/4$ and $\gamma = \delta = -2$ and thus

$$\widetilde{R} = R - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} - 2e^{-\sigma}\nabla^2 e^{\sigma} \qquad [8.57]$$

which is what we set out to show.

8.7 p 233: Eq. (8.1.9) The Kaluza-Klein Action with a Dilaton

Let us now add a dilaton to the Kaluza-Klein theory. The action is the D dimensional Einstein-Hilbert action augmented with a dilaton field Φ.

$$S = \frac{1}{2\kappa_0^2}\int d^D x\,\sqrt{-G}e^{-2\Phi}\left[\boldsymbol{R} + 4\nabla_\mu\Phi\nabla^\mu\Phi\right] \qquad [8.58]$$

Note also that we have $\sqrt{-G}$, the square root of the determinant of G^D_{MN} and not $\sqrt{-G_d}$, the square root of the determinant of the non-compactified metric, as per Joe's errata page. We have also replaced $\partial_\mu\Phi$ by $\nabla_\mu\Phi$ as Φ is a spacetime scalar.

As we are ignoring any dependence on the compactified dimension x^d, we can just integrate it out and find a factor $2\pi R$. We also use [8.19] , i.e. $\sqrt{-G} = e^{2\sigma}\sqrt{-G_d}$, and (8.1.8). This gives

$$S = \frac{\pi R}{\kappa_0^2}\int d^d x\,\sqrt{-G_d}e^{-2\Phi+\sigma}\left[\boldsymbol{R}_d - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} + 4\partial_\mu\Phi\partial^\mu\Phi\right] \qquad [8.59]$$

where we have used the fact that Φ is a spacetime scalar. In order to rewrite this as the second line of (8.1.9) we to replace $-2e^{-\sigma}\nabla^2 e^{\sigma}$ by $-4\partial_\mu\Phi\partial_\sigma$. The solution to this was given to me by "Kosm" on the Physics Stack Exchange and involves the ubiquitous partial integration.

We first note that

$$e^{-\sigma}\nabla^2 e^{\sigma} = e^{-\sigma}\nabla^\mu\left(\nabla_\mu\sigma e^{\sigma}\right) = \nabla^2\sigma + \nabla_\mu\sigma\nabla^\mu\sigma = \nabla^2\sigma + \partial_\mu\sigma\partial^\mu\sigma \qquad [8.60]$$

Next we consider

$$\begin{aligned}\nabla_\mu\left(e^{-2\Phi+\sigma}\nabla^\mu\sigma\right) &= (-2\nabla_\mu\Phi + \nabla_\mu\sigma)\nabla^\mu\sigma e^{-2\Phi+\sigma} + \nabla^2\sigma e^{-2\Phi+\sigma}\\ &= (-2\partial_\mu\Phi\partial^\mu\sigma + \partial_\mu\sigma\partial^\mu\sigma + \nabla^2\sigma)e^{-2\Phi+\sigma}\end{aligned} \qquad [8.61]$$

We now integrate both sides over $\int d^d x\,\sqrt{-G_d}$. The LHS is a total derivative and hence zero. Thus

$$0 = \int d^d x\,\sqrt{-G_d}e^{-2\Phi+\sigma}(-2\partial_\mu\Phi\partial^\mu\sigma + \partial_\mu\sigma\partial^\mu\sigma + \nabla^2\sigma) \qquad [8.62]$$

From which we have

$$\int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} \nabla^2 \sigma = \int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} (2\partial_\mu \Phi \partial^\mu \sigma - \partial_\mu \sigma \partial^\mu \sigma) \qquad [8.63]$$

From this and from [8.60] we then get

$$\begin{aligned}\int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} \left(-2e^{-\sigma}\nabla^2 e^{\sigma}\right) &= \int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} \left(-2\nabla^2 \sigma - 2\partial_\mu \sigma \partial^\mu \sigma\right) \\ &= \int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} \left(-4\partial_\mu \Phi \partial^\mu \sigma + 2\partial_\mu \sigma \partial^\mu \sigma - 2\partial_\mu \sigma \partial^\mu \sigma\right) \\ &= \int d^d x\, \sqrt{-G_d} e^{-2\Phi+\sigma} \left(-4\partial_\mu \Phi \partial^\mu \sigma\right) \qquad [8.64]\end{aligned}$$

so that we can indeed replace $-2e^{-\sigma}\nabla^2 e^{\sigma}$ by $-4\partial_\mu \Phi \partial_\sigma$.

To find the last line of (8.1.9) just write $\Phi = \Phi_d + \sigma/2$ in

$$\begin{aligned}-4\partial_\mu \Phi \partial^\mu \sigma + 4\partial_\mu \Phi \partial^\mu \Phi =& -4\partial_\mu \left(\Phi_d + \frac{\sigma}{2}\right) \partial^\mu \sigma + 4\partial_\mu \left(\Phi_d + \frac{\sigma}{2}\right) \partial^\mu \left(\Phi_d + \frac{\sigma}{2}\right) \\ =& -4\partial_\mu \Phi_d \partial^\mu \sigma - 2\partial_\mu \sigma \partial^\mu \sigma + 4\partial_\mu \Phi_d \partial_\mu \Phi_d + 4\partial_\mu \Phi_d \partial^\mu \sigma + \partial_\mu \sigma \partial^\mu \sigma \\ =& 4\partial_\mu \Phi_d \partial_\mu \Phi_d - \partial_\mu \sigma \partial^\mu \sigma \qquad [8.65]\end{aligned}$$

It looks like the dilation kinetic term has the wrong sign. But as explained in Joe's book we have seen this before. Indeed the lowest order effective action for the bosonic string (3.7.20) also had such a wrong sign. We then performed a Weyl transformation $G_{\mu\nu} \longrightarrow \tilde{G}_{\mu\nu} = e^{2\omega(x)} G_{\mu\nu}$, see (3.722). We also shifted the dilaton $\tilde{\Phi} = \Phi - \Phi_0$ and found in (3.7.25) that the kinetic term of this new dilaton field was $-\frac{4}{D-2}\partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi}$ and so has the right sign. The situation here is identical.

8.8 p 234: Eq. (8.1.11) The Relation Between the Graviton and Gauge Coupling in Kaluza-Klein Theory

We rewrite the action (8.1.9) with the definitions of the couplings

$$S = \int d^d x\, \sqrt{-G_d} \left[\frac{1}{2\kappa_d^2} \boldsymbol{R}_d - \frac{\pi R}{\kappa_0^2} \left[-(\partial\sigma)^2 + 4(\partial\Phi)^2\right] - \frac{1}{4g_d^2} \tilde{F}^2\right] \qquad [8.66]$$

Equating the coefficient of $\boldsymbol{R}_d$ with that in (8.1.9) gives

$$\frac{1}{2\kappa_d^2} = e^{-2\Phi_d} \frac{\pi R}{\kappa_0^2} \quad \Rightarrow e^{2\Phi_d} \kappa_0^2 = 2\pi R \kappa_d^2 \qquad [8.67]$$

Equating the coefficients of the field tensor gives

$$-\frac{1}{4g_d^2} = -e^{-2\Phi_d}\frac{\pi R}{\kappa_0^2}\frac{R^2}{4} \qquad [8.68]$$

The extra R^2 on the RHS coming from the definition $A_\mu = R\tilde{A}_\mu$. We thus have

$$g_d^2 = \frac{e^{2\Phi_d}\kappa_0^2}{\pi R^3} = \frac{2\pi R\kappa_d^2}{\pi R^3} = \frac{2\kappa_d^2}{R^2} \qquad [8.69]$$

8.9 p 234: Eq. (8.1.12) The Relation Between the Graviton Coupling in D and in d Dimensions

Focussing on the Einstein-Hilbert part of the action we have in D dimensional spacetime

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{-G}\boldsymbol{R} + \cdots = \frac{1}{2\kappa^2}2\pi R\int d^d x\sqrt{-G}\boldsymbol{R} + \cdots \qquad [8.70]$$

But if we wouldn't know about the compactified dimension and thought we lived in a d dimensional spacetime we would simply write

$$S = \int d^d x\,\sqrt{-G_d}\frac{1}{2\kappa_d^2}\boldsymbol{R}_d + \cdots \qquad [8.71]$$

Comparing the coupling constants we find

$$\frac{\pi R}{\kappa^2} = \frac{1}{2\kappa_d^2} \quad \Rightarrow \frac{1}{\kappa_d^2} = \frac{2\pi R}{\kappa^2} \qquad [8.72]$$

8.10 p 234: Eq. (8.1.14) The Antisymmetric Tensor in the Kaluza-Klein Theory

This is once more a straightforward albeit tedious calculation, which we will not write out in detail. We will just illustrate how the term with the vector potential arises. From (3.7.20) the contribution from the antisymmetric tensor is

$$\begin{aligned} S_H =& -\frac{1}{24\kappa_0^2}\int d^D x\,\sqrt{-G}e^{-2\Phi}H_{KLM}H^{KLM} \\ =& -\frac{1}{24\kappa_0^2}2\pi R\int d^d x\,\sqrt{-G_d}e^{\sigma}e^{-2\Phi}G^{KN}G^{LO}G^{MP}H_{KLM}H_{NOP} \end{aligned} \qquad [8.73]$$

We extract the terms that have a G^{dd}:

$$\begin{aligned} S_H = -\frac{\pi R}{12\kappa_0^2}\int d^d x\,\sqrt{-G_d}e^{-2\Phi_d}\big(&G^{dd}G^{LO}G^{MP}H_{dLM}H_{dOP} \\ &+ G^{KN}G^{dd}G^{MP}H_{KdM}H_{NdP} + G^{KN}G^{LO}G^{dd}H_{KLd}H_{NOd} + \cdots\big) \end{aligned} \qquad [8.74]$$

and take all other indices to be d-dimensional:

$$S_H = -\frac{\pi R}{12\kappa_0^2}\int d^dx\,\sqrt{-G_d}e^{-2\Phi_d}\big(G^{dd}G^{\lambda\rho}G^{\mu\pi}H_{d\lambda\mu}H_{d\rho\pi} + G^{\kappa\nu}G^{dd}G^{\mu\pi}H_{\kappa d\mu}H_{\nu d\pi} + G^{\kappa\nu}G^{\lambda\rho}G^{dd}H_{\kappa\lambda d}H_{\nu\rho d} + \cdots\big) \qquad [8.75]$$

Now use the inverse metric as derived in [8.21]

$$\begin{aligned}\widetilde{G}^{\mu\nu} &= G^{\mu\nu}\\ \widetilde{G}^{\mu d} &= -A^\mu\\ \widetilde{G}^{dd} &= e^{-2\sigma} + A_\mu A^\mu\end{aligned} \qquad [8.76]$$

This gives

$$S_H = -\frac{\pi R}{12\kappa_0^2}\int d^dx\,\sqrt{-G_d}e^{-2\Phi_d}(e^{-2\sigma}+A^2)\big(G^{\lambda\rho}G^{\mu\pi}H_{d\lambda\mu}H_{d\rho\pi} + G^{\kappa\nu}G^{\mu\pi}H_{\kappa d\mu}H_{\nu d\pi} + G^{\kappa\nu}G^{\lambda\rho}H_{\kappa\lambda d}H_{\nu\rho d} + \cdots\big) \qquad [8.77]$$

We have been sloppy with the notation as we have used $G^{\mu\nu}$ both for the metric in D dimensions as in d dimensions, but as they are the same, we will be forgiven for this sin. The three terms are identical by symmetry and we can thus write

$$S_H = -\frac{\pi R}{12\kappa_0^2}\int d^dx\,\sqrt{-G_d}e^{-2\Phi_d}\left[3e^{-2\sigma}H_{d\lambda\mu}H_d^{\lambda\mu} + \cdots\right] \qquad [8.78]$$

where we have also moved the terms with A^μ into the $\cdots$. Note that this is the only place where the $e^{-2\sigma}$ can occur. We now also have found a term of the form $A^2H_{d\lambda\mu}H_d^{\lambda\mu}$. Hopefully this term will combine with terms that contain the inverse metric $G^{\mu d} = -A^\mu$. We will leave this as an exercise for the reader.

8.11 p 236: Eq. (8.2.5) The Coordinate Change and the Winding Number

If we allow a winding number w then going around the string we have periodicity up to $2\pi Rw$ and the relation

$$2\pi Rw = \oint_{\mathcal{C}}(dz\,\partial X + d\bar{z}\,\bar{\partial}X) \qquad [8.79]$$

Here $\mathcal{C}$ is a closed contour counter-clockwise around the origin. We plug in the Laurent expansion

$$2\pi Rw = -i\sqrt{\frac{\alpha'}{2}}\oint_{\mathcal{C}}\sum_{m=-\infty}^{\infty}\left(dz\,\frac{\alpha_m}{z^{m+1}} + d\bar{z}\,\frac{\tilde{\alpha}_m}{\bar{z}^{m+1}}\right) \qquad [8.80]$$

In the second term we perform a change of variables $\bar{z} = w$, a counter-clockwise closed contour in the z plane $e^{2\pi i\tau}$ with $\tau \in [0,1]$ now becomes $e^{-2\pi i\tau}$ and thus a clock-wise contour. Flipping the direction of the contour introduces a minus sign so we have

$$\begin{aligned} 2\pi R w =& -i\sqrt{\frac{\alpha'}{2}} \oint_{\mathcal{C}} \sum_{m=-\infty}^{\infty} \left(dz\, \frac{\alpha_m}{z^{m+1}} - dw\, \frac{\tilde{\alpha}_m}{w^{m+1}} \right) \\ =& -i\sqrt{\frac{\alpha'}{2}} 2\pi i(\alpha_0 - \tilde{\alpha}_0) = 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0 - \tilde{\alpha}_0) \end{aligned} \qquad [8.81]$$

8.12 p 236: Eq. (8.2.6) The Noether Momentum for the Closed String

We worked this out in (2.7.3), see our [2.114]

$$p^{\mu} = \frac{1}{\sqrt{2\alpha'}}(\alpha_0 + \tilde{\alpha}_0) \qquad [8.82]$$

8.13 p 236: Eq. (8.2.7) The Left and Right Momentum

From (8.2.5) and (8.2.6) we have

$$\begin{aligned} \alpha_0 - \tilde{\alpha}_0 &= \frac{2wR}{\sqrt{2\alpha'}} \\ \alpha_0 + \tilde{\alpha}_0 &= \sqrt{2\alpha'}p \end{aligned} \qquad [8.83]$$

from which we get

$$\begin{aligned} \alpha_0 &= \frac{1}{2}\left(\frac{2wR}{\sqrt{2\alpha'}} + \sqrt{2\alpha'}p \right) \\ \tilde{\alpha}_0 &= \frac{1}{2}\left(-\frac{2wR}{\sqrt{2\alpha'}} + \sqrt{2\alpha'}p \right) \end{aligned} \qquad [8.84]$$

and thus from the definition of $p_L = \sqrt{2/\alpha'}\alpha_0$ and $p_R = \sqrt{2/\alpha'}\tilde{\alpha}_0$,

$$\begin{aligned} p_L &= \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\left(\frac{2Rw}{\sqrt{2\alpha'}} + \sqrt{2\alpha'}p \right) = \frac{wR}{\alpha'} + p = \frac{n}{R} + \frac{wR}{\alpha'} \\ p_R &= \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\left(-\frac{2Rw}{\sqrt{2\alpha'}} + \sqrt{2\alpha'}p \right) = -\frac{wR}{\alpha'} + p = \frac{n}{R} - \frac{wR}{\alpha'} \end{aligned} \qquad [8.85]$$

where we have used the quantisation condition (8.2.2).

8.14 p 237: Eq. (8.2.9) The Partition Function for the Compactified Dimension

We have for a single dimension from (7.2.5)

$$(q\bar{q})^{-1/24}\mathrm{Tr}\; q^{L_0}\bar{q}^{\tilde{L}_0} \tag{8.86}$$

with $q = e^{2\pi i\tau} = e^{2\pi(i\tau_1-\tau_2)}$. The oscillator part and the $(q\bar{q})^{-1/24}$ give the Dedekind function $|\eta(\tau)|^{-2}$. The momentum is now quantized and not continuous and so the integration is replaced by a sum over all possible momentum states, i.e. over all n and w. Plugging in (8.2.7) this gives

$$\begin{aligned}
&\sum_{n,w=-\infty}^{\infty} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2} \\
&= \sum_{n,w=-\infty}^{\infty} e^{(i\tau_1-\tau_2)\frac{\pi\alpha'}{2}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2} e^{(-i\tau_1-\tau_2)\frac{\pi\alpha'}{2}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2} \\
&= \sum_{n,w=-\infty}^{\infty} e^{i\tau_1\frac{\pi\alpha'}{2}\left[\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2-\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2\right]} e^{-\tau_2\frac{\pi\alpha'}{2}\left[\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2+\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2\right]} \\
&= \sum_{n,w=-\infty}^{\infty} e^{i\tau_1\frac{\pi\alpha'}{2}\frac{4nw}{\alpha'}} e^{-\tau_2\frac{\pi\alpha'}{2}2\left(\frac{n^2}{R^2}+\frac{w^2R^2}{\alpha'^2}\right)} = \sum_{n,w=-\infty}^{\infty} e^{2\pi i\tau_1 nw} e^{-\pi\tau_2\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)}
\end{aligned} \tag{8.87}$$

Together with the Dedekind function we find (8.2.9).

8.15 p 237: Eq. (8.2.10) The Poisson Resummation Formula

This formula is also known as the Poisson summation formula and links an infinite sum of a function to an infinite sum of its Fourier transform:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \tag{8.88}$$

where $\hat{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi ixk} f(x)$ is the Fourier transform of $f(x)$. In our case we have

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} dx\, e^{-2\pi ixk} e^{-\pi ax^2+2\pi ibx} = \int_{-\infty}^{\infty} dx\, e^{-\pi ax^2+2\pi i(b-k)x} \\
&= \sqrt{\frac{\pi}{\pi a}} e^{\frac{-4\pi^2(b-k)^2}{4\pi a}} = a^{-1/2} e^{\frac{-\pi(b-k)^2}{a}}
\end{aligned} \tag{8.89}$$

where we have used the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2+\beta x} = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}} \qquad [8.90]$$

Thus

$$\sum_{n=-\infty}^{\infty} \exp\bigl(-\pi a n^2 + 2\pi i b n\bigr) = a^{-1/2} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\pi(m-b)^2}{a}\right] \qquad [8.91]$$

8.16 p 237: Eq. (8.2.11) The Partition Function After the Poisson Resummation Formula

We now use the Poisson summation formula with $a = \alpha'\tau_2/R^2$ and $b = w\tau_1$ on (8.2.9)

$$\begin{aligned}
(q\bar{q})^{-1/24}\mathrm{Tr}\; q^{L_0}\bar{q}^{\tilde{L}_0} &= |\eta(\tau)|^{-2} \sum_{w=-\infty}^{\infty} \exp\left(-\frac{\pi\tau_2 R^2 w^2}{\alpha'}\right) \\
&\quad \times \left(\frac{\alpha'\tau_2}{R^2}\right)^{-1/2} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\pi(m-w\tau_1)^2}{\alpha'\tau_2/R^2}\right] \\
&= |\eta(\tau)|^{-2} \frac{R}{(\alpha'\tau_2)^{1/2}} \sum_{m,w=-\infty}^{\infty} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left[\tau_2^2 w^2 + (m-w\tau_1)^2\right]} \\
&= |\eta(\tau)|^{-2} \frac{R}{(\alpha'\tau_2)^{1/2}} \sum_{m,w=-\infty}^{\infty} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left[w^2(\tau_1^2+\tau_2^2) + m^2 - 2mw\tau_1\right]}
\end{aligned} \qquad [8.92]$$

Note now that

$$\begin{aligned}
|m-w\tau|^2 &= (m-w\tau)(m-w\bar{\tau}) = m^2 + w^2|\tau|^2 - mw(\tau+\bar{\tau}) \\
&= m^2 + w^2(\tau_1^2+\tau_2^2) - 2mw\tau_1
\end{aligned} \qquad [8.93]$$

and thus

$$(q\bar{q})^{-1/24}\mathrm{Tr}\; q^{L_0}\bar{q}^{\tilde{L}_0} = |\eta(\tau)|^{-2} \frac{R}{(\alpha'\tau_2)^{1/2}} \sum_{m,w=-\infty}^{\infty} e^{-\frac{\pi R^2\, |m-w\tau|^2}{\alpha'\tau_2}} \qquad [8.94]$$

We use (7.2.9), i.e. $|\eta(\tau)|^{-2} = (4\pi^2\alpha'\tau_2)^{1/2} Z_X(\tau)$ to find

$$\begin{aligned}
(q\bar{q})^{-1/24}\mathrm{Tr}\; q^{L_0}\bar{q}^{\tilde{L}_0} &= (4\pi^2\alpha'\tau_2)^{1/2} Z_X(\tau) \frac{R}{(\alpha'\tau_2)^{1/2}} \sum_{m,w=-\infty}^{\infty} e^{-\frac{\pi R^2\, |m-w\tau|^2}{\alpha'\tau_2}} \\
&= 2\pi R Z_X(\tau) \sum_{m,w=-\infty}^{\infty} e^{-\frac{\pi R^2\, |m-w\tau|^2}{\alpha'\tau_2}}
\end{aligned} \qquad [8.95]$$

We can view this as the partition function $Z_X(\tau)$ of the non-compact theory times the volume of the compactified dimension $2\pi R$ times correction due to the discrete momentum spectrum and the winding number.

Note that the partition function $Z_X(\tau_2)$ for the uncompactified free scalar has a factor $\tau_2^{-1/2}$. You will recall that this factor comes from the integration over the continuous momentum. This factor $\tau_2^{-1/2}$ is not present in the original form of the partition function (8.2.9) for the compactified free scalar, but it does arise from the resummation formula.

This partition function is modular invariant. We have shown earlier that the non-compactified partition function Z_X is modular invariant, so we need to focus on the infinite sum part only. Under $\tau \longrightarrow \tau + 1$ we have $m - w\tau \longrightarrow m - w(\tau + 1) = m - w - w\tau$ and so replacing m by $n = m - w$ shows invariance of the infinite sum.

Under $\tau \longrightarrow -1/\tau$ we have $\tau_1 \longrightarrow -\tau_1/|\tau|^2$ and $\tau_2 \longrightarrow \tau_2/|\tau|^2$. Thus

$$
\begin{aligned}
\sum_{m,w} \exp\left(-\frac{\pi R^2}{\alpha'}\frac{|m - w\tau|^2}{\tau_2}\right) \longrightarrow & \sum_{m,w} \exp\left(-\frac{\pi R^2}{\alpha'}\frac{|m + w/\tau|^2}{\tau_2/|\tau|^2}\right) \\
&= \sum_{m,w} \exp\left(-\frac{\pi R^2}{\alpha'}\frac{|m\tau + w|^2}{\tau_2}\right) \\
&= \sum_{w,m} \exp\left(-\frac{\pi R^2}{\alpha'}\frac{|-w\tau + m|^2}{\tau_2}\right) \qquad [8.96]
\end{aligned}
$$

In the last lime we have set $w' = -m$ and $m' = w$.

8.17 p 237: Eq. (8.2.13) The Periodicity of the Classical Solution

X_{cl} has the right periodic boundary conditions

$$
X_{\text{cl}}(\sigma^1 + 2\pi, \sigma^2) = (\sigma^1 + 2\pi)wR + \sigma^2(m - w\tau_1)R/\tau_2 = X_{\text{cl}}(\sigma^1, \sigma^2) + 2\pi wR \qquad [8.97]
$$

and

$$
\begin{aligned}
X_{\text{cl}}(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) &= (\sigma^1 + 2\pi\tau_1)wR + (\sigma^2 + 2\pi\tau_2)(m - w\tau_1)R/\tau_2 \\
&= X_{\text{cl}}(\sigma^1, \sigma^2) + 2\pi\tau_1 wR + 2\pi\tau_2(m - w\tau_1)R/\tau_2 \\
&= X_{\text{cl}}(\sigma^1, \sigma^2) + 2\pi mR \qquad [8.98]
\end{aligned}
$$

If we now split the compactified dimension in its classical and quantum part $X = X_{\text{cl}} + \mathsf{X}$ then X satisfies the ordinary boundary conditions $\mathsf{X}(\sigma^1 + 2\pi, \sigma^2) = \mathsf{X}(\sigma^1, \sigma^2)$ and $\mathsf{X}(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) = \mathsf{X}(\sigma^1, \sigma^2)$. So the path integral over X is just like the path integral over a non-compactified dimension and gives Z_X the $2\pi R$ comes from the volume of the compactified dimension. We thus need to evaluate $e^{-S_{cl}}$ where, see (1.2.13),

$$S_{\rm cl} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\gamma^{ab}\partial_a X_{\rm cl}\partial_b X_{\rm cl} \qquad [8.99]$$

We have applied the Faddeev-Popov procedure so we have fixed the gauge $\gamma^{ab} = \delta^{ab}$. We now simply have $\partial_1 X_{\rm cl} = wR$ and $\partial_2 X_{\rm cl} = (m - w\tau_1)R/\tau_2$. Thus

$$e^{-S_{cl}} = \exp\left\{-\frac{1}{4\pi\alpha'}\left[w^2R^2 + (m - w\tau_1)^2 R^2/\tau_2^2\right]\int d^2\sigma\right\} \qquad [8.100]$$

Here $\int d^2\sigma = 4\pi^2\tau_2$ is the surface area of the torus, see [7.10]. Thus

$$\begin{aligned} e^{-S_{cl}} &= \exp\left[-\frac{R^2}{4\pi\alpha'\tau_2^2}\left(w^2\tau_2^2 + m^2 + w^2\tau_1^2 - 2mw\tau_1\right)4\pi^2\tau_2\right] \\ &= \exp\left[-\frac{\pi R^2}{\alpha'\tau_2}\left(w^2|\tau|^2 + m^2 - 2mw\tau_1\right)\right] \\ &= \exp\left(-\frac{\pi R^2|m - w\tau|^2}{\alpha'\tau_2}\right) \end{aligned} \qquad [8.101]$$

In the last line we have used [8.93]. This is the infinite sum of (8.2.11).

8.18 p 238: Eq. (8.2.20) The Phase when a Vertex Operator Circles Another Vertex Operator

To describe how a point z on the complex plane circles the origin, we describe a curve $e^{2\pi i s}z$ with $s \in [0,1]$ a parameter. At $s = 1$ we have come full circle and are back to the original point in the complex plane. To describe how a point z_1 circles another point z_2 we first shift the reference frame so that z_2 is now at the origin and then circle the transformed point z_1 around the origin, thus we describe a curve $e^{2\pi i s}(z_1 - z_2) = e^{2\pi i s}z_{12}$. So encircling it completely once, means replacing z_{12} by $e^{2\pi i}z_{12}$.

Circling z_1 around z_2 in the OPE (8.2.19) thus gives

$$\begin{aligned} \mathcal{V}_{k_L\,k_R}(z_1,\bar z_2)\mathcal{V}_{k'_L\,k'_R}(z_2,\bar z_2) &\longrightarrow \left(e^{2\pi i}z_{12}\right)^{\alpha' k_L k'_L/2}\left(e^{-2\pi i}\bar z_{12}\right)^{\alpha' k_R k'_R/2}\mathcal{V}_{(k+k')_L\,(k+k')_R}(z_2,\bar z_2) \\ &= e^{\pi i\alpha'(k_Lk'_L - k_Rk'_R)}\mathcal{V}_{k_L\,k_R}(z_1,\bar z_2)\mathcal{V}_{k'_L\,k'_R}(z_2,\bar z_2) \end{aligned} \qquad [8.102]$$

and so we pick up a phase $e^{\pi i\alpha'(k_Lk'_L - k_Rk'_R)}$. If $k_L = k_R$ and $k'_L = k'_R$ then the phase is zero. This corresponds to a non-compact dimension. However if we have different momenta, then we have, by (8.2.7), different winding numbers w and w' and so we need to be able to wind the string around something and thus have a compact dimension. Filling in the possible values of the momenta explicitly we have for the phase

$$e^{\pi i\alpha'(k_L k'_L - k_R k'_R)} = \exp\left\{\pi i\alpha'\left[\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)\left(\frac{n'}{R}+\frac{w'R}{\alpha'}\right)-\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)\left(\frac{n'}{R}-\frac{w'R}{\alpha'}\right)\right]\right\}$$
$$= \exp\left[\pi i\alpha' 2\left(\frac{n}{R}\frac{w'R}{\alpha'}+\frac{wR}{\alpha'}\frac{n'}{R}\right)\right] = \exp\left[2\pi i(nw'+n'w)\right] \qquad [8.103]$$

For the OPE, and consequently amplitudes, to be well-defined we need this phase to disappear, i.e. $(nw'+n'w)\in\mathbb{Z}$.

8.19 p 239: Eq. (8.2.21) The Equal Time Commutator $[X_L(z_1), X_L(z_2)]$

We chose the worldsheet coordinate σ^1 to be in the range $[-\pi,\pi]$; it still has periodicity 2π. Defining as usual $z = e^{-iw} = e^{-i\sigma^1+\sigma^2}$ we thus have $\ln z = -i\sigma^1+\sigma^2$ and thus $\sigma^1 = -\mathrm{Im}\,\ln z$. The logarithm has a branch cut that we put on the negative real axis. Working out $[X_L(z_1), X_L(z_2)]$, the only non-zero commutation relations we have are $[x_L, p_L] = i$ and and $[\alpha_m,\alpha_n] = m\delta_{m+n}$. Thus

$$[X_L(z_1), X_L(z_2)] = \Big[x_L - i\frac{\alpha'}{2}p_L\ln z_1 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha_m}{mz_1^m},$$
$$x_L - i\frac{\alpha'}{2}p_L\ln z_2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{nz_2^n}\Big]$$
$$= -i\frac{\alpha'}{2}\ln z_2[x_L,p_L] - i\frac{\alpha'}{2}\ln z_1[p_L,x_L] - \frac{\alpha'}{2}\sum_{m,n\neq 0}\frac{[\alpha_m,\alpha_n]}{mnz_1^m z_2^n}$$
$$= \frac{\alpha'}{2}\left(\ln z_2 - \ln z_1 - \sum_{m,n\neq 0}\frac{m\delta_{m+n}}{mnz_1^m z_2^n}\right)$$
$$= \frac{\alpha'}{2}\left(\ln z_2 - \ln z_1 - \sum_{n\neq 0}\frac{z_1^n}{nz_2^n}\right) \qquad [8.104]$$

Now from $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$ we have

$$\ln\left(1-\frac{z_1}{z_2}\right) - \ln\left(1-\frac{z_2}{z_1}\right) = -\sum_{n=1}^{\infty}\frac{z_1^n}{nz_2^n} + \sum_{n=1}^{\infty}\frac{z_2^n}{nz_1^n} = -\sum_{n=1}^{\infty}\frac{z_1^n}{nz_2^n} - \sum_{m=-\infty}^{-1}\frac{z_1^m}{mz_2^m}$$
$$= -\sum_{n\neq 0}\frac{z_1^n}{nz_2^n} \qquad [8.105]$$

and thus

$$[X_L(z_1), X_L(z_2)] = \frac{\alpha'}{2}\left[\ln z_2 - \ln z_1 + \ln\left(1 - \frac{z_1}{z_2}\right) - \ln\left(1 - \frac{z_2}{z_1}\right)\right]$$
$$= \frac{\alpha'}{2}\left[\ln z_2 - \ln z_1 + \ln\frac{(z_2 - z_1)/z_2}{(z_1 - z_2)/z_1}\right]$$
$$= \frac{\alpha'}{2}\left[\ln z_2 - \ln z_1 + \ln\left(-\frac{z_1}{z_2}\right)\right] \quad [8.106]$$

Using $z = e^{-i\sigma^1 + \sigma^2}$ and the fact that we have equal time commutators, i.e. that $\sigma_1^2 = \sigma_2^2$ we have

$$[X_L(z_1), X_L(z_2)] = \frac{\alpha'}{2}\left[-i\sigma_2^1 + \sigma_2^2 + i\sigma_1^1 - \sigma_1^2 + \ln\left(e^{\pm i\pi}\frac{e^{-i\sigma_1^1+\sigma_1^2}}{e^{-i\sigma_2^1+\sigma_2^2}}\right)\right]$$
$$= \frac{\alpha'}{2}\left(-i\sigma_2^1 + i\sigma_1^2 + \ln e^{\pm i\pi - i\sigma_1^1 + i\sigma_2^1}\right)$$
$$= i\frac{\alpha'}{2}\left(-\sigma_2^1 + \sigma_1^1 \pm \pi - \sigma_1^1 + \sigma_2^1\right) = \pm\frac{\pi i\alpha'}{2} \quad [8.107]$$

Let us now sort out the sign. Let us look at the logarithm in more detail $\ln e^{i(\pm\pi - \sigma_1^1 + \sigma_2^1)}$. If $\sigma_1^1 > \sigma_2^1$ then $-\sigma_1^1 + \sigma_2^1 < 0$ and we need to take the $+$ sign. Indeed the most negative we can have $\sigma_1^1 = +\pi$ and $\sigma_2^1 = -\pi$. This gives $-\sigma_1^1 + \sigma_2^1 = -2\pi$. But this is on the branch cut, and we can bring it back in the range $[-\pi, \pi]$ by adding $+\pi$. Similarly, if $\sigma_1^1 < \sigma_2^1$ then $-\sigma_1^1 + \sigma_2^1 > 0$ and we need to take the $-$ sign. Indeed the most positive we can have $\sigma_1^1 = -\pi$ and $\sigma_2^1 = +\pi$. This gives $-\sigma_1^1 + \sigma_2^1 = 22\pi$ and we can bring it back in the range $[-\pi, \pi]$ by subtracting π. We thus conclude that, indeed,

$$[X_L(z_1), X_L(z_2)] = \frac{\pi i\alpha'}{2}\,\text{sign}\,(\sigma_1^1 - \sigma_2^1) \quad [8.108]$$

8.20 p 239: Eq. (8.2.22) The Correct Oscillator Expression for the Vertex Operator

We need the equal-time commutation relation for X_R. This derivation is very similar to the one of (8.2.21); we only have to change z by $\bar{z}$. This leads us immediately to the similar relation to [8.106]

$$[X_R(z_1), X_R(z_2)] = \frac{\alpha'}{2}\left[\ln\bar{z}_2 - \ln\bar{z}_1 + \ln\left(-\frac{\bar{z}_1}{\bar{z}_2}\right)\right] \quad [8.109]$$

We now use $\bar{z} = e^{+i\sigma^1+\sigma}$ and the fact that we have equal time commutators. to find

$$[X_R(z_1), X_R(z_2)] = \frac{\alpha'}{2}\left[i\sigma_2^1 + \sigma_2^2 - i\sigma_1^1 - \sigma_1^2 + \ln\left(e^{\pm i\pi}\frac{e^{+i\sigma_1^1+\sigma_1^2}}{e^{+i\sigma_2^1+\sigma_2^2}}\right)\right]$$
$$= \frac{\alpha'}{2}\left(+i\sigma_2^1 - i\sigma_1^2 + \ln e^{\pm i\pi + i\sigma_1^1 - i\sigma_2^1}\right) = \pm\frac{\pi i\alpha'}{2} \qquad [8.110]$$

We work out the sign in the same way as before, but now as the argument of the logarithm is $\ln e^{i(\pm\pi+\sigma_1^1-\sigma_2^1)}$ and not $\ln e^{i(\pm\pi-\sigma_1^1+\sigma_2^1)}$ we have the rôle of σ_1^1 and σ_2^1 interchanged. Therefore

$$[X_R(z_1), X_R(z_2)] = -\frac{\pi i\alpha'}{2}\,\mathrm{sign}\,(\sigma_1^1 - \sigma_2^1) \qquad [8.111]$$

We now wish to see find the relation between

$$\mathcal{V}_{k_{L_1}k_{R_1}}(z_1,\bar{z}_1)\mathcal{V}_{k_{L_2}k_{R_2}}(z_2,\bar{z}_2) \quad \text{and} \quad \mathcal{V}_{k_{L_2}k_{R_2}}(z_2,\bar{z}_2)\mathcal{V}_{k_{L_1}k_{R_1}}(z_1,\bar{z}_1) \qquad [8.112]$$

with the vertex operator as defined in (8.2.22). Let us reflect a little bit on this before we start calculating anything. The commutator of an X_L and an X_R vanishes and the commutators of two X_L's or two X_R's is a scalar. So, from the Baker-Campbell-Haussdorf formula formula we have $e^Ae^B = e^{[A,B]}e^Be^A$ if $[A,B]$ is a scalar. We thus have

$$e^{ik_{L_1}X_L(z_1)}e^{ik_{L_2}X_L(z_2)} = e^{ik_{L_1}k_{L_2}[X_L(z_1),X_L(z_2)]}e^{ik_{L_2}X_L(z_2)}e^{ik_{L_1}X_L(z_1)}$$
$$= e^{ik_{L_1}k_{L_2}\frac{\pi i\alpha'}{2}\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}e^{ik_{L_2}X_L(z_2)}e^{ik_{L_1}X_L(z_1)}$$
$$= e^{-(\pi\alpha'/2)k_{L_1}k_{L_2}\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}e^{ik_{L_2}X_L(z_2)}e^{ik_{L_1}X_L(z_1)} \qquad [8.113]$$

and similarly

$$e^{ik_{R_1}X_R(z_1)}e^{ik_{R_2}X_R(z_2)} = e^{+(\pi\alpha'/2)k_{R_1}k_{R_2}\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}e^{ik_{R_2}X_R(z_2)}e^{ik_{R_1}X_R(z_1)} \qquad [8.114]$$

Together we have

$$e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)} = e^{-(\pi\alpha'/2)k_{L_1}k_{L_2}\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}$$
$$\times\, e^{+(\pi\alpha'/2)k_{R_1}k_{R_2}\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)}$$
$$= e^{-(\pi\alpha'/2)(k_{L_1}k_{L_2}-k_{R_1}k_{R_2})\,\mathrm{sign}\,(\sigma_1^1-\sigma_2^1)}$$
$$\times\, e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \qquad [8.115]$$

Now

$$k_{L_1}k_{L_2} - k_{R_1}k_{R_2} = \left(\frac{n_1}{R}+\frac{w_1R}{\alpha'}\right)\left(\frac{n_2}{R}+\frac{w_2R}{\alpha'}\right) - \left(\frac{n_1}{R}-\frac{w_1R}{\alpha'}\right)\left(\frac{n_2}{R}-\frac{w_2R}{\alpha'}\right)$$
$$= 2\frac{n_1w_2 + w_1n_2}{\alpha'} \qquad [8.116]$$

and thus

$$
\begin{aligned}
e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)} &= e^{-\pi(n_1w_2+w_1n_2)\,\text{sign}\,(\sigma_1^1-\sigma_2^1)} \\
&\quad\times e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \\
&= (-)^{n_1w_2+w_1n_2}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \qquad [8.117]
\end{aligned}
$$

We thus see that if we define the vertex operators as in (8.2.18) then they would commute only if $n_1w_2 + w_1n_2$ is even and anti-commute if $n_1w_2 + w_1n_2$ is odd. This vindicates the remark under (8.2.21).

Let us now add the cocyles. We use $\mathcal{C}_{12}$, with the subscripts denoting the order of the vertex operators, i.e.

$$
\begin{aligned}
\mathcal{C}_{12} &= e^{i\pi(k_{L_1}-k_{R_1})(p_L+p_R)\alpha'/4}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \\
&\quad\times e^{i\pi(k_{L_2}-k_{R_2})(p_L+p_R)\alpha'/4}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)} \qquad [8.118]
\end{aligned}
$$

The p are momentum operators, so they pick up the momentum component of everything that is to their right. This thus becomes

$$
\begin{aligned}
\mathcal{C}_{12} &= e^{i\pi(k_{L_1}-k_{R_1})(k_{L_1}+k_{L_2}+k_{R_1}+k_{R_2})\alpha'/4}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \\
&\quad\times e^{i\pi(k_{L_2}-k_{R_2})(k_{L_2}+k_{R_2})\alpha'/4}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)} \\
&= e^{i\pi\left[(k_{L_1}-k_{R_1})(k_{L_1}+k_{L_2}+k_{R_1}+k_{R_2})+(k_{L_2}-k_{R_2})(k_{L_2}+k_{R_2})\right]\alpha'/4} \\
&\quad\times e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)} \qquad [8.119]
\end{aligned}
$$

Use [8.117]

$$
\begin{aligned}
\mathcal{C}_{12} &= e^{i\pi\left[(k_{L_1}-k_{R_1})(k_{L_1}+k_{L_2}+k_{R_1}+k_{R_2})+(k_{L_2}-k_{R_2})(k_{L_2}+k_{R_2})\right]\alpha'/4} \\
&\quad\times (-)^{n_1w_2+w_1n_2}e^{ik_{L_2}X_L(z_2)+ik_{R_2}X_R(\bar{z}_2)}e^{ik_{L_1}X_L(z_1)+ik_{R_1}X_R(\bar{z}_1)} \qquad [8.120]
\end{aligned}
$$

and re-introduce the cocycles and their inverse. This gives

$$
\begin{aligned}
\mathcal{C}_{12} &= e^{i\pi\left[(k_{L_1}-k_{R_1})(k_{L_1}+k_{L_2}+k_{R_1}+k_{R_2})+(k_{L_2}-k_{R_2})(k_{L_2}+k_{R_2})\right]\alpha'/4} \\
&\quad\times (-)^{n_1w_2+w_1n_2}e^{-i\pi(k_{L_2}-k_{R_2})(k_{L_1}+k_{L_2}+k_{R_1}+k_{R_2})}e^{-i\pi(k_{L_1}-k_{R_1})(k_{L_1}+k_{R_1})\alpha'/4}\mathcal{C}_{21} \qquad [8.121]
\end{aligned}
$$

It remains to work out the phase:

$$
\begin{aligned}
\mathcal{C}_{12} &= (-)^{n_1w_2+w_1n_2}e^{i\pi\left[(k_{L_1}-k_{R_1})(k_{L_2}+k_{R_2})-(k_{L_2}-k_{R_2})(k_{L_1}+k_{R_1})\right]\alpha'/4}\mathcal{C}_{21} \\
&= (-)^{n_1w_2+w_1n_2}e^{i\pi(+2k_{L_1}k_{R_2}-2k_{R_1}k_{L_2})\alpha'/4}\mathcal{C}_{21}
\end{aligned}
$$

[8.122]

We find

$$k_{L_1}k_{R_2} - k_{R_1}k_{L_2} = \left(\frac{n_1}{R} + \frac{w_1 R}{\alpha'}\right)\left(\frac{n_2}{R} - \frac{w_2 R}{\alpha'}\right) - \left(\frac{n_1}{R} - \frac{w_1 R}{\alpha'}\right)\left(\frac{n_2}{R} + \frac{w_2 R}{\alpha'}\right)$$
$$= -2\frac{n_1 w_2 - w_1 n_2}{\alpha'} \qquad [8.123]$$

and thus

$$\mathcal{C}_{12} = (-)^{n_1 w_2 + w_1 n_2} e^{-i\pi(n_1 w_2 - w_1 n_2)}\mathcal{C}_{21} = (-)^{n_1 w_2 + w_1 n_2}(-)^{n_1 w_2 - w_1 n_2}\mathcal{C}_{21}$$
$$= (-)^{2 n_1 w_2}\mathcal{C}_{21} = \mathcal{C}_{21} \qquad [8.124]$$

as $2n_1 w_2$ is always even. We thus see that adding the cocycle indeed ensures that the vertex operators commute.

8.21 p 240: Eq. (8.2.26) The OPEs in the Light-Cone Reference Frame

Ignoring the L and R we have the OPEs $X^\mu(z)X^\nu(0) = -(\alpha'/2)\eta^{\mu\nu}\ln z$ so that

$$X^+(z)X^-(w) \sim \frac{1}{2}\left[X^0(z) + X^1(z)\right]\left[X^0(0) - X^1(0)\right] = \frac{1}{2}\left[X^0(z)X^0(0) - X^1(z)X^1(0)\right]$$
$$= -\frac{\alpha'}{2}\ln z$$
$$X^\pm(z)X^\pm(w) \sim \frac{1}{2}\left[X^0(z) \pm X^1(z)\right]\left[X^0(0) \pm X^1(0)\right] = \frac{1}{2}\left[X^0(z)X^0(0) + X^1(z)X^1(0)\right]$$
$$= 0 \qquad [8.125]$$

Recall that spacetime has Lorentz signature $(- + \cdots +)$.

8.22 p 240: Eq. (8.2.27) $V^i(nk_0, z)$ is a Primary Field

We can split the energy-momentum tensor as

$$T(z) = -\frac{1}{\alpha'}\left[-\partial X^0\partial X^0(z) + \partial X^1\partial X^1(z) + \partial X^i\partial X^i(z)\right]$$
$$= T^{(2\cdots D)}(z) - \frac{1}{2\alpha'}\left[-(\partial X^+ + \partial X^-)(\partial X^+ + \partial X^-)(z) + (\partial X^+ - \partial X^-)(\partial X^+ - \partial X^-)\right]$$
$$= T^{(2\cdots D)}(z) + \frac{2}{\alpha'}\partial X^+\partial X^-(z) \qquad [8.126]$$

where $T^{(2\cdots D}(z) = -\frac{1}{\alpha'}\partial X^i\partial X^i(z)$ is the energy-momentum tensor for all but the first two space-time fields. As $T^{(2\cdots D)}(z)X^+(0)$ is regular we can break down the calculation in parts. We clearly have, ignoring the normalisation $\sqrt{2/\alpha'}$,

$$\begin{aligned} T^{(2\cdots D)}(z)V^i(nk_0,0) =& T^{(2\cdots D)}(z)\partial X^i e^{ink_0X^+}(0) \\ &\sim \frac{\partial X^i e^{ink_0X^+}(0)}{z^2} + \frac{\partial^2 X^i e^{ink_0X^+}(0)}{z} \end{aligned} \tag{8.127}$$

We also have have

$$\begin{aligned} \frac{2}{\alpha'}\partial X^+\partial X^-(z)V^i(nk_0,0) &= \frac{2}{\alpha'}\partial X^+\partial X^-(z)\partial X^i e^{ink_0X^+}(0) \\ &\sim \frac{2}{\alpha'}\partial X^+(z)\partial X^i(0)(ink_0)\frac{\alpha'}{2z}e^{ink_0X^+}(0) \\ &\sim \frac{\partial X^i ink_0 \partial X^+ e^{ink_0X^+}(0)}{z} = \frac{\partial X^i \partial e^{ink_0X^+}(0)}{z} \end{aligned} \tag{8.128}$$

where we have used $\partial X^-(z)X^+(0) \sim \alpha'/2z$ and have expanded $\partial X^+(z)$ around $z = 0$. Therefore

$$\begin{aligned} T(z)V^i(nk_0,0) =& T(z)\partial X^i e^{ink_0X^+}(0) \\ &\sim \frac{\partial X^i e^{ink_0X^+}(0)}{z^2} + \frac{\partial^2 X^i e^{ink_0X^+}(0)}{z} + \frac{\partial X^i \partial e^{ink_0X^+}(0)}{z} \\ &= \frac{\partial X^i e^{ink_0X^+}(0)}{z^2} + \frac{\partial\big(\partial X^i e^{ink_0X^+}\big)(0)}{z} \\ &= \frac{V^i(nk_0,0)}{z^2} + \frac{\partial V^i(nk_0,0)(0)}{z} \end{aligned} \tag{8.129}$$

and so $V^i(nk_0,z)$ is indeed a primary field with weight one.

8.23 p 240: Eq. (8.2.28) The $V^i(nk_0,z)V^j(mk_0,z)$ OPE

This is straightforward as the only non-singular terms come from the $\partial X^i(z)\partial X^j(0) = -\alpha'\delta^{ij}/2z^2$ part:

$$\begin{aligned} V^i(nk_0,z)V^j(mk_0,0) &= \frac{2}{\alpha'}\partial X^i e^{ink_0X^+}(z)\partial X^j e^{imk_0X^+}(0) \\ &= -\frac{2}{\alpha'}\frac{\alpha'\delta^{ij}e^{ink_0X^+}(z)e^{imk_0X^+}(0)}{2z^2} \\ &= -\frac{\delta^{ij}e^{i(n+m)k_0X^+}(0)}{2z^2} - \frac{ink_0\partial X^+\delta^{ij}e^{i(n+m)k_0X^+}(0)}{z} \end{aligned} \tag{8.130}$$

8.24 p 240: Eq. (8.2.30) The Commutation Relation of the DDF Operators

It's been a while since we did one of these, so we go slowly. Our staring point is (2.6.14)

$$\begin{aligned}
[A^i_m, A^j_n] &= \oint \frac{dz_2}{2\pi i}\text{Res}_{z_1\to z_2} V^i(nk_0, z_1)V^j(mk_0, z_2) \\
&= \oint \frac{dz_2}{2\pi i}\text{Res}_{z_1\to z_2}\left[-\frac{\delta^{ij}e^{i(n+m)k_0X^+}(z_2)}{2(z_1-z_2)^2} - \frac{ink_0\partial X^+\delta^{ij}e^{i(n+m)k_0X^+}(z_2)}{z_1-z_2}\right] \\
&= \oint \frac{dz_2}{2\pi i}\left(-ink_0\partial X^+\delta^{ij}e^{i(n+m)k_0X^+}\right) \\
&= -\, ink_0\delta^{ij}\oint \frac{dz_2}{2\pi i}\partial X^+ e^{i(n+m)k_0X^+}(z_2)
\end{aligned} \quad [8.131]$$

Let us work this out. For $n+m=0$ we simply have

$$\begin{aligned}
-ink_0\delta^{ij}\oint \frac{dz_2}{2\pi i}\partial X^+ &= +\, imk_0\delta^{ij}\left(-i\sqrt{\frac{\alpha'}{2}}\alpha_0^+\right) = mk_0\delta^{ij}\sqrt{\frac{\alpha'}{2}}\alpha_0^+ \\
&= mk_0\delta^{ij}\sqrt{\frac{\alpha'}{2}}\sqrt{\frac{\alpha'}{2}}p^+ = \frac{mk_0\alpha' p^+}{2}\delta^{ij}
\end{aligned} \quad [8.132]$$

where we have used (2.7.2), i.e. $\alpha^\mu_m = \sqrt{2/\alpha'}\oint(dz/2\pi)\, z^m\partial X^\mu$ and (2.7.3), i.e. $p^\mu = \sqrt{2/\alpha'}\alpha^\mu_u$.

If $m+n\neq 0$ then we can write $\partial X^+ e^{i(n+m)k_0X^+}(z_2) = \partial e^{i(n+m)k_0X^+}(z_2)$. But by definition $e^{i(n+m)k_0X^+}(z_2)$ is normal ordered and all its singularities are subtracted, so that it is regular, and so is its derivative. Thus in that case $\oint X^+e^{i(n+m)k_0X^+}(z_2) = 0$.

We conclude that

$$[A^i_m, A^j_n] = \frac{mk_0\alpha' p^+}{2}\delta^{ij}\delta_{m+n} \quad [8.133]$$

8.25 p 240: The DDF Operators as Building Blocks for Physical States

A physical state of momentum q of the form $|\psi\rangle = f(\partial^i\bar\partial^j X^k)e^{iq\cdot X}|0\rangle$ with f a function of the derivatives of the transverse fields, such that $L_m|\psi\rangle = 0$ for $m\geq 0$. The vertex operator creating that state is $\mathcal{V} = f(\partial^i\bar\partial^j X^k)e^{iq\cdot X}$. If we take the OPE with a DDF operator $V^i(nk_0, z)$ then we see that the exponential in that operator only contracts with the exponential in the vertex operator giving something proportional to

$$\begin{aligned}
\partial X^i e^{ink_0X^+}(z) f(\partial^i\bar\partial^j X^k)e^{iq\cdot X}(0) &= \partial X^i e^{ink_0X^+}(z) f(\partial^i\bar\partial^j X^k)e^{i(-q^+X^- - q^-X^+ + q^iX^i)}(0) \\
&\sim z^{-\alpha' nk_0q^+/2} : \partial X^i e^{ink_0X^+}(z) f(\partial^i\bar\partial^j X^k)e^{i(-q^+X^- - q^-X^+ + q^iX^i)}(0) : \\
&\quad + \text{other contractions}
\end{aligned} \quad [8.134]$$

Here the "other contractions" are terms that arise from contractions of the transverse coordinates. The important point here is that all these other terms are by construction single valued. The only possible non-single-valued terms is thus the one above. So if we require the OPE of a DDF operator with a vertex operator of momentum q to be single-valued we necessarily need $\alpha' n k_0 q^+/2 \in \mathbb{Z}$. In particular, the choice $k_0 = 2/\alpha q^+$ gives a single-valued OPE.[3] This means that the action of a DDF operator on a physical state, that is, after all, created by action of a vertex operator on the ground state, is well-defined.

To show that Virasoro operators and DDF operators commute we use (2.6.24), which says that for a primary operator $\mathcal{O}$ of weight h we have $[L_m, \mathcal{O}_n] = \big[(h-1)m - n\big]\mathcal{O}_{m+n}$, with the mode $\mathcal{O}_m = \oint (dz/2\pi i) z^{m+h-1}\mathcal{O}(z)$. As a $V^i(nk_0, z)$ is a primary weight one operator its zero mode is

$$V_0^i(nk_0) = \oint \frac{dz}{2\pi i} z^{0+1-1} V^i(nk_0, z) = -i \oint \frac{dz}{2\pi} V^i(nk_0, z) = -iA_n^i \tag{8.135}$$

and so

$$[L_m, A_n^i] = -[L_m, V_0^i(nk_0)] = \big[(1-1)m - 0\big]\mathcal{V}_{m+0}^i(nk_0) = 0 \tag{8.136}$$

8.26 p 241: Eq. (8.3.1) The Mass-Shell Condition with a Compactified Dimension, I

This starts from our all time favourites (4.3.31) and (4.3.32) which we just repeat here

$$L_0 = \frac{\alpha'}{4}(p^2 + m^2) \tag{8.137}$$

with

$$\frac{\alpha'}{4} m^2 = \sum_{n=1}^{\infty} n \left(N_{bn} + N_{cn} + \sum_{\mu=0}^{25} N_{\mu n} \right) - 1 \tag{8.138}$$

The mass-shell condition for the matter sector $L_0 |\psi\rangle = 0$ thus becomes $m^2 = -p^2$. But now we have to split this in the non-compactified and the compactified dimensions. For the non-compactified dimensions we can use the above equation. As we are looking at the matter sector only we can ignore the ghost contributions and set the total level of the

3

Joe has $k_0 = 2/\alpha q^-$. I believe this is an error as the contribution clearly comes from the $X^+(z)X^-(0)$ OPE and the $X^-(0)$ comes with q^+. It is a bit strange that this is not on Joe's errata page.

matter excitations as $N = \sum_{n=1}^{\infty}\sum_{\mu=0}^{25} N_{\mu n}$. For the compactified dimension we need to treat the momentum separately as it is quantised. We thus have

$$m^2 = -p^2 = \frac{4}{\alpha'}(N-1) + \left(k_L^{25}\right)^2 \tag{8.139}$$

We have reintroduced the fact that we were working in the left-moving sector. There is a similar equation for the right sector.

8.27 p 241: Eq. (8.3.2) The Mass-Shell Condition with a Compactified Dimension, II

We rewrite the mass-shell conditions (8.3.1) into two new equations. For the first one, we use the quantisation (8.2.7) and take the average of both sectors

$$\begin{aligned} m^2 &= \frac{1}{2}\left[\left(k_L^{25}\right)^2 + \left(k_R^{25}\right)^2 + \frac{4}{\alpha'}(N-1) + \frac{4}{\alpha'}(\tilde{N}-1)\right] \\ &= \frac{1}{2}\left[\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2 + \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N+\tilde{N}-2)\right] \\ &= \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\tilde{N}-2) \end{aligned} \tag{8.140}$$

For the second equation we take the difference of the two mass-shell conditions:

$$\begin{aligned} 0 &= \left(k_L^{25}\right)^2 - \left(k_R^{25}\right)^2 + \frac{4}{\alpha'}(N-1) - \frac{4}{\alpha'}(\tilde{N}-1) \\ &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2 - \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N-\tilde{N}) \\ &= 4\frac{n}{R}\frac{wR}{\alpha'} + \frac{4}{\alpha'}(N-\tilde{N}) \end{aligned} \tag{8.141}$$

From this we get

$$0 = nw + N - \tilde{N} \tag{8.142}$$

8.28 p 241: Eq. (8.3.3) The Massless States

The identification of the different states should be obvious. The only clarification I want to make is about the scalar $\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\left|0;k\right\rangle$. This the modulus for the radius of the compact dimension. What does this exactly mean? A modulus is a flat background field. Here is corresponds to the compactified dimension in both the left- and right-handed sector, so it is clearly linked to the compactified dimension.

But what does it exactly mean to say that it is "the modulus for the **radius** of the compactified direction"? That is not clear to me.

The second statement, that "its vertex operator, $:\partial X^{25}\bar{\partial} X^{25} e^{ik\cdot X}:$ is a perturbation of the metric $G_{25,25}$ follows from the discussion in section 3.7 about strings in curved backgrounds. There we saw that the interaction with different states created by the vertex operators can be viewed as perturbations of the spacetime metric. As an example, expanding the spacetime metric (assuming an uncompactified string) $G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$ the integrand in the worldsheet path integral is, see (3.7.3) and (3.7.4)

$$e^{-S_P}\left[1 - \frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{g}\,g^{ab}\chi_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \cdots\right] \tag{8.143}$$

where S_P is the Polyakov action. This corresponds to the introduction in a correlation function of a vertex operator $g^{ab}\chi_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$ with $\chi_{\mu\nu}(X) = -4\pi g_c e^{ik\cdot X} s_{\mu\nu}$. And so the introduction of a vertex operator $:\partial X^{25}\bar{\partial} X^{25} e^{ik\cdot X}:$ does indeed correspond to a perturbation of $G_{25,25}$.

8.29 p 242: Eq. (8.3.7) The Gauge Coupling

We need to evaluate

$$\begin{aligned}\mathcal{S}_3^{\mu[\pm]} = \Bigg\langle \frac{\sqrt{2}g_{c,25}}{\alpha'} :\left(\partial X^\mu \bar{\partial} X^{25} \pm \partial X^{25}\bar{\partial} X^\mu\right) e^{ik_1\cdot X}(z_1,\bar{z}_1):\; g_{c,25} : e^{ik_{2L}\cdot X_L(z_2) + ik_{2R}\cdot X_R(\bar{z}_2)} : \\ \times\, g_{c,25} : e^{ik_{3L}\cdot X_L(z_3) + ik_{3R}\cdot X_R(\bar{z}_3)} : \Bigg\rangle\end{aligned} \tag{8.144}$$

Let us consider the term with $\partial X^\mu \bar{\partial} X^{25}$. The other term will follow from this immediately.

$$\begin{aligned}\mathcal{S}_{3\,(a)}^{\mu[\pm]} = c\Big\langle :\partial X^\mu \bar{\partial} X^{25} e^{ik_1\cdot X}(z_1,\bar{z}_1) :: e^{ik_{2L}\cdot X_L(z_2) + ik_{2R}\cdot X_R(\bar{z}_2)} : \\ \times : e^{ik_{3L}\cdot X_L(z_3) + ik_{3R}\cdot X_R(\bar{z}_3)} : \Big\rangle\end{aligned} \tag{8.145}$$

where $c = \sqrt{2}g_{c,25}^3/\alpha'$. We now use (6.6.14), or even better, our derivation of that equation, so it might be worthwhile revisiting that. The first equation we use is [6.241]. We repeat it here for convenience. What we did there is calculate

$$\begin{aligned}S_{s_2}(k_1,\epsilon_1;k_2;k_3) = g_c^2 g_c' e^{-2\lambda}\epsilon^1_{\mu\nu}\Big\langle :\tilde{c}c\partial X^\mu \bar{\partial} X^\nu e^{ik_1\cdot X} : (z_1,\bar{z}_1) \\ \times : \tilde{c}ce^{ik_2\cdot X} : (z_2,\bar{z}_2) : \tilde{c}ce^{ik_3\cdot X} : (z_3,\bar{z}_3)\Big\rangle\end{aligned} \tag{8.146}$$

and this lead to

$$S_{s_2}(k_1,\epsilon_1;k_2;k_3) = -\frac{i\alpha'^2}{4}g_c^2 g_c' e^{-2\lambda}C^X_{S_2}C^g_{S_2}(2\pi)^{26}\delta^{26}(\sum_i k_i)\epsilon^1_{\mu\nu} \times |z_{12}|^{\alpha' k_1\cdot k_2+2}|z_{13}|^{\alpha' k_1\cdot k_3+2}|z_{23}|^{\alpha' k_2\cdot k_3+2}\left(\frac{k_2^\mu}{z_{12}}+\frac{k_3^\mu}{z_{13}}\right)\left(\frac{k_2^\nu}{\bar z_{12}}+\frac{k_3^\nu}{\bar z_{13}}\right) \quad [8.147]$$

Let us now see how we need to adapt this result. First we note that we have 25 non-compact dimensions in stead of 26. The factor $(2\pi)^{26}\delta^{26}(\sum_i k_i)$ came from the zero mode contribution, see the derivation of (6.2.13). The compact dimension has no zero mode as such a mode cannot satisfy the boundary conditions. So these factors become $(2\pi)^{25}\delta^{25}(\sum_i k_i)$. Moreover the holomorphic and anti-holomorphic sectors split and as $X = X_L + X_R$ these two sectors only talk to one another. Ignoring the details of the overall constant, for now, we thus find

$$\mathcal{S}^{\mu[\pm]}_{3\,(a)} = \tilde c(2\pi)^{25}\delta^{25}(\sum_i k_i) z_{12}^{\alpha' k_1\cdot k_{2L}/2+1}\bar z_{12}^{\alpha' k_1\cdot k_{2R}/2+1} z_{13}^{\alpha' k_1\cdot k_{3L}/2+1}\bar z_{13}^{\alpha' k_1\cdot k_{3R}/2+1} \times z_{23}^{\alpha' k_{2L}\cdot k_{3L}/2+1}\bar z_{23}^{\alpha' k_{2R}\cdot k_{3R}/2+1}\left(\frac{k_{2L}^\mu}{z_{12}}+\frac{k_{3L}^\mu}{z_{13}}\right)\left(\frac{k_{2R}^{25}}{\bar z_{12}}+\frac{k_{3R}^{25}}{\bar z_{13}}\right) \quad [8.148]$$

The remainder is analogous to the derivation of (6.6.14). The tachyon mass shell condition implies that $k_1\cdot k_{2L,R} = k_1\cdot k_{3L,R} = 0$ and the gauge boson mass shell condition implies that $k_{2L,R}\cdot k_{3L,R} = -4/\alpha'$. We thus get

$$\mathcal{S}^{\mu[\pm]}_{3\,(a)} = \tilde c(2\pi)^{25}\delta^{25}(\sum_i k_i)|z_{12}|^2|z_{13}|^2|z_{23}|^{-2}\left(\frac{k_{2L}^\mu}{z_{12}}+\frac{k_{3L}^\mu}{z_{13}}\right)\left(\frac{k_{2R}^{25}}{\bar z_{12}}+\frac{k_{3R}^{25}}{\bar z_{13}}\right) \quad [8.149]$$

This enables us to write down immediately the corresponding result from [6.247]

$$\mathcal{S}^{\mu[\pm]}_{3\,(a)} = \frac{\tilde c}{4}(2\pi)^{25}\delta^{25}(\sum_i k_i)k^\mu_{23L}k^{25}_{23R} \quad [8.150]$$

From this it follows that the second term is

$$\mathcal{S}^{\mu[\pm]}_{3\,(b)} = \frac{\tilde c}{4}(2\pi)^{25}\delta^{25}(\sum_i k_i)k^{25}_{23L}k^{\mu}_{23R} \quad [8.151]$$

and thus

$$\mathcal{S}^{\mu[\pm]}_{3} = \frac{\tilde c}{4}(2\pi)^{25}\delta^{25}(\sum_i k_i)\left(k^\mu_{23L}k^{25}_{23R} \pm k^{25}_{23L}k^{\mu}_{23R}\right) \quad [8.152]$$

But from momentum conservation in the non-compact dimensions we have

$$k^\mu_{23L} = k^\mu_{2L} - k^\mu_{3L} = k^\mu_{2L} + k^\mu_1 + k^\mu_{2L} \to 2k^\mu_{2L} = 2k^\mu_2 \quad [8.153]$$

where we can ignore the k_1^μ due to the Ward identity for a gauge boson $\epsilon \cdot k_1 = 0$ as the gauge boson vertex operator will be contracted with a polarisation vector. We also use the fact that for the non-compact dimensions we don't have a split of the momenta in a left- and right handed part and $k_{2L}^\mu = k_{2R}^\mu = k_2^\mu$. We have, of course, similarly $k_{23R}^\mu \to 2k_2^\mu$. Therefore

$$\mathcal{S}_3^{\mu[\pm]} = \frac{\tilde{c}}{2}(2\pi)^{25}\delta^{25}(\sum_i k_i)k_2^\mu \left(k_{23R}^{25} \pm k_{23L}^{25}\right) \qquad [8.154]$$

Finally we also write for the compact dimension

$$k_{23L}^{25} = k_{2L}^{25} - k_{3L}^{25} = k_{2L}^{25} + k_1^{25} + k_{2L}^{25} \to 2k_{2L}^{25} \qquad [8.155]$$

where this time we have taken $k_1 \to 0$. Similarly we have $k_{23R}^{25} \to 2k_{2R}^{25}$. This gives our final result

$$\mathcal{S}_3^{\mu[\pm]} = -\tilde{c}(2\pi)^{25}\delta^{25}(\sum_i k_i)k_2^\mu \left(k_{2L}^{25} \pm k_{2R}^{25}\right) \qquad [8.156]$$

We will leave it to the reader to work out the proportionality constant. Filling in the momenta (8.2.7) this gives

$$\begin{aligned}\mathcal{S}_3^{\mu[+]} &= -2\tilde{c}(2\pi)^{25}\delta^{25}(\sum_i k_i)k_2^\mu \frac{n_2}{R}\\ \mathcal{S}_3^{\mu[-]} &= -2\tilde{c}(2\pi)^{25}\delta^{25}(\sum_i k_i)k_2^\mu \frac{w_2 R}{\alpha'}\end{aligned} \qquad [8.157]$$

So one couples to the compact quantised compact momentum and the other one to the winding number.

8.30 p 242: Eq. (8.3.8) The Mass-Shell Condition at $R = \sqrt{\alpha'}$

We start from (8.3.1) and use $k_{L,R}^{25} = (\alpha')^{-1/2}(n \pm w)$:

$$\begin{aligned}0 &= (k_L^{25})^2 + \frac{4}{\alpha'}(N-1) = \frac{1}{\alpha'}\left[(n+w)^2 + 4N - 4\right] \quad \Rightarrow \quad 4 = (n+w)^2 + 4N\\ 0 &= (k_R^{25})^2 + \frac{4}{\alpha'}(\tilde{N}-1) = \frac{1}{\alpha'}\left[(n-w)^2 + 4\tilde{N} - 4\right] \quad \Rightarrow \quad 4 = (n-w)^2 + 4\tilde{N}\end{aligned} \qquad [8.158]$$

Note that there are also additional massless states at other values for the compactification radius. Set $R = 2(p/q)\sqrt{\alpha'}$ for integers p and q with $p \mod q = 0$. The mass-shell conditions then become

$$1 = \frac{p^2}{q^2}(n+w)^2 + N \qquad \text{and} \qquad 1 = \frac{p^2}{q^2}(n-w)^2 + \tilde{N} \qquad [8.159]$$

This shows that we cannot have massless states for $N, \tilde{N} > 1$. For $N = \tilde{N} = 1$ the only massless states have necessarily $n = w = 0$. For $N = \tilde{N} = 0$ we get

$$1 = \frac{p^2}{q^2}(n+w)^2 \qquad \text{and} \qquad 1 = \frac{p^2}{q^2}(n-w)^2 \qquad [8.160]$$

From this we get that $(n+w)^2 = (n-w)^2$, which implies that either n or w is zero. If $w = 0$ then $n^2 = q^2/p^2$. As n needs to be an integer we need $p = 1$ and s solution is given by $n = \pm q$. We have similar solutions for w when $n = 0$. In summary we have $R = 2/q\sqrt{\alpha'}$ then we have the following massless states written as $(n, w; N\tilde{N})$:

$$(\pm q, 0; 0, 0); \quad (0, \pm q; 0, 0); \quad (0, 0, 1, 1) \qquad [8.161]$$

8.31 p 242: Eq. (8.3.9)-(8.3.10) The Special Massless States at $R = \sqrt{\alpha'}$

We put the extra massless states at that level of the compactification radius in a table for our convenience

n	w	N	$\tilde{N}$	$\sqrt{\alpha'}k_L^{25}$	$\sqrt{\alpha'}k_R^{25}$	**vertex operator**
$+1$	$+1$	0	1	$+2$	0	$\bar{\partial}X^\mu e^{ik\cdot X}e^{+2i(\alpha')^{-1/2}X_L^{25}}$
-1	-1	0	1	-2	0	$\bar{\partial}X^\mu e^{ik\cdot X}e^{-2i(\alpha')^{-1/2}X_L^{25}}$
$+1$	-1	1	0	0	$+2$	$\partial X^\mu e^{ik\cdot X}e^{+2i(\alpha')^{-1/2}X_R^{25}}$
-1	$+1$	1	0	0	-2	$\partial X^\mu e^{ik\cdot X}e^{-2i(\alpha')^{-1/2}X_R^{25}}$
$+2$	0	0	0	$+2$	0	$e^{ik\cdot X}e^{+2i(\alpha')^{-1/2}X_L^{25}}$
-2	0	0	0	-2	0	$e^{ik\cdot X}e^{-2i(\alpha')^{-1/2}X_L^{25}}$
0	$+2$	0	0	0	-2	$e^{ik\cdot X}e^{-2i(\alpha')^{-1/2}X_R^{25}}$
0	-2	0	0	0	$+2$	$e^{ik\cdot X}e^{+2i(\alpha')^{-1/2}X_R^{25}}$

Table 8.2: Special massless states at $R = \sqrt{\alpha'}$

In order to see what couples with what, let us take the $(n, w, N, \tilde{N}) = (0, 0, 0, 1)$ state in (8.3.5). It is proportional to $\partial X^{25}\bar{\partial}X^\mu e^{ik\cdot X}$. Because there is an ∂X^{25} it can only have a non-zero amplitude with another state if that state also has an X_L^{25} in its vertex operator. This means that it can only couple to the states $(n, w, N, \tilde{N}) = (\pm 1, \pm 1, 0, 1)$ and $(n, w, N, \tilde{N}) = (\pm 2, 0, 0, 0)$ from the above tables. Under that gauge vector, these states have a charge of ± 1. They also have a charge zero under the gauge vector corresponding to the X_R symmetry. We thus have three operators that couple with one another on the holomorphic side with charge $0, \pm 1$. The anti-holomorphic side has similarly three operators that couple with one another and have charge $0, \pm 1$. Each sector is neutral under the action of the other one. The gauge group we have uncovered is thus $SU(2) \times SU(2)$.

8.32 p 243: Eq. (8.3.12) The $SU(2) \times SU(2)$ Current Algebra

The three operators involving a $\bar{\partial}X^\mu$ from an $SU(2)$. The operator $\partial X^{25}\bar{\partial}X^\mu e^{ik\cdot X}$ has zero charge and so we can take this to be the j^3. The operators $\bar{\partial}X^\mu e^{ik\cdot X}e^{\pm 2i(\alpha')^{-1/2}X_L^{25}}$ have charge ±1 and hence are the $j^\pm$. We can combine the latter to create the standard $SU(2)$ generators. Including appropriate normalisation these are (8.3.12).

8.33 p 243: Eq. (8.3.13) The $SU(2)$ Current Algebra OPEs

We will just show this for a couple of OPEs. As a warm-up let us first do an easy one, the normalisation of $j^3(z)$

$$j^3(z)j^3(w) = -\frac{1}{\alpha'}\partial X_L^{25}(z)\partial X_L^{25}(w) \sim -\frac{1}{\alpha'}\left(-\frac{\alpha'/2}{(z-w)^2}\right) = \frac{1}{2(z-w)^2} \qquad [8.162]$$

Marginally increasing the level of complexity and calling $a = (\alpha')^{-1/2}$.

$$j^3(z)j^1(z) = ia\partial X_L^{25}(z)\cos 2aX_L^{25}(w) = ia\partial X_L^{25}(z)\frac{1}{2}\left(e^{+2iaX_L^{25}} + e^{-2iaX_L^{25}}\right)(w) \qquad [8.163]$$

Now

$$\begin{aligned}\partial X_L^{25}(z)e^{\pm 2iaX_L^{25}}(w) &= \partial X_L^{25}(z)\sum_{k=0}^{\infty}\frac{(\pm 2ia)^k}{k!}(X_L^{25})^k(w) \\ &= \partial_z \sum_{k=1}^{\infty}\frac{(\pm 2ia)^k}{k!}k\left[-\frac{\alpha'}{2}\ln(z-w)\right](X_L^{25})^{k-1}(w) \\ &= \mp\frac{ia\alpha' e^{\pm 2iaX_L^{25}}(w)}{z-w}\end{aligned} \qquad [8.164]$$

and thus

$$\begin{aligned}j^3(z)j^1(z) &= \frac{ia}{2}\left(-\frac{ia\alpha' e^{+2iaX_L^{25}}(w)}{z-w} + \frac{ia\alpha' e^{-2iaX_L^{25}}(w)}{z-w}\right) \\ &= \frac{a^2\alpha'}{2}\frac{e^{+2iaX_L^{25}}(w) - e^{-2iaX_L^{25}}(w)}{z-w} = \frac{i\sin 2iaX_L^{25}(w)}{z-w} = \frac{ij^2(w)}{z-w}\end{aligned} \qquad [8.165]$$

We will leave the rest to the reader if he/she is bored enough to do this.

8.34 p 243: Eq. (8.3.14) The Affine Lie Algebra Commutation Relations

This should be entirely standard by now, but in the spirit of being as complete as possible we will do this. Our starting point is, of course, (2.6.14) which becomes in this case

$$\begin{aligned}
[j^i_m, j^j_n] &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m z_2^n j^i(z_1) j^j(z_2) \\
&= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m z_2^n \left[\frac{\delta^{ij}}{2(z_1 - z_2)^2} + \frac{i\epsilon^{ijk} j^k(z_2)}{z_1 - z_2} \right] \\
&= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m z_2^n \left[-\frac{1}{2}\delta^{ij} \partial_1 \frac{1}{z_1 - z_2} + \frac{i\epsilon^{ijk} j^k(z_2)}{z_1 - z_2} \right] \\
&= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} \frac{\frac{1}{2}\delta^{ij} m z_1^{m-1} z_2^n + z_1^m z_2^n i\epsilon^{ijk} j^k(z_2)}{z_1 - z_2} \\
&= \oint \frac{dz_2}{2\pi i} \left[\frac{m}{2} \delta^{ij} z_2^{m+n-1} + i\epsilon^{ijk} j^k(z_2) z_2^{m+n} \right] \\
&= \frac{m}{2} \delta^{ij} \delta_{m+n} + i\epsilon^{ijk} j^k_{m+n}
\end{aligned} \qquad [8.166]$$

8.35 p 244: Eq. (8.3.15) The Relation Between the Gauge and the Gravitational Coupling in the Compactified Dimension

(8.1.11) is $g_d^2 = 2\kappa_d^2/\rho^2$, where $\rho = R = \sqrt{\alpha'}$ is the compactified dimension. Using $d = 25$, this immediately gives (8.3.15).

8.36 p 244: Eq. (8.3.16) The Magnitude of the String Length $\sqrt{\alpha'}$

Here we are back in traditional Kaluza-Klein with five dimensions. The compactified fifth dimension, confusingly x^{25}, gives rise to the gauge field with gauge-coupling g_4. This gauge coupling is, give or take, of the order of one. Indeed for QED the coupling is the fine structure constant $\alpha \approx 1/137$. The weak gauge coupling is also smaller than one, hence perturbation theory works there as well. For QCD the coupling constant is larger and perturbation theory fails, but as we increase the energy of the system, we need to take into account the renormalisation of the coupling constants. For QED and the weak interaction, the coupling constant slowly increase, for QCD it decreases – asymptotic freedom. So in terms of order of magnitudes, saying that the coupling constants are of the order of one, isn't too far of the mark. Setting $g_4^2 \approx 1$ we get $\sqrt{\alpha'} \approx \sqrt{2}\kappa_4$.

8.37 p 244: Eq. (8.3.17) The Effective Gauge Coupling

We refer to table 8.3 for the mass dimensions of the different fields. As $[\partial] = [A] = +1$ and we have $D_\mu = \partial_\mu + igA_\mu$ we thus have $[g] = 0$. The gauge group coupling constant is dimensionless (and hence a gauge theory is renormalisable). The gravitational coupling appears in the Einstein-Hilbert action $\kappa^{-2} \int d^4x \sqrt{g} R$. With $[d^4x] = -4$, $[g] = 0$ and $[R] = 2$ we have $-[\kappa_4^2] - 4 + 2 = 0$ or $[\kappa_4^2] = -2$ (and hence quantum gravity is not renormalizable). As energy has mass-dimension one; recall $E = mc^2$, we can make a dimensionless effective gravitational coupling

$$g_{G,4}^2(E) = \kappa_4^2 E^2 \tag{8.167}$$

with indeed $[g_{G,4}^2(E)] = [\kappa_4^2] + [E^2] = -2 + 2 = 0$.

The gauge coupling g_4 gets renormalised as well so has an energy dependence, but for Yang-Mills type theory this change in running of the coupling constant is at a snail's pace and involves the logarithm of the energy; see your favourite book on QFT, or even better, my QFT Notes. Using (8.3.16) we also have

$$\kappa_4^2 = E^{-2} g_{G,4}^2(E) = \frac{\alpha'}{2} g_4^2 \quad \Rightarrow \quad g_{G,4}^2(E) = \frac{\alpha'}{2} E^2 g_4^2 \tag{8.168}$$

The string mass scale is where the string effects become relevant, i.e. when the energy of the system is of the order $\alpha^{-1/2}$, in which case $g_{G,4}^2(E) \approx g_4^2$, i.e. when the gravitational coupling is of the order of the gauge coupling.

8.38 p 245: Eq. (8.3.20) The Gauge Boson Mass for near the Enhanced Symmetry $SU(2) \times SU(2)$

The general mass-shell formula is given by (8.3.2)

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{8.169}$$

The enhanced symmetry states at $(n, w, N, \tilde{N}) = (\pm 1, \pm 1, 0, 1)$ or $(\pm 1, \mp 1, 1, 0)$ and therefore

$$m^2 = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'} \tag{8.170}$$

and we see that they are massless as $R = \sqrt{\alpha'}$. We now take $R^2 = \alpha' + \varepsilon$ with ε small. This gives

$$\begin{aligned} m^2 &= \frac{1}{\alpha' + \varepsilon} + \frac{\alpha' + \varepsilon}{\alpha'^2} - \frac{2}{\alpha'} \\ &= \frac{1}{\alpha'}\left(1 - \frac{\varepsilon}{\alpha'} + \frac{\varepsilon^2}{\alpha'^2} + o(\varepsilon^3) + 1 + \frac{\varepsilon}{\alpha'} - 2\right) \\ &= \frac{\varepsilon^2}{\alpha'^3} + o(\varepsilon^3) = \frac{(R^2 - \alpha')^2}{\alpha'^3} + o(\varepsilon^3) \end{aligned} \quad [8.171]$$

We can rewrite this, to lowest order as, as $m^2 = (R^2 - \alpha')^2 / R^2 \alpha'^2$. Indeed

$$\frac{(R^2 - \alpha')^2}{R^2 \alpha'^2} = \frac{\varepsilon^2}{\alpha'^2(\alpha' + \varepsilon)} = \frac{\varepsilon^2}{\alpha'^3}\left(1 - \frac{\varepsilon}{\alpha'}\right) = \frac{\varepsilon^2}{\alpha'^3} + o(\varepsilon^3) \quad [8.172]$$

Taking the square root we find

$$m = \frac{|R^2 - \alpha'|}{R\alpha'} \quad [8.173]$$

In order to show that this is the same as the second equation of (8.3.20), let us square both sides. We need to show that, to lowest order,

$$\frac{(R^2 - \alpha')^2}{R^2} = 4\left(R - \sqrt{\alpha'}\right)^2 \quad [8.174]$$

Now $R^2 = \alpha' + \varepsilon$ so that

$$R = \sqrt{\alpha' + \varepsilon} = \sqrt{\alpha'}\sqrt{1 + \frac{\varepsilon}{\alpha'}} = \sqrt{\alpha'}\left(1 + \frac{\varepsilon}{2\alpha'}\right) \quad [8.175]$$

The LHS of [8.174] becomes

$$\text{LHS} = \frac{\varepsilon^2}{\alpha'(1 + \varepsilon/2\alpha')^2} = \frac{\varepsilon^2}{\alpha'(1 + \varepsilon/\alpha')} = \frac{\varepsilon^2}{\alpha'}\left(1 - \frac{\varepsilon}{\alpha'}\right) = \frac{\varepsilon^2}{\alpha'} + o(\varepsilon^3) \quad [8.176]$$

The RHS of [8.174] is

$$\text{RHS} = 2\left[\sqrt{\alpha'}\left(1 + \frac{\varepsilon}{2\alpha'}\right) - \sqrt{\alpha'}\right]^2 = 4\left(\frac{\varepsilon}{2\sqrt{\alpha'}}\right)^2 = \frac{\varepsilon^2}{\alpha'} + o(\varepsilon^3) \quad [8.177]$$

so that $\text{LHS} = \text{RHS}$, showing (8.3.20).

8.39 p 245: The Ten Massless Scalars at the Enhanced Symmetry Compactification Radius

The first two massless scalars are from the list (8.3.3). They are the dilaton, which is the traceless part of $\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\,|0;k\rangle$ and the modulus for the radius of compactification, $\alpha^{25}_{-1}\alpha^{25}_{-1}\,|0;k\rangle$. The enhanced symmetry providers eight more scalars. Four come from the states (8.3.9) with vertex operators (8.3.11). These split into a spacetime vector in the non-compactified dimension with $\mu = 0, \cdots, 24$ and a spacetime scalar for the compactified component $\mu = 25$. This gives four spacetime scalars

$$\bar{\partial}X^{25}\,e^{ik\cdot X}e^{\pm 2i\alpha^{-1/2}X^{25}_L} \quad \text{and} \quad \partial X^{25}\,e^{ik\cdot X}e^{\pm 2i\alpha^{-1/2}X^{25}_R} \qquad [8.178]$$

Finally, there are four more massless scalars coming from (8.3.10). These have vertex operator

$$e^{ik\cdot X}e^{\pm 2i\alpha^{-1/2}X^{25}_L}\,e^{\pm 2i\alpha^{-1/2}X^{25}_R} \qquad [8.179]$$

8.40 p 245: Eq. (8.3.21) The $(3, 3)$ of $SU(2) \times SU(2)$

We consider the vertex operators $V^{ij} = j^i\tilde{\jmath}^j e^{ik\cdot X}$. From the definition of the j^i and $\tilde{\jmath}^j$ in (8.3.1) it is clear that these 9 vertex operators create 9 out of the 10 massless scalars. The one not created by these operators is the dilaton.

The operators V^{1j}, V^{2j} and V^{3j} transform under the left $SU(2)$. Each such operator has three components, e.g. V^{1j} consists of V^{11}, V^{12} and V^{13}. It thus forms a three-dimensional representation of the first $SU(2)$ of $SU(2) \times SU(2)$ which is denoted by $\mathbf{3}$. Similarly the V^{i1}, V^{i2} and V^{i3} from a three-dimensional representation of the second $SU(2)$ of $SU(2) \times SU(2)$. The currents V^{ij} thus form a $(\mathbf{3}, \mathbf{3})$ of $SU(2) \times SU(2)$.

8.41 p 246: Eq. (8.3.22) The Invariance of the Potential $U(M)$ under $SU(2) \times SU(2)$

We wish to show that the potential $u(m) = \epsilon^{ijk}m_i m_j m_k$ is invariant under an $SU(2)$ transformation. We are using small letters here as we are only considering the left-handed side of the potential. The situation for the right-handed side is entirely similar. An $SU(2)$ transformation is obtained by acting with a generator t^{ℓ} on such a field. Here $k = 1, 2, 3$ the dimension of $SU(2)$. This means that we wish to show that the potential is invariant under a transformation $m_i \longrightarrow (\mathrm{t}^{\ell}m)_i$. As m is in the $\mathbf{3}$ representation of $SU(2)$, i.e. the adjoint representation, the generators are given by the structure constants: $\left(\mathrm{t}^{\ell}\right)_{ij} = \epsilon^{\ell ij}$. We are being somewhat sloppy with upstairs and downstairs indices, but they are raised and lowered with a Kronecker delta, so it doesn't matter. We thus need to check the potential under the transformation[4] $m_i \longrightarrow (\delta^{ij} + i\varepsilon\epsilon^{\ell ij})m_j$. To lowest order in ε we have

$$\begin{aligned}
u(m) \longrightarrow\ & \epsilon^{ijk}(\delta^{im} + i\varepsilon\epsilon^{\ell im})m_m(\delta^{jn} + i\varepsilon\epsilon^{\ell jn})m_n(\delta^{kp} + i\varepsilon\epsilon^{\ell kp})m_p \\
&= \epsilon^{ijk}m_i m_j m_k + i\varepsilon\epsilon^{ijk}\left(\epsilon^{\ell im}\delta^{jn}\delta^{kp} + \delta^{im}\epsilon^{\ell jn}\delta^{kp} + \delta^{im}\delta^{jn}\epsilon^{\ell kp}\right)m_m m_n m_p \\
&= u(m) + i\varepsilon\left(\epsilon^{inp}\epsilon^{\ell im} + \epsilon^{mjp}\epsilon^{\ell jn} + \epsilon^{mnk}\epsilon^{\ell kp}\right)m_m m_n m_p \\
&= u(m) + i\varepsilon\left(\epsilon^{inp}\epsilon^{im\ell} + \epsilon^{ipm}\epsilon^{in\ell} + \epsilon^{imn}\epsilon^{ip\ell}\right)m_m m_n m_p \\
&= u(m) + i\varepsilon\left(\delta^{nm}\delta^{p\ell} - \delta^{n\ell}\delta^{pm} + \delta^{pn}\delta^{m\ell} - \delta^{p\ell}\delta^{mn} + \delta^{mp}\delta^{n\ell} - \delta^{m\ell}\delta^{np}\right)m_m m_n m_p \\
&= u(m)
\end{aligned} \tag{8.180}$$

Adding the right-handed side we see that $U(M)$ is invariant under an $SU(2) \times SU(2)$ transformation.

8.42 p 246: Eq. (8.3.24) The Equations for the Scalar Fields M_{ij}

If only the diagonal elements of M are nonzero then $U(M) = \det M = M_{11}M_{22}M_{33}$ and so the equation of motion $U(M) = 0$ implies $M_{11}M_{22}M_{33} = 0$. In addition, $\partial U(M)/\partial M_{ij}$ is zero automatically for $i \neq j$. For $i = j$ we have, e.g. $0 = \partial U(M)/\partial M_{11} = M_{22}M_{33}$. Similarly, we have $M_{11}M_{33} = 0$ and $M_{22}M_{33} = 0$.

To solve these equations, we note that a first solution is $M_{11} = M_{22} = M_{33} = 0$; this solution does not have any of the nine scalars. Another solution is $M_{11} = M_{22} = 0$ and $M_{33} \neq 0$, and similar solutions for $M_{22} \neq 0$ and $M_{33} \neq 0$ by symmetry. There are no other solutions.

8.43 p 247: Eq. (8.3.27) The Momenta under T-Duality

Under $n \longleftrightarrow w$ and $R \longleftrightarrow \alpha'/R$ we have

$$\begin{aligned}
p_L^{25} &= \frac{n}{R} + \frac{wR}{\alpha'} \longrightarrow w\frac{R}{\alpha'} + \frac{n}{R} = p_L^{25} \\
p_R^{25} &= \frac{n}{R} - \frac{wR}{\alpha'} \longrightarrow w\frac{R}{\alpha'} - \frac{n}{R} = -\left(\frac{n}{R} - \frac{wR}{\alpha'}\right) = -p_R^{25}
\end{aligned} \tag{8.181}$$

[4]The ϵ^{ijk} are the generators of the Lie algebra $\mathfrak{su}(2)$ to find the group generator we need to exponentiate the Lie algebra generators. We are only considering an infinitesimal transformation with parameter ε, but the group is connected.

8.44 p 247: Eq. (8.3.28) *T*-Duality gives Equivalent Theories

We have from (8.2.16), dropping the superscript ^{25} for convenience,

$$\begin{aligned} +X_L &= +x_L - \frac{i\alpha'}{2} p_L \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m z^m} \\ -X_R &= -x_R + \frac{i\alpha'}{2} p_R \ln \bar{z} - i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\tilde{\alpha}_m}{m \bar{z}^m} \end{aligned} \qquad [8.182]$$

In X_R we can change $\tilde{\alpha}_m$ into $-\tilde{\alpha}_m$. This doesn't change the commutation relations $[\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+2}$ or anything else. Similarly x_R is just the c.o.m. momentum. The only real impact is that p_R comes with the wrong sign. Now p_R is a combination of momentum and winding number, but we know from (8.3.27) that we can change p_R into $-p_R$ by interchanging n and w and replacing R by α'/R. So the theory with $X' = X_L - X_R$ is just the theory with $X = X_L + X_R$ after the T-duality transformation. But the theory with X and X' are exactly the same CFTs, and this is a non-perturbative statement, as they have the same OPE and energy-momentum tensor. The two theories related by T-duality are thus the same, also at a non-perturbative level.

8.45 p 248: Eq. (8.3.30) The Gravitational Coupling under *T*-Duality

Consider the scattering amplitudes of gravitons with $n = w = 0$. The amplitude must be the same under $n \longleftrightarrow w$ and $\rho \longrightarrow \alpha'/\rho$. In this particular case, it is only the compactification radius that changes. The amplitude of these gravitons depends on the gravitational coupling constant in the non-compactified dimensions – as these are the dimensions in which we define the graviton – i.e. κ_{25}. Under T-duality that coupling constant must be unchanged, i.e. $\kappa_{25} \longrightarrow \kappa'_{25} = \kappa_{25}$. We know the relation between the gravitational coupling in the non-compactified dimensions and the gravitational coupling in the full theory: $\kappa_{26}^2 = 2\pi\rho\kappa_{25}^2$.[5] After a T-duality transformation this becomes $\kappa'_{26} = \sqrt{2\pi\alpha'/\rho}\kappa'_{25}$. Setting $\kappa'_{25} = \kappa_{25}$ then gives

$$\frac{\kappa_{26}}{\sqrt{2\pi\rho}} = \frac{\kappa'_{26}}{\sqrt{2\pi\alpha'/\rho}} \quad \Rightarrow \quad \kappa'_{26} = \frac{\sqrt{\alpha'}}{\rho}\kappa_{26} \qquad [8.183]$$

[5] Recall that this is due to the fact that we assume that there is no dependence on the compactified dimension, so it can be integrated out. It just sits there and does nothing.

8.46 p 248: Eq. (8.3.31) The Dilaton under T-Duality

The low-energy effective action (8.1.9) is of the form

$$S \propto \int d^d x \sqrt{-G_d}\, e^{-2\Phi}(R_d + \cdots) \qquad [8.184]$$

where Φ is the dilaton field. The Einstein-Hilbert action for the full theory is

$$S \propto \kappa_{26}^{-2} \int d^d x \sqrt{-G_d}\, R_d \qquad [8.185]$$

and thus $\kappa_{26}^{-2} \propto e^{-2\Phi}$ or $\kappa_{26} \propto e^{\Phi}$. Plugging this in (8.3.30) gives (8.3.31).

8.47 p 249: Eq. (8.4.2) The Low Energy Action for k Compactified Dimensions

This is an extension of (8.1.9) and (8.1.13) for the case that more than one dimension is compactified. We will not give a full derivation, but only check a number of things.

Let us first recall the origin of this equation. Go back to section 3.7 of Joe's book, where he analysed Weyl invariance of the non-linear sigma model $\int d^2\sigma\, G^{\mu\nu}\partial_a X_\mu \partial^a X_\nu$ and showed how this lead to the requirement of the vanishing of the β function. This in turn gave the Einstein equations plus string corrections. He then wrote the most general non-linear sigma model, by introducing, next to the spacetime metric $G^{\mu\nu}$, also the anti-symmetric tensor $B^{\mu\nu}$ and the dilaton Φ. This gives the more general equations for the vanishing of the β functions, viz. (3.7.14). Next, he claimed, and we checked, that these equations followed from the variation principle with a low energy action given by (3.7.20).

Let us now check the number of scalars. The $D = 26$-dimensional metric splits into $G_{MN} = (G_{\mu\nu}, G_{\mu m}, G_{mn})$ with $\mu, \nu = 0, \cdots, d = 26 - k$ and $m, n = 1, \cdots, k$. The $G_{\mu\nu}$ is the (spacetime) graviton. The $G_{\mu m}$ are k spacetime vector fields and the G_{mn} are spacetime scalars. G_{mn} is symmetric so there are $k(k+1)/2$ such scalars. Similarly, the anti-symmetric tensor splits into $B_{MN} = (B_{\mu\nu}, B_{\mu m}, B_{mn})$. Here again, B_{mn} are spacetime scalars and due to antisymmetry there are $k(k-1)/2$ of them. In total we thus have $k(k+1)/2 + k(k-1)/2 = k^2$ scalars from both these fields.

Next, let us first check that for one compactified dimension (8.4.2) reduces to (8.1.9) and (8.1.13). In that case m and n can only take the value d and $G_{dd} = e^{2\sigma}$ and $B_{dd} = 0$ by symmetry. The dilaton is defined as $\Phi_d = \Phi - \frac{1}{4}\ln\det G_{mn}$ which becomes

$$\Phi_d = \Phi - \frac{1}{4}\ln\det e^{2\sigma} = \Phi - \frac{1}{4}\ln\, e^{2\sigma} = \Phi - \frac{\sigma}{2} \qquad [8.186]$$

which is the same formula as used in (8.1.9). We thus get in this case for (8.4.2)

$$\begin{aligned}S=&\frac{(2\pi R)^1}{2\kappa_0^2}\int d^dx\,\sqrt{-G_d}\,e^{-2\Phi_d}\Big[R_d+4\partial_\mu\Phi_d\partial^\mu\Phi_d-\frac{1}{4}e^{-2\sigma}e^{-2\sigma}\partial_\mu e^{2\sigma}\partial^\mu e^{2\sigma}\\&-\frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}e^{-2\sigma}H_{d\mu\nu}H_d^{\ \mu\nu}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big]\\=&\frac{\pi R}{\kappa_0^2}\int d^dx\,\sqrt{-G_d}\,e^{-2\Phi_d}\Big[R_d+4\partial_\mu\Phi_d\partial^\mu\Phi_d-\partial_\mu\sigma\partial^\mu\sigma\\&-\frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}e^{-2\sigma}H_{d\mu\nu}H_d^{\ \mu\nu}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big]\end{aligned}\qquad [8.187]$$

and this is indeed (8.1.9) plus (8.1.13).

Let us now turn to the longer stuff. We first find the relationship between the determinants of the D and the d-dimensional metric. As in the derivation of (8.1.9), we use a $\tilde{}$ to denote D dimensional quantities. The D dimensional metric is given by the generalisation of (8.1.2), or in our case [8.22],

$$\begin{aligned}\widetilde{G}_{\mu\nu}=&G_{\mu\nu}+G_{mn}A_\mu^mA_\nu^n\\\widetilde{G}_{\mu m}=&G_{mn}A_\mu^n\\\widetilde{G}_{mn}=&G_{mn}\end{aligned}\qquad [8.188]$$

Note that we now have k vector fields A_μ^m for $m=1,\cdots,k$. The inverse metric is

$$\begin{aligned}\widetilde{G}^{\mu\nu}=&G^{\mu\nu}\\\widetilde{G}^{\mu m}=&-G^{\mu\nu}A_\nu^m\\\widetilde{G}^{mn}=&G^{mn}+G^{\mu\nu}A_\mu^mA_\nu^n\end{aligned}\qquad [8.189]$$

One easily checks this:

$$\begin{aligned}\widetilde{G}_{\mu N}\widetilde{G}^{N\rho}=&\widetilde{G}_{\mu\nu}\widetilde{G}^{\nu\rho}+\widetilde{G}_{\mu n}\widetilde{G}^{n\rho}=(G_{\mu\nu}+G_{mn}A_\mu^mA_\nu^n)G^{\nu\rho}+G_{mn}A_\mu^m(-G^{\rho\sigma}A_\sigma^m)\\=&G_{\mu\nu}G^{\nu\rho}+G_{mn}A_\mu^mA_\nu^nG^{\nu\rho}-G_{mn}G^{\rho\sigma}A_\mu^mA_\sigma^m=G_{\mu\nu}G^{\nu\rho}=\delta_\mu^\rho\\\widetilde{G}_{\mu N}\widetilde{G}^{N\ell}=&\widetilde{G}_{\mu\nu}\widetilde{G}^{\nu\ell}+\widetilde{G}_{\mu n}\widetilde{G}^{n\ell}=(G_{\mu\nu}+G_{mn}A_\mu^mA_\nu^n)(-G^{\nu\sigma}A_\sigma^\ell)+G_{kn}A_\mu^n(G^{k\ell}+G^{\sigma\nu}A_\sigma^kA_\nu^\ell)\\=&-G_{\mu\nu}G^{\nu\sigma}A_\sigma^\ell-G_{mn}G^{\nu\sigma}A_\mu^mA_\nu^nA_\sigma^\ell+G_{kn}G^{k\ell}A_\mu^n+G_{kn}G^{\sigma\nu}A_\mu^nA_\sigma^kA_\nu^\ell\\=&-A_\mu^\ell+A_{\mu n}A^{\sigma n}A_\sigma^\ell+A_\mu^\ell+A_{k\mu}A^{k\nu}A_\nu^\ell=0\end{aligned}\qquad [8.190]$$

and finally

$$\begin{aligned}\widetilde{G}_{mN}\widetilde{G}^{N\ell}=&\widetilde{G}_{m\nu}\widetilde{G}^{\nu\ell}+\widetilde{G}_{mn}\widetilde{G}^{n\ell}=G_{mn}A_\nu^n(-G^{\mu\nu}A_\mu^\ell)+G_{mn}(G^{n\ell}+G^{\mu\nu}A_\mu^nA_\nu^\ell)\\=&-G_{mn}G^{\mu\nu}A_\nu^nA_\mu^\ell+G_{mn}G^{n\ell}+G_{mn}G^{\mu\nu}A_\mu^nA_\nu^\ell\\=&-A_m^\mu A_\mu^\ell+\delta_m^\ell+A_m^\nu A_\nu^\ell=\delta_m^\ell\end{aligned}\qquad [8.191]$$

As is the case for a single compactified dimension, the specific choice of metric implies that the determinant of the D dimensional metric factorises:

$$\det \tilde{G} = \det G \, \det G_{mn} \qquad [8.192]$$

Just to remind ourselves here G is the D dimensional metric G_D, G is the d dimensional metric G_d and G_{mn} is the metric in the compactified dimensions. This property can either be worked out explicitly, or as in the case of one compactified dimension, by noting the structure of the rows. We can rewrite this as

$$\sqrt{\tilde{G}} = \sqrt{G}\,\sqrt{G_{mn}} = \sqrt{G}\,\exp\ln\sqrt{\det G_{mn}} = \sqrt{G}\,\exp\frac{1}{2}\ln\det G_{mn} \qquad [8.193]$$

and thus

$$\sqrt{\tilde{G}}e^{-2\Phi} = \sqrt{\tilde{G}}e^{-2\Phi}e^{\frac{1}{2}\ln\det G_{mn}} = \sqrt{\tilde{G}}e^{-2(\Phi-\frac{1}{4}\ln\det G_{mn})} = \sqrt{\tilde{G}}e^{-2\Phi_d} \qquad [8.194]$$

where $\Phi_d = \Phi - \frac{1}{4}\ln\det G_{mn}$. This explains in (8.4.2) the coefficients of the integrand in front of the square brackets.

It remains to derive what is in the square brackets. The first two terms, the first part of the second line and the first term of the third line are merely the decomposition of $\tilde{R}$. We will not do the detailed calculation, but remind ourselves that we just showed that it reduces to the correct formula for $k = 1$ and that it is the natural generalisation of one compactified dimension to k compactified dimensions. Note that we now have k field strengths $F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu$.

Let us now turn to the antisymmetric product in (3.7.20), i.e. $-\frac{1}{12}H_{MNL}H^{MNL}$. This is

$$\begin{aligned}-\frac{1}{12}H_{MNL}H^{MNL} =& \tilde{G}^{MP}\tilde{G}^{NQ}\tilde{G}^{LR}H_{MNL}H_{PQR} \\ =& -\frac{1}{12}\tilde{G}^{MP}\tilde{G}^{NQ}\tilde{G}^{LR}(\partial_M B_{NL} + \partial_N B_{LM} + \partial_L B_{MN}) \\ & \times(\partial_P B_{QR} + \partial_Q B_{RP} + \partial_R B_{PQ})\end{aligned} \qquad [8.195]$$

Clearly if we consider all the indices to be in the non-compact dimensions we find a contribution

$$-\frac{1}{12}H_{MNL}H^{MNL} = -\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \cdots \qquad [8.196]$$

Where we have also used the fact that $\tilde{G}^{\mu\nu} = G^{\mu\nu}$. We now focus on the terms that have the same index in the derivative, i.e.

$$\begin{aligned} H_{MNL}H^{MNL} &= \tilde{G}^{MP}\tilde{G}^{NQ}\tilde{G}^{LR}(\partial_M B_{NL}\partial_P B_{QR} + \partial_N B_{LM}\partial_Q B_{RP} + \partial_L B_{MN}\partial_R B_{PQ}) + \cdots \\ &= \tilde{G}^{NQ}\tilde{G}^{LR}\partial_M B_{NL}\partial^M B_{QR} + \tilde{G}^{MP}\tilde{G}^{LR}\partial_N B_{LM}\partial^N B_{RP} \\ &\quad + \tilde{G}^{MP}\tilde{G}^{NQ}\partial_L B_{MN}\partial^L B_{PQ} + \cdots \\ &= 3\tilde{G}^{NQ}\tilde{G}^{LR}\partial_M B_{NL}\partial^M B_{QR}\partial^M B_{QR} + \cdots \\ &= 3\tilde{G}^{NQ}\tilde{G}^{LR}\partial_\mu B_{NL}\partial^\mu B_{QR} + \cdots \end{aligned} \quad [8.197]$$

In the last two lines we have renamed the summation indices and used the fact that we assume that there is no dependence on the coordinates of the compact dimensions, $\partial_m = 0$. If we now take the remaining indices those of the compact dimensions, we find, in addition to [8.196], a contribution[6]

$$-\frac{1}{12}H_{MNL}H^{MNL} = -\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{4}G^{nq}G^{\ell r}\partial_\mu B_{n\ell}\partial^\mu B_{qr} + \cdots \quad [8.198]$$

and we have already recovered two of the terms in the square bracket of (8.4.2).

One can proceed in the same way with all the possible combinations of indices to recover (8.4.2).

8.48 p 249: Eq. (8.4.3) The Antisymmetric Tensor in the Worldsheet Lagrangian

The antisymmetric tensor appears in the worldsheet Lagrangian in the nonlinear sigma model in (3.7.6)

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\left(\cdots + i\epsilon^{ab}B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + \cdots\right) \quad [8.199]$$

We now have

$$\begin{aligned} B_{mn}\partial_a\Big(\sqrt{g}\epsilon^{ab}X^m\partial_b X^n\Big) &= B_{mn}\Big(\partial_a\sqrt{g}\Big)\epsilon^{ab}X^m\partial_b X^n + \sqrt{g}\epsilon^{ab}\partial_a X^m\partial_b X^n + \sqrt{g}\epsilon^{ab}X^m\partial_a\partial_b X^n \\ &= B_{mn}\sqrt{g}\epsilon^{ab}\partial_a X^m\partial_b X^n \end{aligned} \quad [8.200]$$

The last term vanishes due to symmetry considerations and the first because we have fixed the gauge to Euclidean spacetime.

[6] $\widetilde{G}^{mn} = G^{mn} + G^{\mu\nu}A^m_\mu A^n_\nu$ so there will also be contributions quadratic in A, but we are focussing on the terms that don't have any A's.

8.49 p 249: Eq. (8.4.4) The Zero Mode of the Compactified String

We have

$$X^m(\sigma^1 + 2\pi, \sigma^2) = x^m(\sigma^2) + w^m R(\sigma^1 + 2\pi) = X^m(\sigma^1, \sigma^2) + 2\pi w^m R \qquad [8.201]$$

Here w^m is the winding number in the m-th dimension and not some winding number to the m-th power. We see that we recover the boundary condition (8.2.3) as we should.

8.50 p 249: Eq. (8.4.5) The Worldsheet Action for the Zero-Mode of the Compactified Dimensions

We plug the zero-mode contribution (8.4.4) in the nonlinear worldsheet action (3.7.6), i.e.

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[\left(g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X) \right] \qquad [8.202]$$

We fix the gauge $g^{ab} = \delta^{ab}$ and ignore the dilaton term because we are considering the low energy action, i.e the lowest order in α'. We focus on the compactified dimensions only and find for the Lagrangian

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \Big[G_{mn} \big(\partial_1 X^m \partial_1 X^n + \partial_2 X^m \partial_2 X^n \big) + 2iB_{mn} \partial_1 X^m \partial_2 X^n + \cdots \Big] \qquad [8.203]$$

Now

$$\begin{aligned} \partial_1 X^m &= \partial_1 \big[x^m(\sigma^2) + w^m R\sigma^1 \big] = w^m R \\ \partial_2 X^m &= \partial_2 \big[x^m(\sigma^2) + w^m R\sigma^1 \big] = \dot{x}^m \end{aligned} \qquad [8.204]$$

and thus

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi\alpha'} \Big[G_{mn} \big(w^m R w^n R + \dot{x}^m \dot{x}^n \big) + 2iB_{mn} w^m R \dot{x}^n \Big] \\ &= \frac{1}{2\pi} \left[\frac{1}{2\alpha'} G_{mn} \big(w^m R w^n R + \dot{x}^m \dot{x}^n \big) - \frac{i}{\alpha'} B_{mn} \dot{x}^m w^n R \right] \end{aligned} \qquad [8.205]$$

We have interchanged the dummy indices m and n in the last term to obtain a minus sign. This is in line with the correction on Joe's errata page. I am not sure what happens with the factor $1/2\pi$ but it is not relevant for our purposes.

8.51 p 249: Eq. (8.4.6) The Canonical Momenta of the Zero Modes

We have by definition

$$p_m = \frac{\partial \mathcal{L}}{\partial(\partial_{\sigma^2_{(M)}} x^m)} = \frac{\partial \mathcal{L}}{\partial(\partial_{i\sigma^2} x^m)} = -\frac{\partial \mathcal{L}}{i\partial \dot{x}^m} \qquad [8.206]$$

where $\sigma^2_{(M)} = \sigma^0$ is in Minkowski signature and $\sigma^2_{(M)} = i\sigma^2$ is the Wick rotation. With $v^m = i\dot{x}^m$ we thus have

$$\begin{aligned} p_m &= -\frac{\partial \mathcal{L}}{\partial v^m} = -\frac{1}{i}\left(\frac{1}{2\alpha'}G_{kn}2\delta^k_m\dot{x}^n - \frac{i}{\alpha'}B_{kn}\delta^k_m w^n R\right) \\ &= \frac{1}{\alpha'}\left(iG_{mn}\dot{x}^n + B_{mn}w^nR\right) = \frac{1}{\alpha'}\left(G_{mn}v^n + B_{mn}w^nR\right) \end{aligned} \quad [8.207]$$

8.52 p 250: Eq. (8.4.7) The Quantisation of the Canonical Momenta of the Zero Mode

Using $p_m = n_m/R$ with $n_m \in \mathbb{Z}$ in (8.4.6) we have

$$\frac{n_m}{R} = \frac{1}{\alpha'}\left[v_m + B_{mn}w^nR\right] \quad [8.208]$$

with $v_m = G_{mn}v^n$. From this it follows that

$$v_m = \alpha'\frac{n_m}{R} - B_{mn}w^nR \quad [8.209]$$

8.53 p 250: Eq. (8.4.8) The Zero Mode Contribution to the Hamiltonian

The Hamiltonian $\mathcal{H}$ of a system with generalised coordinates q^i and canonical momenta p_i for a given Lagrangian is $\mathcal{H} = \sum_i \dot{q}^i p_i - \mathcal{L}$. As the canonical pair of coordinates is $(-v^m, p_m)$, see (8.4.6), we thus find for the contribution of the zero modes to the Hamiltonian

$$\begin{aligned} \mathcal{H} &= -\left(-\dot{v}^m p_m\right) + \mathcal{L} \\ &= \dot{v}^m\frac{1}{\alpha'}\left[G_{mn}v^n + B_{mn}w^nR\right] + \left[\frac{1}{2\alpha'}G_{mn}\big(w^mRw^nR + \dot{x}^m\dot{x}^n\big) - \frac{i}{\alpha'}B_{mn}\dot{x}^m w^nR\right] \\ &= \frac{1}{\alpha'}\Big[v^mG_{mn}v^n + v^mB_{mn}w^nR + \frac{1}{2}G_{mn}w^mw^nR^2 + \frac{1}{2}G_{mn}(-iv^m)(-iv^n) \\ &\qquad - iB_{mn}(-iv^m)w^nR\Big] \\ &= \frac{1}{2\alpha'}G_{mn}(v^mv^n + w^mw^nR^2) \end{aligned} \quad [8.210]$$

There seems to be an overall minus sign that I can't trace, but this form is positive definite, so it must be correct.

8.54 p 250: Eq. (8.4.9) The Compactified Closed String Mass Formula

From (8.4.9b) we have

$$\begin{aligned} v_L^m v_L^n + v_R^m v_R^n &= (v^m + w^m R)(v^n + w^n R) + (v^m - w^m R)(v^n - w^n R) \\ &= 2(v^m v^n + w^m w^n R^2) \end{aligned} \qquad [8.211]$$

The mass formula (4.3.31) and (4.3.32) gives $\alpha' m^2/4 = L_0 + N - 1 = \tilde{L}_0 + \tilde{N} - 1$ so that $\alpha' m^2/2 = L_0 + \tilde{L}_0 + N + \tilde{N} - 2$. Now $L_0 + \tilde{L}_0$ is nothing but the Hamiltonian so we have

$$\begin{aligned} \frac{\alpha'}{2} m^2 &= H + N + \tilde{N} - 2 = \frac{1}{2\alpha'} G_{mn}(v^m v^n + w^m w^n R^2) + N + \tilde{N} - 2 \\ &= \frac{1}{2\alpha'} G_{mn} \frac{1}{2}(v_L^m v_L^n + v_R^m v_R^n) + N + \tilde{N} - 2 \end{aligned} \qquad [8.212]$$

or

$$m^2 = \frac{1}{2\alpha'^2} G_{mn}(v_L^m v_L^n + v_R^m v_R^n) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \qquad [8.213]$$

What is the interpretation of this? Let us recall the case of one compactified dimension where the momenta split in a left- and a right-moving part, (8.2.7), i.e. $p_{L,R} = n/R \pm wR/\alpha'$. We now have a momentum

$$\begin{aligned} v_{L,R}^m &= v^m \pm w^m R = \alpha' \frac{n_m}{R} - B_{mn} w^n R \pm w^m R \\ &= \alpha' \left(\frac{n_m}{R} \pm \frac{w^m R}{\alpha'} \right) - B_{mn} w^n R = \alpha' p_{L,R}^{[B=0]} - B_{mn} w^n R \end{aligned} \qquad [8.214]$$

We see that the appearance of the the spacetime scalars B_{mn} shifts the momenta of states with non-zero winding numbers, and accordingly also the mass of these states.

8.55 p 250: Eq. (8.4.10) The $L_0 - \tilde{L}_0$ Constraint for the Compactified Closed String

The second constraint from (4.3.31) and (4.3.32) is that $L_0 + N = \tilde{L}_0 + \tilde{N}$. This gives with $L_0 = \alpha'(p_L^2 + m^2)/4$ and $\tilde{L}_0 = \alpha'(p_R^2 + m^2)/4$

$$\begin{aligned} 0 &= \alpha'(p_L^2 + m^2)/4 + N - \alpha'(p_R^2 + m^2)/4 - \tilde{N} \\ &= \frac{1}{4\alpha'}(v_L^2 - v_R^2) + N - \tilde{N} \end{aligned} \qquad [8.215]$$

or

$$v_L^2 - v_R^2 + 4\alpha'(N - \tilde{N}) = 0 \qquad [8.216]$$

As $v_{L,R}^2 = G_{mn} v_{L,R}^m v_{L,R}^n$ this is the first line of (8.4.10). To get the second line, we work out

$$\begin{aligned} v_L^2 - v_R^2 &= (v_m + w_m R)^2 - (v_m - w_m R)^2 = 4 v_m w^m R \\ &= 4\left(\alpha' \frac{n_m}{R} - B_{mn} w^n R\right) w^m R = 4\alpha' n_m w^m \end{aligned} \tag{8.217}$$

In the last line we have used (8.4.7) and the antisymmetry of B_{mn}. The constraints [8.216] thus becomes

$$0 = 4\alpha' n_m w^m + 4\alpha'(N - \tilde{N}) = 4\alpha'(n_m w^m + N - \tilde{N}) \tag{8.218}$$

8.56 p 250: Eq. (8.4.12) The World-Sheet Action for the B_{mn} Field on the Torus

The Lagrangian contribution in this case is, using (8.4.3)

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi\alpha'} i\epsilon^{ab} B_{mn} \partial_a X^m \partial_b X^n + \cdots = \frac{1}{4\pi\alpha'} i2B_{mn} \partial_1 X^m \partial_2 X^n + \cdots \\ &= \frac{1}{2\pi} \frac{iB_{mn}}{\alpha'} w_1^m R w_2^n R + \cdots = \frac{1}{2\pi} i b_{mn} w_1^m w_2^n + \cdots \end{aligned} \tag{8.219}$$

The world-sheet action is obtained by integrating over $d^2\sigma$ with both σ^1 and σ^2 ranging from 0 to 2π. In the cae of constant B_{mn}, which is what we are considering, this gives a factor $(2\pi)^2$ and thus

$$\mathcal{S} = 2\pi i b_{mn} w_1^m w_2^n + \cdots \tag{8.220}$$

We now estimate the partition function from the canonical approach and show that we recover the same phase. So we need to work out $Z_k = (q\bar{q})^{-1/24} \mathrm{tr}\ q^{L_0} \bar{q}^{\tilde{L}_0}$. For the compactified spacetime dimensions we have

$$\begin{aligned} L_0 &= \frac{1}{4\alpha'} v_L^2 + \sum_{k=1}^{\infty} G_{mn} \alpha_{-k}^m \alpha_k^n \\ \tilde{L}_0 &= \frac{1}{4\alpha'} v_R^2 + \sum_{k=1}^{\infty} G_{mn} \tilde{\alpha}_{-k}^m \tilde{\alpha}_k^n \end{aligned} \tag{8.221}$$

and we can repeat the calculation of (8.2.9) for more than one compact dimension

$$Z_k = (q\bar{q})^{-1/24} \mathrm{tr}\ q^{\frac{1}{4\alpha'} v_L^2 + \sum_{k=1}^{\infty} G_{mn} \alpha_{-k}^m \alpha_k^n} q^{\frac{1}{4\alpha'} v_R^2 + \sum_{k=1}^{\infty} G_{mn} \tilde{\alpha}_{-k}^m \tilde{\alpha}_k^n} \tag{8.222}$$

The oscillator part and the $(q\bar{q})^{-1/24}$ gives as usual the Dedekind function for each compactified dimension. Thus

$$\begin{aligned} Z_k &= |\eta(\tau)|^{-2k} \mathrm{tr}\, q^{\frac{1}{4\alpha'}v_L^2} q^{\frac{1}{4\alpha'}v_R^2} \\ &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{i\pi\tau\frac{1}{2\alpha'}v_L^2} e^{-i\pi\tau\frac{1}{2\alpha'}v_R^2} \\ &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{\frac{\pi}{2\alpha'}(i\tau_1-\tau_2)v_L^2} e^{\frac{\pi}{2\alpha'}(-i\tau_1-\tau_2)v_R^2} \\ &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{\frac{i\pi\tau_1}{2\alpha'}(v_L^2-v_R^2)} e^{-\frac{\pi\tau_2}{2\alpha'}(v_L^2+v_R^2)} \end{aligned} \qquad [8.223]$$

We already found that $v_L^2 - v_R^2 = 4\alpha' n_m w^m$ in [8.217]. We also have

$$v_L^2 + v_R^2 = (v_m + w_m R)^2 + (v_m - w_m R)^2 = 2(v_m^2 + w_m^2 R^2) \qquad [8.224]$$

Therefore

$$Z_k = |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{2\pi i\tau_1 n_m w^m} e^{-\frac{\pi\tau_2}{\alpha'}(v_m^2+w_m^2R^2)} \qquad [8.225]$$

We now have

$$\begin{aligned} v_m^2 &= G^{mn} v_m v_n = G^{mn}\left(\frac{\alpha' n_m}{R} - B_{mk}w^k R\right)\left(\frac{\alpha' n_n}{R} - B_{n\ell}w^\ell R\right) \\ &= \frac{\alpha'^2}{R^2} n_m n^m - 2\alpha' G^{mn} B_{n\ell} w^\ell n_m + R^2 G^{mn} B_{mk} B_{n\ell} w^k w^\ell \end{aligned} \qquad [8.226]$$

and thus

$$\begin{aligned} Z_k &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{2\pi i\tau_1 n_m w^m} e^{-\frac{\pi\tau_2}{\alpha'}(\frac{\alpha'^2}{R^2}n_m n^m - 2\alpha' G^{mn}B_{n\ell}w^\ell n_m + R^2 G^{mn}B_{mk}B_{n\ell}w^k w^\ell + w_m^2 R^2)} \\ &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{-\pi\tau_2\frac{\alpha'}{R^2}n_m n^m + 2\pi i(\tau_1 w^m - i\tau_2 G^{mn}B_{n\ell}w^\ell)n_m - \frac{\pi\tau_2 R^2}{\alpha'}(G^{mn}B_{mk}B_{n\ell}w^k w^\ell + w_m^2)} \\ &= |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{-\pi\tau_2\frac{\alpha'}{R^2}n_m n^m + 2\pi i\left(\tau_1\delta^m_\ell - i\tau_2 G^{mn}B_{n\ell}\right)w^\ell n_m} \\ &\qquad\qquad \times e^{-\frac{\pi\tau_2 R^2}{\alpha'}(G^{mn}B_{mk}B_{n\ell} + G_{k\ell})w^k w^\ell} \end{aligned} \qquad [8.227]$$

As a quick check, let us set $k = 1$ and hence also $B_{mn} = 0$. Then

$$Z_1 = |\eta(\tau)|^{-2} \sum_{n,w\in\mathbb{Z}} e^{-\pi\tau_2\frac{\alpha'}{R^2}n^2 + 2\pi i\tau_1 wn - \frac{\pi\tau_2 R^2}{\alpha'}w^2} \qquad [8.228]$$

and we recover the case of a single compactified dimension [8.87] as we should indeed.

In order to show that we recover from the path integral the partition function for the shifted spectrum, we need a generalisation of the Poisson resummation formula (8.2.10) which is

$$\sum_{n\in\mathbb{Z}^k} e^{-\pi S^{\alpha\beta}n_\alpha n_\beta+2\pi i b^\alpha n_\beta} = S^{1/2}\sum_{n\in\mathbb{Z}^k} e^{-\pi S_{\alpha\beta}(n^\alpha-b^\alpha)(n^\beta-b^\beta)} \qquad [8.229]$$

Here $S^{\alpha\beta}$ is a symmetric invertible matrix with inverse $S_{\alpha\beta}$ and $S=\det S_{\alpha\beta}$. We will not prove this, but just check the $k=1$ case, when it becomes

$$\sum e^{-\pi S n^2+2\pi i b n} = S^{-1/2}\sum e^{-\pi(m-b)^2/S} \qquad [8.230]$$

which is precisely (8.2.10). We now apply this to [8.227]. We have

$$\begin{aligned} S^{mn} &= \frac{\alpha'}{R^2}\tau_2 G^{mn} \\ b^m &= \left(\tau_1\delta^m_\ell - i\tau_2 G^{mn}B_{n\ell}\right)w^\ell \end{aligned} \qquad [8.231]$$

S^{mn} is obviously symmetric and invertible with

$$S_{mn} = \frac{R^2}{\alpha'\tau_2}G_{mn} \qquad [8.232]$$

and

$$S = \det S_{mn} = \left(\frac{R^2}{\alpha'\tau_2}\right)^k \det G_{mn} \qquad [8.233]$$

Applying the Poisson resummation formula then gives

$$\begin{aligned} Z_k = |\eta(\tau)|^{-2k}\,\frac{R^k}{(\alpha'\tau_2)^{k/2}}\det G_{mn} &\sum_{n,w\in\mathbb{Z}^k} \\ &e^{-\pi\frac{R^2}{\alpha'\tau_2}G_{mn}\left[n^m-\left(\tau_1\delta^m_\ell-i\tau_2G^{mk}B_{k\ell}\right)w^\ell\right]\left[n^n-\left(\tau_1\delta^n_p-i\tau_2G^{nr}B_{rp}\right)w^p\right]} \\ &\times e^{-\pi\frac{\tau_2R^2}{\alpha'}(G^{mn}B_{mk}B_{n\ell}+G_{k\ell})w^kw^\ell} \end{aligned} \qquad [8.234]$$

Let us work out the argument of the exponential

$$\begin{aligned} \mathfrak{a} = -\frac{\pi R^2}{\alpha'\tau_2}\Big[&G_{mn}(n^mn^n-\tau_1n^m\delta^n_pw^p+i\tau_2G^{nr}B_{rp}w^pn^m \\ &-\tau_1\delta^m_\ell w^\ell n^n+\tau_1^2\delta^m_\ell\delta^n_pw^\ell w^p-i\tau_1\tau_2\delta^m_\ell G^{nr}B_{rp}w^\ell w^p \\ &+i\tau_2G^{mk}B_{k\ell}w^\ell n^n-i\tau_1\tau_2G^{mk}B_{k\ell}\delta^n_pw^\ell w^p-\tau_2^2G^{mk}B_{k\ell}G^{nr}B_{rp}w^\ell w^p \\ &+\tau_2^2G^{mn}B_{mk}B_{n\ell}w^kw^\ell+\tau_2^2G_{k\ell}w^kw^\ell\Big] \end{aligned} \qquad [8.235]$$

Denoting $a \cdot b = G_{mn} a^m b^n$ and $a^2 = a \cdot a$ this gives

$$\begin{aligned}
\mathfrak{a} =& -\frac{\pi R^2}{\alpha' \tau_2}\Big(n^2 - 2\tau_1 n \cdot w + \tau_1^2 w^2 + 2i\tau_2 G_{mn} G^{nr} B_{rp} w^p n^m - 2i\tau_1\tau_2 G_{mn}\delta^m_\ell G^{nr} B_{rp} w^\ell w^p \\
&\qquad - \tau_2^2 G_{mn} G^{mk} B_{k\ell} G^{nr} B_{rp} w^\ell w^p + \tau_2^2 G^{mn} B_{mk} B_{n\ell} w^k w^\ell + \tau_2^2 w^2\Big) \\
=& -\frac{\pi R^2}{\alpha' \tau_2}\Big[n^2 - 2\tau_1 n \cdot w + (\tau_1^2 + \tau_2^2) w^2 + 2i\tau_2 B_{mp} w^p n^m + 2\tau_1\tau_2 B_{\ell p} w^\ell w^p \\
&\qquad + \tau_2^2(-B_{n\ell} G^{nr} B_{rp} w^\ell w^p + G^{mn} B_{mk} B_{n\ell} w^k w^\ell)\Big] \\
=& -\frac{\pi R^2}{\alpha' \tau_2}\Big[n^2 - 2\tau_1 n \cdot w + (\tau_1^2 + \tau_2^2) w^2 + 2i\tau_2 B_{mp} w^p n^m\Big]
\end{aligned} \qquad [8.236]$$

We use [8.93]. i.e. $|m - w\tau|^2 = m^2 + w^2(\tau_1^2 + \tau_2^2) - 2mw\tau_1$ and $B_{mn} = \alpha' b_{mn}/R^2$ to get

$$\begin{aligned}
\mathfrak{a} =& -\frac{\pi R^2}{\alpha' \tau_2}\Big(|n - w\tau|^2 + 2i\tau_2 \frac{\alpha'}{R^2} b_{mp} w^p n^m\Big) \\
=& -\frac{\pi R^2}{\alpha' \tau_2}|n - w\tau|^2 - 2\pi i b_{mp} w^p n^m
\end{aligned} \qquad [8.237]$$

and so our partition function becomes

$$Z_k = \frac{R^k}{(\alpha'\tau_2)^{k/2}} \det G_{mn}\, |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{-\frac{\pi R^2}{\alpha'\tau_2}|n-w\tau|^2 - 2\pi i b_{mp} w^p n^m} \qquad [8.238]$$

Finally, using the partition function for the uncompactified dimension (7.2.9), i.e. $|\eta(\tau)|^{-2} = (4\pi^2\alpha'\tau_2)^{1/2} Z_X(\tau)$ we find

$$\begin{aligned}
Z_k =& \frac{R^k}{(\alpha'\tau_2)^{k/2}} \det G_{mn}\ \Big[(4\pi^2\alpha'\tau_2)^{1/2} Z_X(\tau)\Big]^k \sum_{n,w\in\mathbb{Z}^k} e^{-\frac{\pi R^2}{\alpha'\tau_2}|n-w\tau|^2 - 2\pi i b_{mp} w^p n^m} \\
=& (2\pi R)^k \det G_{mn}\, Z_X(\tau)^k \sum_{n,w\in\mathbb{Z}^k} e^{-\frac{\pi R^2}{\alpha'\tau_2}|n-w\tau|^2 - 2\pi i b_{mp} w^p n^m}
\end{aligned} \qquad [8.239]$$

and we indeed see that the the calculation of the partition function from the canonical approach also gives the phase (8.4.12), i.e $2\pi i b_{mp} w^p n^m$ that we see in the path integral.

8.57 p 250: Eq. (8.4.13) Introducing the Spacetime Tetrad

If we write $G_{m_n} = e^r_m e^r_n$, with $r = 1, \cdots, k$, then the corresponding term in the worldsheet Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, e^r_m e^r_n \partial_a X^m \partial_b X^n + \cdots = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_a X^r \partial_b X^r \qquad [8.240]$$

where $X^r = e^r_m X^m$. Here e^r_m is the spacetime tetrad; it lives in the tangent space of the spacetime coordinates X^m and so does not depend on the worldsheet coordinate, i.e $\partial_a e^r_m = 0$ by construction. In terms of the X^r the action is now that of the free scalars and so we can use that, including their standard OPE.

8.58 p 250: Eq. (8.4.14) The Momentum for the Vertex Operator with $X^r = e^r_m X^m$

We can write a vertex operator for the left-moving field X^m_L as

$$e^{ip_L\cdot X_L} = e^{ip_{Lm}X^m_L} = e^{ip_{Lm}e^m_r X^r_L} = e^{i(v_{Lm}/\alpha')e^m_r X^r_L} = e^{ik_{Lr}X^r_L} \tag{8.241}$$

where

$$k_{Lr} = \frac{e^m_r v_{Lm}}{\alpha'} \tag{8.242}$$

Here we have used e^m_r as the inverse of e^r_m, i.e. $e^m_r e^r_n = \delta^m_n$ and have used [8.214], i.e. $v_L = \alpha' p_L$. A similar relation holds, of course for the right-moving part.

8.59 p 251: Eq. (8.4.15) The Mass Shell Conditions for the Vertex Operator with $X^r = e^r_m X^m$

For the mass-shell conditions, we just rewrite (8.4.9a) in terms of the "tetradic" spacetime coordinates

$$\begin{aligned} m^2 &= \frac{1}{2\alpha'^2}G_{mn}(v^m_L v^n_L + v^m_R v^n_R) + \frac{2}{\alpha'}(N+\tilde{N}-2) \\ &= \frac{1}{2\alpha'^2}e^r_m e^r_n(\alpha' e^m_s k^s_L \alpha' e^n_t k^t_L + \alpha' e^m_s k^s_R \alpha' e^n_t k^t_R) + \frac{2}{\alpha'}(N+\tilde{N}-2) \\ &= \frac{1}{2}e^r_m e^r_n e^m_s e^n_t(k^s_L k^t_L + k^s_R k^t_R) + \frac{2}{\alpha'}(N+\tilde{N}-2) \\ &= \frac{1}{2}\delta^r \delta^r_t(k^s_L k^t_L + k^s_R k^t_R) + \frac{2}{\alpha'}(N+\tilde{N}-2) \\ &= \frac{1}{2}(k^r_L k^r_L + k^r_R k^r_R) + \frac{2}{\alpha'}(N+\tilde{N}-2) \end{aligned} \tag{8.243}$$

We have been a bit sloppy with the location of the indices r, s, t but these live in a tangent space and so the metric is flat and indices can be raised and lowered at will.

For the second condition we use [8.216]

$$\begin{aligned} 0 &= v_L^2 - v_R^2 + 4\alpha'(N-\tilde{N}) = G_{mn} v_L^m v_L^n - G_{mn} v_R^m v_R^n + 4\alpha'(N-\tilde{N}) \\ &= e_m^r e_n^r \alpha' e_s^m k_L^s \alpha' e_t^n k_L^t - e_m^r e_n^r \alpha' e_s^m k_R^s \alpha' e_t^n k_R^t + 4\alpha'(N-\tilde{N}) \\ &= \alpha'^2 e_m^r e_n^r e_s^m e_t^n (k_L^s k_L^t - k_R^s k_R^t) + 4\alpha'(N-\tilde{N}) \\ &= \alpha'^2 \delta_s^r \delta_t^r (k_L^s k_L^t - k_R^s k_R^t) + 4\alpha'(N-\tilde{N}) \\ &= \alpha'^2 (k_L^r k_L^r - k_R^r k_R^r) + 4\alpha'(N-\tilde{N}) \end{aligned} \qquad [8.244]$$

8.60 p 251: Eq. (8.4.16) The OPE of Vertex Operators for Winding States

$$\begin{aligned} &: e^{ik_L\cdot X_L(z) + ik_R\cdot X_R(z)} :: e^{ik'_L\cdot X_L(0) + ik'_R\cdot X_R(0)} : \\ &\qquad = z^{\alpha' k_L\cdot k'_L/2} \bar{z}^{\alpha' k_R\cdot k'_R/2} : e^{i(k_L+k'_L)\cdot X_L(z) + i(k_R+k'_R)\cdot X_R(z)} : \\ &\qquad = z^{\ell_L\cdot \ell'_L} \bar{z}^{\ell_R\cdot \ell'_R} : e^{i(k_L+k'_L)\cdot X_L(z) + i(k_R+k'_R)\cdot X_R(z)} : \end{aligned} \qquad [8.245]$$

8.61 p 251: Eq. (8.4.17) The Phase for One Vertex Operator Encircling Another One

This was already worked out in (8.2.20). The net phase when z_1 circles z_2 is

$$e^{2\pi i\varphi} = e^{\pi i\alpha'(k_L k'_L - k_R k'_R)} = e^{\pi i\alpha' \frac{2}{\alpha'}(\ell_L\ell'_L - \ell_R\ell'_R)} = e^{2\pi i(\ell_L\ell'_L - \ell_R\ell'_R)} \qquad [8.246]$$

(8.4.17), i.e. $\ell \circ \ell' \in \mathbb{Z}$ is the requirement that this OPE is single-valued.

8.62 p 251: Eq. (8.4.19) The Condition $\ell \circ \ell \in 2\mathbb{Z}$

The requirement that $L_0 - \tilde{L}_o \in \mathbb{Z}$ was derived in (7.2.29). We now use $L_0 = \alpha'(p_L^2 + m^2)/4$, $v_L = \alpha' p_L$, (8.4.14) and $\ell_L = (\alpha'/2)^{1/2} k_\ell$, and the same for the right-handed components

$$\alpha'(p_L^2 - p_R^2) = \frac{1}{\alpha'}(v_L^2 - v_R^2) = \frac{1}{\alpha'}\alpha'^2(k_L^2 - k_R^2) = \alpha'\frac{2}{\alpha'}(\ell_L^2 - \ell_R^2) = 2(\ell_L^2 - \ell_R^2) \qquad [8.247]$$

So the condition $L_0 - \tilde{L}_o \in \mathbb{Z}$ becomes

$$2(\ell_L^2 - \ell_R^2)/4 \in \mathbb{Z} \quad \Rightarrow \quad \ell_L^2 - \ell_R^2 = \ell \circ \ell \in 2\mathbb{Z} \qquad [8.248]$$

8.63 p 251: Eq. (8.4.20) $\tau \longrightarrow \tau + 1$ Implies Single-Valuedness

We have

$$\begin{aligned} &(\ell + \ell') \circ (\ell + \ell') - \ell \circ \ell - \ell' \circ \ell' \\ =&(\ell_L + \ell'_L) \cdot (\ell_L + \ell'_L) - (\ell_R + \ell'_R) \cdot (\ell_R + \ell'_R) - \ell_L \cdot \ell_L - \ell'_L \cdot \ell'_L - \ell_R \cdot \ell_R - \ell'_R \cdot \ell'_R \\ =&2\ell_L \cdot \ell'_L - 2\ell_R \cdot \ell_R = 2\ell \circ \ell' \end{aligned} \qquad [8.249]$$

Modular invariance of the partition function under $\tau \longrightarrow \tau + 1$ implies, according to (8.4.19), that $\ell\circ\ell \in 2\mathbb{Z}$ for all $\ell \in \Gamma$. But then we also have that $(\ell+\ell')\circ(\ell+\ell')-\ell\circ\ell-\ell'\circ\ell' \in 2\mathbb{Z}$ and thus that $2\ell\circ\ell' \in 2\mathbb{Z}$ and hence $\ell\circ\ell' \in \mathbb{Z}$. Wich is the condition for single-valuedness of the OPE of two vertex operators (8.4.17).

8.64 p 251: Eq. (8.4.21) The Partition Function for Compactification on a Lattice

The partition function for compactification of k dimensions is given by [8.223]

$$Z_k = |\eta(\tau)|^{-2k} \sum_{n,w\in\mathbb{Z}^k} e^{\frac{i\pi\tau_1}{2\alpha'}(v_L^2 - v_R^2)} e^{-\frac{\pi\tau_2}{2\alpha'}(v_L^2 + v_R^2)} \qquad [8.250]$$

From (8.4.14) and the definition $\ell_L = (\alpha'/2)^{1/2} k_L$ we have $v_L^2 = \alpha'^2 k_L^2 = 2\alpha' \ell_L^2$. Thus

$$\begin{aligned} Z_\Gamma &= |\eta(\tau)|^{-2k} \sum_{\ell\in\Gamma} e^{\frac{i\pi\tau_1}{2\alpha'}(2\alpha'\ell_L^2 - 2\alpha'\ell_R^2)} e^{-\frac{\pi\tau_2}{2\alpha'}(2\alpha'\ell_L^2 - 2\alpha'\ell_R^2)} \\ &= |\eta(\tau)|^{-2k} \sum_{\ell\in\Gamma} e^{i\pi\tau_1(\ell_L^2 - \ell_R^2) - \pi\tau_2(\ell_L^2 + \ell_R^2)} \\ &= |\eta(\tau)|^{-2k} \sum_{\ell\in\Gamma} e^{i\pi(\tau_1 + i\tau_2)\ell_L^2 - i\pi(\tau_1 - i\tau_2)\ell_R^2} \\ &= |\eta(\tau)|^{-2k} \sum_{\ell\in\Gamma} e^{i\pi\tau\ell_L^2 - i\pi\bar{\tau}\ell_R^2} \end{aligned} \qquad [8.251]$$

8.65 p 252: Eq. (8.4.22) The Delta Function Sum over the Lattice

Consider the function

$$f(\ell) = \sum_{k\in\Gamma^*} e^{2\pi i k\circ\ell} \qquad [8.252]$$

Let us first consider the case where $\ell \in \Gamma$. From the definition of the dual lattice Γ^* we know that $\ell \circ k = n \in \mathbb{Z}$ for any $k \in \Gamma^*$. Note that n is here an integer number, not a point on the lattice or the dual lattice. Thus, in this case,

$$f(\ell) = \sum_{k\in\Gamma^*} e^{2\pi i n} = \sum_{k\in\Gamma^*} 1 \qquad [8.253]$$

Now if $\ell \notin \Gamma$, then $\ell \circ k \notin \mathbb{Z}$ for any $k \in \Gamma^*$. If $\ell \circ k$ would be an integer, then ℓ would be in Γ^*, which is a contradiction. We then have a non-vanishing phase and the sum averages to zero. We thus have that $f(\ell)$ is non-zero if and only if ℓ is in Γ. We can write this more precisely as

$$f(\ell) = \sum_{k\in\Gamma^*} e^{2\pi i k\circ\ell} = \mathcal{N}\sum_{k\in\Gamma} \delta(\ell - k) \qquad [8.254]$$

Indeed $\sum_{k\in\Gamma} \delta(\ell - k)$ is zero unless ℓ is a point in Γ. Here $\mathcal{N}$ is a normalisation constant, which turns out to be the volume of a unit cell of Γ.

8.66 p 252: Eq. (8.4.23) The Change in Partition Function under $\tau \longrightarrow -1/\tau$

Let us first consider the RHS of the first line of (8.4.23). On the one hand, we can write this, using (8.4.22), as

$$\begin{aligned} RHS &= |\eta(\tau)|^{-2k} \int d^{2k}\ell \left(V_\Gamma^{-1} \sum_{\ell''\in\Gamma^*} e^{2\pi i \ell''\circ\ell} \right) e^{i\pi\tau\ell_L^2 - i\pi\bar\tau\ell_R^2} \\ &= |\eta(\tau)|^{-2k} \int d^{2k}\ell \sum_{\ell'\in\Gamma} \delta(\ell - \ell') e^{i\pi\tau\ell_L^2 - i\pi\bar\tau\ell_R^2} \\ &= |\eta(\tau)|^{-2k} \sum_{\ell'\in\Gamma} e^{i\pi\tau{\ell'_L}^2 - i\pi\bar\tau{\ell'_R}^2} = Z_\Gamma(\tau) \end{aligned} \qquad [8.255]$$

On the other hand we can perform the ℓ integration first

$$\begin{aligned} Z_\Gamma(\tau) &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} \sum_{\ell''\in\Gamma^*} \int d^{2k}\ell\, e^{2\pi i\ell''\circ\ell} e^{i\pi\tau\ell_L^2 - i\pi\bar\tau\ell_R^2} \\ &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} \sum_{\ell''\in\Gamma^*} \int d^{2k}\ell\, e^{2\pi i(\ell''_L\ell_L - \ell''_R\ell_R) + i\pi\tau\ell_L^2 - i\pi\bar\tau\ell_R^2} \\ &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} \sum_{\ell''\in\Gamma^*} \int d^k\ell_L\, e^{\pi i(\tau\ell_L^2 + 2\ell''_L\ell_L)} \int d^k\ell_R\, e^{-\pi i(\bar\tau\ell_R^2 + 2\ell''_R\ell_R)} \end{aligned} \qquad [8.256]$$

We now use the Gaussian integral $\int_{-\infty}^{+\infty} e^{-ax^2+bx} = (\pi/a)^{1/2} e^{b^2/4a}$:

$$\begin{aligned} Z_\Gamma(\tau) &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} \sum_{\ell'' \in \Gamma^*} \left(\frac{\pi}{-\pi i \tau}\right)^{k/2} e^{-4\pi^2 \ell_L''^2/(-4\pi i \tau)} \left(\frac{\pi}{\pi i \bar\tau}\right)^{k/2} e^{-4\pi^2 \ell_R''^2/(4\pi i \bar\tau)} \\ &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} \left(\frac{1}{-i\tau i\bar\tau}\right)^{k/2} \sum_{\ell'' \in \Gamma^*} e^{-\pi i \ell_L''^2/\tau + \pi i \ell_R''^2/\bar\tau} \\ &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} (\tau\bar\tau)^{-k/2} Z_{\Gamma^*}(-1/\tau) |\eta(-1/\tau)|^{2k} \end{aligned} \quad [8.257]$$

Using (7.2.44b), i.e. $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$ we get

$$\begin{aligned} Z_\Gamma(\tau) &= |\eta(\tau)|^{-2k} V_\Gamma^{-1} (\tau\bar\tau)^{-k/2} Z_{\Gamma^*}(-1/\tau) \left|(-i\tau)^{1/2}\eta(\tau)\right|^{2k} \\ &= V_\Gamma^{-1} Z_{\Gamma^*}(-1/\tau) \end{aligned} \quad [8.258]$$

8.67 p 252: Eq. (8.4.24) The Lattice Must be Self-Dual

If $\Gamma = \Gamma^*$ then (8.4.23) becomes

$$Z_\Gamma(\tau) = V_\Gamma^{-1} Z_\Gamma(-1/\tau) \quad [8.259]$$

From $V_\Gamma = V_{\Gamma^*}^{-1}$ we have for a self-dual lattice that $V_\Gamma^2 = 1$, and so $V_\Gamma = 1$. Therefore, for a self-dual lattice we have indeed modular invariance $Z_\Gamma(\tau) = Z_\Gamma(-1/\tau)$. Joe claims that "a little thought shows that it is also necessary if modular invariance is to hold for all τ". Unfortunately that little thought has escaped me so far.

8.68 p 252: Eq. (8.4.25) The Lorentz Invariance of Even Self-Dual Lattices

We need to show that if Γ is and even self-dual lattice then so is $\Gamma' = \Lambda\Gamma$ where Λ is an $O(k,k;\mathbb{R})$ rotation, i.e. it satisfies $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$, with η the (k,k) Minkowski metric $\eta = (+1,\cdots,+1,-1,\cdots,-1)$.

If we write

$$\Lambda = \begin{pmatrix} \Lambda_{LL} & \Lambda_{LR} \\ \Lambda_{RL} & \Lambda_{RR} \end{pmatrix} \quad [8.260]$$

then the fact that $\Lambda \in O(k,k;\mathbb{R})$ means that

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} \Lambda_{LL} & \Lambda_{RL} \\ \Lambda_{LR} & \Lambda_{RR} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Lambda_{LL} & \Lambda_{LR} \\ \Lambda_{RL} & \Lambda_{RR} \end{pmatrix} \\ &= \begin{pmatrix} \Lambda_{LL}^2 - \Lambda_{RL}^2 & \Lambda_{LL}\Lambda_{LR} - \Lambda_{RL}\Lambda_{RR} \\ \Lambda_{LR}\Lambda_{LL} - \Lambda_{RR}\Lambda_{RL} & \Lambda_{LR}^2 - \Lambda_{RR}^2 \end{pmatrix} \end{aligned} \quad [8.261]$$

This gives three equations

$$\begin{aligned}1 &= \Lambda_{LL}^2 - \Lambda_{RL}^2\\ 1 &= \Lambda_{RR}^2 - \Lambda_{LR}^2\\ 0 &= \Lambda_{LL}\Lambda_{LR} - \Lambda_{RR}\Lambda_{RL}\end{aligned} \qquad [8.262]$$

We now have $\ell' = \Lambda\ell$ or

$$\begin{pmatrix}\ell'_L\\ \ell'_R\end{pmatrix} = \begin{pmatrix}\Lambda_{LL} & \Lambda_{LR}\\ \Lambda_{RL} & \Lambda_{RR}\end{pmatrix}\begin{pmatrix}\ell_L\\ \ell_R\end{pmatrix} = \begin{pmatrix}\Lambda_{LL}\ell_L + \Lambda_{LR}\ell_R\\ \Lambda_{RL}\ell_L + \Lambda_{RR}\ell_R\end{pmatrix} \qquad [8.263]$$

Let us now check that if Γ is even, i.e. $\ell \circ \ell \in 2\mathbb{Z}$ then also $\Lambda\Gamma$ is even, i.e. $\ell' \circ \ell' \in 2\mathbb{Z}$. Indeed

$$\begin{aligned}\ell' \circ \ell' &= \left(\ell'_L\right)^2 - \left(\ell'_R\right)^2 = \left(\Lambda_{LL}\ell_L + \Lambda_{LR}\ell_R\right)^2 - \left(\Lambda_{RL}\ell_L + \Lambda_{RR}\ell_R\right)^2\\ &= \left(\Lambda_{LL}^2 - \Lambda_{RL}^2\right)\ell_L^2 + \left(\Lambda_{LR}^2 - \Lambda_{RR}^2\right)\ell_R^2 + 2(\Lambda_{LL}\Lambda_{LR} - \Lambda_{RL}\Lambda_{RR})\ell_L\ell_R\\ &= \left(\ell_L\right)^2 - \left(\ell_R\right)^2 = \ell \circ \ell \in 2\mathbb{Z}\end{aligned} \qquad [8.264]$$

I am not sure how I can show that $\Gamma' = \Lambda\Gamma$ is also self-dual. Indeed in order to show this I need to work out $(\Lambda\Gamma)^*$ and I am not sure how to do that. Any help is welcome

8.69 p 252: Eq. (8.4.26) The Narain Momenta for One Compactified Dimension

We first establish the relation between $\ell_{L,R}$ and p_{L_R}. Dropping the subscripts $_{L,R}$ for simplicity we have

$$\ell_r = \sqrt{\frac{\alpha'}{2}}\frac{e_r^m v_m}{\alpha'} = \sqrt{\frac{\alpha'}{2}}\frac{e_r^m \alpha' p_m}{\alpha'} = \sqrt{\frac{\alpha'}{2}}e_r^m p_m = \sqrt{\frac{\alpha'}{2}}p_r \qquad [8.265]$$

Using (8.2.7) we thus find the quantisation

$$\ell_{Lr} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) = \sqrt{\frac{\alpha'}{2}}\frac{1}{R}\left(n + \frac{wR^2}{\alpha'}\right) = \sqrt{\frac{\alpha'}{2}}\sqrt{\frac{2}{\alpha'}}\frac{1}{r}\left(n + \frac{w\alpha'}{2\alpha'}\right) \qquad [8.266]$$

and thus

$$\ell_{Lr} = \frac{n}{r} + \frac{wr}{2} \qquad [8.267]$$

We have, entirely similarly,

$$\ell_{Rr} = \frac{n}{r} - \frac{wr}{2} \qquad [8.268]$$

Let us check that this is an even self-dual lattice. First we check

$$\ell \circ \ell = \ell_L^2 - \ell_R^2 = \left(\frac{n}{r} + \frac{wr}{2}\right)^2 - \left(\frac{n}{r} - \frac{wr}{2}\right)^2 = 2nw \in 2\mathbb{Z} \qquad [8.269]$$

Next we find the dual lattice. It consists of pairs $k = (k_L, k_R)$ such that $k \circ \ell \in \mathbb{Z}$ for all ℓ in the original lattice. I.e.

$$k_L \left(\frac{n}{r} + \frac{wr}{2}\right) - k_R \left(\frac{n}{r} - \frac{wr}{2}\right) \in \mathbb{Z} \qquad [8.270]$$

or

$$(k_L - k_R)\frac{n}{R} + (k_L + k_R)\frac{wr}{2} \in \mathbb{Z} \qquad [8.271]$$

and this must be valid for all integers n and w and for all radii r. Thus we need to have

$$\begin{aligned} k_L - k_R &= rp \\ k_L + k_R &= \frac{2}{r}q \end{aligned} \qquad [8.272]$$

for some integers p and q. Thus

$$\begin{aligned} k_L &= \frac{q}{r} + \frac{pr}{2} \\ k_R &= \frac{q}{r} - \frac{pr}{2} \end{aligned} \qquad [8.273]$$

and so $k = (k_L, k_R)$ spans exactly the same lattice as ℓ and hence the lattice is self-dual.

8.70 p 253: Eq. (8.4.27) Lorentz Boosts for One Compactified Dimension

It should be obvious that this is a $O(1,1)$ transformation, but let us check for completeness. We have

$$\Lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \qquad [8.274]$$

and

$$\begin{aligned} \Lambda^{\mathrm{T}}\eta\Lambda &= \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \\ &= \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} \cosh\lambda\lambda & \sinh\lambda \\ -\sinh\lambda & -\cosh\lambda \end{pmatrix} \\ &= \begin{pmatrix} \cosh^2\lambda - \sinh^2\lambda & 0 \\ 0 & \sinh^2\lambda - \cosh^2\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta \end{aligned} \qquad [8.275]$$

Now we write the transformed momenta in terms of n and m:

$$\begin{aligned}\ell'_L &= \left(\frac{n}{r}+\frac{mr}{2}\right)\cosh\lambda + \left(\frac{n}{r}-\frac{mr}{2}\right)\sinh\lambda \\ &= \frac{n}{r}(\cosh\lambda+\sinh\lambda) + \frac{mr}{2}(\cosh\lambda-\sinh\lambda) \\ &= \frac{n}{r}e^{+\lambda} + \frac{mr}{2}e^{-\lambda} = \frac{n}{re^{-\lambda}} + \frac{mre^{-\lambda}}{2} = \frac{n}{r'} + \frac{mr'}{2}\end{aligned} \qquad [8.276]$$

with $r' = re^{-\lambda}$. A similar relation holds for ℓ'_R as is easily checked.

8.71 p 253: Eq. (8.4.28) The Space of Inequivalent Even Self-Dual Lattices, I

When we set $B_{mn} = 0$ and are at the compactification radius of the enhanced gauge symmetry, i.e. $R = \sqrt{\alpha'}$, for each of the k compactified dimensions, then each such dimension has an $SU(2) \times SU(2)$ symmetry as we have seen. Assuming these compactified dimensions are not mixed up in some way – that is probably what is meant with them being orthogonal – then we have k copies of this, i.e we have a $(SU(2) \times SU(2))^k = SU(2)^{2k}$ gauge symmetry.

We know that if if perform a Lorentz rotation $O(k,\mathbb{R})\times O(k,\mathbb{R})$ on the quantum numbers $(n_L, w_L) \times (n_R, w_R)$ then the transformed theory has the same spectrum and is thus the the same as the original theory. Thus, if we have a given even self-dual lattice Γ with a given signature (k,k) then we can get all other even self-dual lattices by acting with an $O(k,k,\mathbb{R})$ rotation on the lattice $\Lambda_{O(k,k,\mathbb{R})}\Gamma$. But all these lattices that are related by an $O(k,\mathbb{R}) \times O(k,\mathbb{R})$ rotation $\Lambda_{O(k,\mathbb{R})\times O(k,\mathbb{R})}$ give the same theory. The space of different theories is this given by

$$\frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R}) \times O(k,\mathbb{R})} \qquad [8.277]$$

8.72 p 253: Eq. (8.4.28) The Space of Inequivalent Even Self-Dual Lattices, II

Assume that we have a given even self-dual lattice Γ and that this lattice has a (discrete) symmetry group, which we call $O(k,k,\mathbb{Z})$. This means that Γ and $\Lambda_{O(k,k,\mathbb{Z})}\Gamma$ are the same lattice. We then obtain all even self-dual lattices by acting with a $\Lambda_{O(k,k,\mathbb{R})}$ on this original lattice Γ, or equivalently on $\Lambda_{O(k,k,\mathbb{Z})}\Gamma$. All these new even-self-dual lattices that are related by an $O(k,\mathbb{R}) \times O(k,\mathbb{R})$ transformation are equivalent. Thus we conclude that the lattices

$$\Lambda_{O(k,k,\mathbb{R})}\Gamma; \quad \Lambda_{O(k,\mathbb{R})\times O(k,\mathbb{R})}\Lambda_{O(k,k,\mathbb{R})}\Gamma; \quad \Lambda_{O(k,\mathbb{R})\times O(k,\mathbb{R})}\Lambda_{O(k,k,\mathbb{R})}\Lambda_{O(k,k,\mathbb{Z})}\Gamma \qquad [8.278]$$

are all equivalent. Taking into account the discrete symmetries of the even self-dual lattice Γ the space of inequivalent theories is thus in fact

$$\frac{O(k,k,\mathbb{R})}{O(k,\mathbb{R})\times O(k,\mathbb{R})\times O(k,k,\mathbb{Z})} \tag{8.279}$$

8.73 p 254: Eq. (8.4.32) The $SL(k,\mathbb{Z})$ Part of $O(k,k,\mathbb{Z})$

Joe says that " [...] large spacetime coordinate transformations respecting the periodicity

$$x'^m = L^m{}_n x^n$$

with $L^m{}_n$ integers and $\det L = 1$ for invertibility" are transformations of the T-duality group $O(k,k,\mathbb{Z})$. This rather cryptic statement deserves some explanation. Remember the most general worldsheet action is (3.7.6)

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\left[\left(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\right] \tag{8.280}$$

This action is by construction invariant under $GL(1,d,\mathbb{R})$ transformations of X^μ, i.e. $X^\mu \longrightarrow X'^\mu = L^\mu{}_\nu X^\nu$. We can normalise this and restrict ourselves to those transformations with unit determinant so the symmetry group is $SL(1,d,\mathbb{R})$. If we compactify k dimensions then we have a symmetry under the subgroup $SL(k,\mathbb{R})$ on these compactified dimensions, i.e $X^m \longrightarrow X'^m = L^m{}_n X^n$. However, we need to ensure that these transformed spacetime fields satisfy the boundary conditions. Recall that the compactified coordinates are given by (8.2.15)

$$X^m(z,\bar z) = x^m - i\frac{\alpha'}{2}p_L^m \ln z - i\frac{\alpha'}{2}p_R^m \ln\bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde\alpha_m}{\bar z^m}\right) \tag{8.281}$$

with $x^m = x_L^m + x_R^m$ and the c.o.m. coordinate. But we still need to ensure that the transformed coordinates X'^m satisfy the appropriate boundary conditions (8.2.1) $X'^m \equiv X'^m + 2\pi R$ together with the condition (8.2.3) that one can wind the string around a compact dimension, i.e. $X(\sigma+2\pi) = X(\sigma) + 2\pi R w$ with $w \in \mathbb{Z}$. Recall from the discussion in section 8.2. that the former condition requires the total c.o.m. momentum to be quantised, $p = n/R, n \in \mathbb{Z}$ whereas the latter condition allows to split this total momentum in a left- and right-moving sector $p = p_L + p_R$, with p_L and p_R taking values on an even self-dual lattice as per the Narain analysis of modular invariance of the torus partition function. Let us now check that these conditions are also satisfied for the transformed coordinate. We have

$$X'^m(z,\bar z) = L^m{}_n\left[x^n - i\frac{\alpha'}{2}p_L^n \ln z - i\frac{\alpha'}{n}p_R^2 \ln\bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\frac{\alpha_n}{z^n} + \frac{\tilde\alpha_n}{\bar z^n}\right)\right] \tag{8.282}$$

The total c.o.m. momentum after the transformation needs to be quantised

$$p'^m = L^m{}_n(p_L^n + p_R^n) = \frac{k^m}{R}, \quad k^m \in \mathbb{Z} \tag{8.283}$$

or $RL^m{}_n(p_L^n + p_R^n) \in \mathbb{Z}$. We know that $R(p_L^n + p_R^n) = \ell^n \in \mathbb{Z}$, so that the requirement becomes that $L^m{}_n\ell^n \in \mathbb{Z}$ for all $\ell^n \in \mathbb{Z}$. This can only be the case if the $L^m{}_n$ themselves are integers and so in order to satisfy the boundary condition $X'^m \equiv X'^m + 2\pi R$ the group $SL(k,\mathbb{R})$ is actually reduced to $SL(k,\mathbb{Z})$.

We still need to check that an $SL(k,\mathbb{Z})$ transformation satisfies the second boundary condition, i.e. $X(\sigma + 2\pi) = X(\sigma) + 2\pi Rw$. Recall that we have $z = e^{iw} = e^{i\sigma^1 - \sigma^2}$. Under $\sigma^1 \longrightarrow \sigma^1 + 2\pi$ we thus have $z \longrightarrow z$ and similarly $\bar{z} \longrightarrow \bar{z}$. We need to be careful with the momentum terms as they contain logarithms. We have $\ln z = i\sigma^1 - \sigma^2 \longrightarrow 2\pi i + i\sigma^1 - \sigma^2 = 2\pi i + \ln z$ and similarly $\ln \bar{z} = -i\sigma^1 + \sigma^2 \longrightarrow -2\pi i - i\sigma^1 + \sigma^2 = -2\pi i + \ln \bar{z}$. Therefore

$$\begin{aligned} X'^m(\sigma^1 + 2\pi, \sigma^2) &= L^m{}_n\Bigg[x^n - i\frac{\alpha'}{2}p_L^n(2\pi i + \ln z) - i\frac{\alpha'}{n}p_R^2(-2\pi i + \ln \bar{z}) \\ &\qquad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\frac{\alpha_n}{z^n} + \frac{\tilde{\alpha}_n}{\bar{z}^n}\right)\Bigg] \\ &= X'^m(\sigma^1, \sigma^2) + \alpha'\pi L^m{}_n(p_L^n - p_R^n) \end{aligned} \tag{8.284}$$

Here we also know that $\alpha'\pi(p_L^n - p_R^n) = 2\pi Rw^m, w^m \in \mathbb{Z}$ as X^m satisfies these boundary conditions. Thus $\alpha'\pi L^m{}_n(p_L^n - p_R^n) = L^m{}_n 2\pi Rw^m = 2\pi Rw'^m$ with $w'^m = L^m{}_n w^n \in \mathbb{Z}$ as all the $L^m{}_n$ are integers as we have already established.

In summary we have shown that an $SL(k,\mathbb{Z})$ is indeed a symmetry of the theory that preserves the boundary conditions, both for the quantisation of the momentum as for the winding of the string around the compactified dimension. This transformation mixes up the n and w numbers into new ones, $n^m \longrightarrow L^m{}_n n^n$ and $w^m \longrightarrow L^m{}_n w^n$, and so necessarily transforms one point of the even self-dual lattice into another such point and is thus part of the T-duality. I.e. it is a part of the $O(k,k,\mathbb{Z})$ symmetry. Notice that we did not have to use the explicit form of the momenta p_L and p_R, i.e. the explicit form of the even self-dual lattice. This reasoning is valid for any G_{mn} and B_{mn}.

8.74 p 254: Eq. (8.4.33) Integer Shifts of the Antisymmetric Tensor

We consider the shift $b_{mn} \longrightarrow b_{mn}+N_{mn}$ with N_{nm} integers on $v_m = i\dot{x}^m$ given in (8.4.7.2). Using $B_{mn} = b_{mn}\alpha'/R^2$ we find

$$\begin{aligned} v_m =& \alpha' \frac{n_m}{R} - \left(\frac{\alpha'}{R^2} b_{mn} \right) w^n R = \frac{\alpha'}{R} \left(n_m - b_{mn} w^n \right) \\ & \frac{\alpha'}{R} \longrightarrow \left[n_m - (b_{mn} + N_{nm}) w^n \right] = \frac{\alpha'}{R} \left(n'_m - b_{mn} w^n \right) \end{aligned} \qquad [8.285]$$

with $n'_m = n_m - N_{nm} w^n \in \mathbb{Z}$, which is another point on the lattice and hence a symmetry.

Alternatively, putting this in the phase (8.4.12) we get

$$2\pi i b_{mn} w_1^m w_2^n \longrightarrow 2\pi i (b_{mn} + N_{mn}) w_1^m w_2^n = 2\pi i N_{mn} w_1^m w_2^n + 2\pi i b_{mn} w_1^m w_2^n \qquad [8.286]$$

The first term is 2π times an integer and so that phase doesn't contribute. leaving the partition function invariant. As the partition function "measures" the spectrum of the theory, recall $Z \propto \text{tr } q_0^L \bar{q}^{\tilde{L}_0}$, such a transformation leaves the spectrum unchanged.

8.75 p 254: Eq. (8.4.35) The Kinetic Terms of the Moduli

We will not perform the detailed calculation but show that it is a reasonable result by comparing to previous calculations. Let us remind ourselves of what the Weyl transformation is and why it is needed. We go back to the low energy action of the uncompactified closed string in (3.7.20)

$$S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[- \frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi \right] \qquad [8.287]$$

plus terms of order α' and higher. Here $H_{\mu\nu\lambda}$ is the field tensor of the antisymmetric tensor $B_{\mu\nu}$. Recall that this action gives the field equations necessary for Weyl invariance at the quantum level, at first order, of the most general worldsheet action, i.e. the vanishing of their β functions. The dilation factor $e^{-2\Phi}$ is inconvenient and we could simplify the action by making a field redefinition $\tilde{G}_{\mu\nu} = e^{2\omega(x) G_{\mu\nu}(x)}$ of the spacetime metric. This is effectively a Weyl transformation. Under such a transformation the spacetime Ricci scalar transforms as (3.7.23)

$$\tilde{\boldsymbol{R}} = e^{-2\omega} \left[\boldsymbol{R} - 2(D-1)\nabla^2 \omega - (D-2)(D-1)\partial_\mu \omega \partial^\mu \omega \right] \qquad [8.288]$$

By choosing $\omega = 2(\Phi_0 - \Phi)/(D-2)$, for some constant Φ_0, and defining a new dilaton field $\tilde{\Phi} = \Phi - \Phi_0$ all the terms in the action recombine as

$$S = \frac{1}{2\kappa^2}\int d^D X\sqrt{-\tilde{G}}\Bigg[-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}} - \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + o(\alpha')\Bigg] \qquad [8.289]$$

The $\tilde{H}$ denoting that spacetime indices are here raised with $\tilde{G}^{\mu\nu}$. We see that the Weyl transformation has gotten rid of the nasty $e^{-2\Phi}$ factor in the action and has left us with nice kinetic terms coming from $\boldsymbol{R}$ or $\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}$.

We now apply the same reasoning to the low energy action with compactified dimension (8.4.2). Let us focus on the terms that will contribute to the kinetic terms:

$$S = \frac{(2\pi R)^k}{2\kappa_0^2}\int d^d x\sqrt{-G_d}e^{-2\Phi_d}\Big[4\partial_\mu\Phi_d\partial^\mu\Phi_d - \frac{1}{4}G^{mn}G^{pq}(\partial_\mu G_{mp}\partial^\mu G_{nq} + \partial_\mu B_{mp}\partial^\mu B_{nq}) + \cdots\Big] \qquad [8.290]$$

One sees by comparison with the uncompactified case that these kinetic terms become

$$S \propto \int d^d x\sqrt{-G_d}\Bigg[-\frac{4}{D-2}\partial_\mu\Phi\partial^\mu\tilde{\Phi} - \frac{1}{4}G^{mn}G^{pq}(\partial_\mu G_{mp}\partial^\mu G_{nq} + \partial_\mu B_{mp}\partial^\mu B_{nq})\Bigg] \qquad [8.291]$$

which is (8.4.35).

8.76 p 255: Preliminary Considerations to the $d = 2$ Example

Before we look at the $d = 2$ example it is useful to put the preceding results in a more formal framework. We are looking for a general formula of how the moduli G and B transform under an $O(k, k, \mathbb{R})$ transformation and what transformations leave the spectrum invariant. In order to achieve this we follow Giveon et al hep-th/9401139v1, section 2.

We start by rescaling the spacetime coordinates to a dimensionless one.

$$X \longrightarrow \hat{X} = RX \qquad [8.292]$$

The periodicity of the coordinate is now 2π. The worldsheet action is the

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,\Big[\Big(g^{ab}G_{mn} + i\epsilon^{ab}B_{mn}\Big)R^2\partial_a X^m\partial_b X^n + \alpha'\boldsymbol{R}\Phi(RX)\Big] \\ &= \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,\Big[\Big(g^{ab}\hat{G}_{mn} + i\epsilon^{ab}\hat{B}_{mn}\Big)\partial_a X^m\partial_b X^n + \boldsymbol{R}\Phi(RX)\Big] \end{aligned} \qquad [8.293]$$

where we have defined the spacetime metric and antisymmetric field

$$\hat{G}_{mn} = \frac{R^2 G_{mn}}{\alpha'} \qquad ; \qquad \hat{B}_{mn} = \frac{R^2 B_{mn}}{\alpha'} \qquad [8.294]$$

Note that the hatted fields are dimensionless, just as the unhatted fields are. In the above equation $\boldsymbol{R}$ is the worldsheet Ricci scalar, not to be confused with the compactification radius R. We also rescale the compactification radius as

$$R \longrightarrow \hat{R} = \sqrt{\alpha'} R \qquad [8.295]$$

Under T-duality the rescaled R now simply transforms as $R \longrightarrow 1/R$.

We now rewrite the mass-shell formula (8.4.9) in a convenient way. We start using the unhatted fields. Focussing on the terms without the oscillator contribution we have

$$\begin{aligned} m^2 &= \frac{1}{2\alpha'^2} G_{mn}(v_L^m v_L^n + v_R^m v_R^n) \\ &= \frac{1}{2\alpha'^2} G_{mn}\left[(v^m + w^m R)(v^n + w^n R) + (v^m - w^m R)(v^n - w^n R)\right] \\ &= \frac{1}{\alpha'^2} G_{mn}(v^m v^n + R^2 w^m w^n) \end{aligned} \qquad [8.296]$$

We need to be careful about the location of the indices. In order to use (8.4.7) we need to lower the indices on the vs.

$$\begin{aligned} m^2 &= \frac{1}{\alpha'^2} G_{mn}(G^{mp}G^{nq} v_p v_q + R^2 w^m w^n) \\ &= \frac{1}{\alpha'^2} G_{mn}\left[G^{mp}G^{nq}\left(\frac{\alpha'}{R} n_p - R B_{pr} w^r\right)\left(\frac{\alpha'}{R} n_q - R B_{qs} w^s\right) + R^2 w^m w^n)\right] \end{aligned} \qquad [8.297]$$

First, consider the terms in n^2:

$$\frac{1}{\alpha'^2} G_{mn}\left(G^{mp}G^{nq}\frac{\alpha'}{R} n_p \frac{\alpha'}{R} n_q\right) = \frac{1}{R^2} n_p G^{pq} n_q \qquad [8.298]$$

Next, take the terms in w^2

$$\begin{aligned} &\frac{1}{\alpha'^2} G_{mn}\left(G^{mp}G^{nq} R B_{pr} w^r R B_{qs} w^s + R^2 w^m w^n\right) \\ &= \frac{R^2}{\alpha'^2} w^r \left(G^{pq} B_{pr} B_{qs} + G_{rs}\right) w^s = \frac{R^2}{\alpha'^2} w^r \left(G_{rs} - B_{rp} G^{pq} B_{qs}\right) w^s \end{aligned} \qquad [8.299]$$

Finally, the term in nw gives

$$\begin{aligned} &-\frac{1}{\alpha'^2} G_{mn} G^{mp} G^{nq}\left(\frac{\alpha'}{R} n_p R B_{qs} w^s + R B_{pr} w^r \frac{\alpha'}{R} n_q\right) \\ &= -\frac{1}{\alpha'} G^{pq}(n_p B_{qs} w^s + n_q B_{pr} w^r) = \frac{2}{\alpha'} w^s B_{sq} G^{qp} n_p \end{aligned} \qquad [8.300]$$

We bring the contributions together and use a matrix notation

$$G \equiv G_{mn} \quad ; \quad G^{-1} \equiv G^{mn} \quad ; \quad B \equiv B_{mn} \tag{8.301}$$

and see that we can write the mass-shell condition in the simple form

$$m^2 = \frac{1}{R^2} n G^{-1} n + \frac{R^2}{\alpha'^2} w (G - B G^{-1} B) w + \frac{2}{\alpha'} w B G^{-1} n \tag{8.302}$$

We now go to the hatted fields $G_{mn} = (\alpha'/R^2)\hat{G}_{mn}$, $B_{mn} = (\alpha'/R^2)\hat{B}_{mn}$ and $(G^{-1})^{mn} = (R^2/\alpha')(\hat{G}^{-1})^{mn}$ and obtain

$$m^2 = \frac{1}{R^2} n \frac{R^2}{\alpha'} \hat{G}^{-1} n + \frac{R^2}{\alpha'^2} w (\frac{\alpha'}{R^2} G - \frac{\alpha'}{R^2} B \frac{R^2}{\alpha'} G^{-1} \frac{\alpha'}{R^2} B) w + \frac{2}{\alpha'} w \frac{\alpha'}{R^2} B \frac{R^2}{\alpha'} G^{-1} n \tag{8.303}$$

which simplifies to

$$\alpha' m^2 = n G^{-1} n + w (G - B G^{-1} B) w + 2 w B G^{-1} n \tag{8.304}$$

where we have here, and in the forthcoming expressions, deleted the hats for convenience.

We can simplify this even further. We put the momentum quanta and the winding numbers in a $2k$-dimensional row matrix

$$Z^t = (w^1, \cdots, w^k, n_1, \cdots n_k) \tag{8.305}$$

and introduce the $2k \times 2k$ matrix

$$M = \begin{pmatrix} G - B G^{-1} B & B G^{-1} \\ -G^{-1} B & G^{-1} \end{pmatrix} \tag{8.306}$$

The mass-shell condition then becomes simply

$$m^2 = Z^t M Z \tag{8.307}$$

We will also denote $E = G + B$ where the matrix E has as symmetric part G and an antisymmetric part B.

The moduli space for the toroïdal compactification is (8.4.28), i.e. $O(k, k, \mathbb{R})/O(k, \mathbb{R}) \times O(k, \mathbb{R})$. The group $O(k, k, \mathbb{R})$ acting on a given even self-dual lattice, generates all possible even self-dual lattices of the same signature.[7] Let us work out how this group acts on the moduli G and B. We represent an element $g \in O(k, k, \mathbb{R})$ by the $2d \times 2d$ matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8.308}$$

[7]This is a standard mathematical result, see e.g. J.P. Serre, A Course in Arithmetic, Springer-Verlag 1973.

such that it preserves

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad [8.309]$$

i.e. $J = g^t J g$:

$$\begin{aligned} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} &= \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} a^t c + c^t a & a^t d + c^t b \\ b^t c + d^t a & b^t d + d^t b \end{pmatrix} \end{aligned} \quad [8.310]$$

Hence

$$a^t c + c^t a = b^t d + d^t b = 0; \quad a^t d + c^t b = \mathbb{1} \quad [8.311]$$

Under an orthogonal transformation g, the matrix M in [8.306] transforms as

$$M \longrightarrow m_g = gMg^t \quad [8.312]$$

Note that from $J = g^t J g$ we have $J^{-1} = (g^t J g)^{-1} = g^{-1} J^{-1} (g^t)^{-1}$. Using $J^{-1} = J$ this gives $J = g^{-1} J (g^t)^{-1}$. Multiply both sides with g on the left and g^t on the right to get $J = gJg^t$ which shows that if $g \in O(k,k,\mathbb{R})$ then so is g^t.

We now write the moduli G and B in an $O(k,k,\mathbb{R})$ form as

$$g_E = \begin{pmatrix} e & B(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix} \quad [8.313]$$

Here e is the vielbein (in matrix form) satisfying $G = ee^t$. One easily checks that $g_E^t J g_E = J$. Indeed

$$\begin{aligned} &\begin{pmatrix} e^t & 0 \\ e^{-1}B^t & (e)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} e & B(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ e^{-1}B^t & e^{-1} \end{pmatrix} \begin{pmatrix} 0 & (e^t)^{-1} \\ e & B(e^t)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^t(e^t)^{-1} \\ e^{-1}e & e^{-1}B^t(e^t)^{-1} + e^{-1}B(e^t)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \end{aligned} \quad [8.314]$$

as B is antisymmetric, $B^t = -B$ and $ee^{-1} = e^t(e^t)^{-1} = 1$.

Define the action of an element of $g \in O(k,k,\mathbb{R})$ on a k-dimensional matrix F as a fractional linear transformation

$$g(F) = (aF + b)(cF + d)^{-1} \quad [8.315]$$

Note the multiplication of the inverse is form the right. Using this we find (with here $\mathbb{1}$ the k-dimensional unit matrix)

$$\begin{aligned} g_E(\mathbb{1}_k) &= (e\mathbb{1}_k + B(e^t)^{-1})(0 + (e^t)^{-1})^{-1} = (e + B(e^t)^{-1})e^t = ee^t + B \\ &= (G + B) = E \end{aligned} \quad [8.316]$$

This is how we can extract G and B from an $O(k,k,\mathbb{R})$ transformation.

Furthermore we have

$$\begin{aligned} g_E g_E^t &= \begin{pmatrix} e & B(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ e^{-1}B^t & e^{-1} \end{pmatrix} \\ &= \begin{pmatrix} ee^t + B(e^t)^{-1}e^{-1}B^t & B(e^t)^{-1}e^{-1} \\ (e^t)^{-1}e^{-1}B^t & (e^t)^{-1}e^{-1} \end{pmatrix} \\ &= \begin{pmatrix} G + BG^{-1}B^t & BG^{-1} \\ G^{-1}B^t & G^{-1} \end{pmatrix} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = M \end{aligned} \tag{8.317}$$

where we have again used $B^t = -B$.

Using all this it follows that under a transformation $g \in O(k,k,\mathbb{R})$

$$M \longrightarrow M_g = gMg^t = gg_E g_E^t g^t = (gg_E)(gg_E)^t = g_{E'}g_{E'}^t \tag{8.318}$$

where $g_{E'} = gg_E$ is also an $O(k,k,\mathbb{R})$ transformation. From [8.316] we have that

$$E' = g_{E'}(\mathbb{1}_k) = gg_E(\mathbb{1}_k) = g(E) = (aE+b)(cE+d)^{-1} \tag{8.319}$$

We have found how the moduli transform under an $O(k,k,\mathbb{R})$ transformation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, i.e.

$$E \longrightarrow E' = (aE+b)(cE+d)^{-1} \tag{8.320}$$

By taking the symmetric and antisymmetric parts of both sides, we find the transformation rule for G and B respectively.

Now that we have found how an $O(k,k,\mathbb{R})$ transformation acts on the moduli, we will look at such special transformations that leave the spectrum invariant.

1. Consider the element

$$g_\Theta = \begin{pmatrix} \mathbb{1} & \Theta \\ 0 & \mathbb{1} \end{pmatrix} \tag{8.321}$$

with Θ and antisymmetric $k \times k$ matrix with integers as entries, i.e. $\Theta_{ij} = -\Theta_{ji} \in \mathbb{Z}$. One easily checks that, because Θ is antisymmetric, $g_\Theta^t J g_\Theta = J$ so that $g_\Theta \in O(k,k,\mathbb{R})$. The moduli are transformed as per [8.320]:

$$E' = (E+\Theta)(0+\mathbb{1})^{-1} = E + \Theta \tag{8.322}$$

Taking the symmetric and antisymmetric side we find that

$$G' = G \quad \text{and} \quad B' = B + \Theta \tag{8.323}$$

This transformation thus leaves G invariant and shifts B by integers. Let us now check the mass spectrum [8.306]. The hurried reader can check that this transformation leaves the mass spectrum invariant if we change n into $n - \Theta w$ and leave w unchanged. and then rush to the second symmetry, about two pages from here. The more assiduous reader can follow the derivation below of how we come to this solution.

The mass spectrum for the transformed moduli is given by

$$\begin{aligned}
\alpha' m'^2 =& nG^{-1}n + w[G - (B+\Theta)G^{-1}(B+\Theta)]w + 2w(B+\Theta)G^{-1}n \\
=& nG^{-1}n + w(G - BG^{-1}B)w + 2wBG^{-1}n \\
&+ w\left[GG^{-1}\Theta - BG^{-1}\Theta - \Theta G^{-1}B - \Theta G^{-1}\Theta\right]w + 2w\Theta G^{-1}n \\
=& nG^{-1}n + w(G - BG^{-1}B)w + 2wBG^{-1}n \\
&- 2wBG^{-1}\Theta w - w\Theta G^{-1}\Theta w + 2w\Theta G^{-1}n
\end{aligned} \qquad [8.324]$$

We have used the fact that $w\Theta G^{-1}Bw = wBG^{-1}\Theta w$ and that $w\Theta w = 0$ by antisymmetry of Θ. We now wish to see if we can rewrite this as [8.306] with n replaced by $n' = n + \Delta n$ and w replaced by $w' = w + \Delta w$ with Δn and Δw integers, i.e.

$$\begin{aligned}
\alpha' m'^2 =& (n+\Delta n)G^{-1}(n+\Delta n) + (w+\Delta w)(G - BG^{-1}B)(w+\Delta w) \\
&+ 2(w+\Delta w)BG^{-1}(n+\Delta n)
\end{aligned} \qquad [8.325]$$

Let us first analyse the terms in the middle. This gives

$$\begin{aligned}
0 =& 2w(G - BG^{-1}B)\Delta w + \Delta w(G - BG^{-1}B)\Delta w \\
=& (2w + \Delta w)\Delta w(G - BG^{-1}B)
\end{aligned} \qquad [8.326]$$

This is one equation for k components $(\Delta w^1, \cdots, \Delta w^k)$, but this must be valid for any possible background G and B. We must thus have $\Delta w = 0$ or $\Delta w = -2w$. Both give integers so both need to be taken into consideration.

Next consider the last term in [8.325]:

$$0 = 2wBG^{-1}\Theta w + 2wBG^{-1}\Delta n + 2\Delta wBG^{-1}n + 2\Delta wBG^{-1}\Delta n \qquad [8.327]$$

Let us firs look at the $\Delta w = 0$ solution. This equation then becomes

$$0 = 2wBG^{-1}\Theta w + 2wBG^{-1}\Delta n = 2wBG^{-1}(\Theta w + \Delta n) \qquad [8.328]$$

Once more this needs to be valid for all G and B and thus this implies that $\Delta n = -\Theta w$, and this is an integer. For the other solution $\Delta w = -2w$ we have

$$\begin{aligned}
0 =& 2wBG^{-1}\Theta w + 2wBG^{-1}\Delta n - 4wBG^{-1}n - 4wBG^{-1}\Delta n \\
=& 2wBG^{-1}(\Theta w - \Delta n - 2n)
\end{aligned} \qquad [8.329]$$

which implies $\Delta n = \Theta w + 2n$ which is also an integer.

We thus have two sets of possible solutions:

$$(\Delta n, \Delta w) = (-\Theta w, 0) \quad \text{or} \quad (\Theta w + 2n, -2w) \qquad [8.330]$$

It remains to check the first term in [8.325]:

$$0 = w\Theta G^{-1}\Theta w - 2w\Theta G^{-1}n + 2nG^{-1}\Delta n + \Delta n G^{-1}\Delta n \qquad [8.331]$$

For the first solution the RHS becomes

$$RHS = w\Theta G^{-1}\Theta w - 2w\Theta G^{-1}n - 2nG^{-1}\Theta w + \Theta w G^{-1}\Theta w \qquad [8.332]$$

Let us look at the last term

$$\begin{aligned}\Theta w G^{-1}\Theta w &= \Theta_{ij}w^j\left(G^{-1}\right)^{ik}\Theta_{k\ell}w^\ell = w^\ell\Theta_{k\ell}\left(G^{-1}\right)^{ik}\Theta_{ij}w^j\\ &= -\,w^\ell\Theta_{\ell k}\left(G^{-1}\right)^{ki}\Theta_{ij}w^j = -w\Theta G^{-1}\Theta w \qquad [8.333]\end{aligned}$$

and so the last term cancels the first term. Similarly we find that the third term cancels the second term:

$$nG^{-1}\Theta w = n_i\left(G^{-1}\right)^{ij}\Theta_{jk}w^k = -w^k\Theta_{kj}\left(G^{-1}\right)^{ji}n_i = -w\Theta G^{-1}n \qquad [8.334]$$

We have thus found that we can rewrite the mass-shell condition [8.306] as

$$\alpha' m'^2 = n'G^{-1}n' + w'(G - BG^{-1}B)w' + 2w'BG^{-1}n' \qquad [8.335]$$

with

$$n' = n - \Theta w \quad \text{and} \quad w' = w \qquad [8.336]$$

As Θ consists of integers n' and w' range over all integers and this solutions does leaves the spectrum invariant. In other words, the transformation [8.323], i.e. $G' = G, B' = B + \Theta$ is a symmetry of the spectrum.[8]

2. Next, we consider a transformation

$$g_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1}\end{pmatrix} \qquad [8.337]$$

with $A \in GL(k, \mathbb{Z})$, i.e. a matrix of integers. We leave it as an exercise that $g_A^t J g_A = J$. From [8.320] we find that this corresponds with a change of moduli

$$E \longrightarrow E' = (aE + b)(cE + d)^{-1} = AEA^t \qquad [8.338]$$

Let us first work out how the moduli change under this transformation

$$\begin{aligned}G' &= \frac{1}{2}(E' + E'^t) = \frac{1}{2}(AEA^t + (AEA^t)^t) = \frac{1}{2}(AEA^t + AE^tA^t) = A\frac{1}{2}(E + E^t)A^t\\ &= AGA^t \qquad [8.339]\end{aligned}$$

[8] What happens with the second solution? It doesn't matter we already have one working solution. Never mind about the extra work.

and similarly

$$B' = \frac{1}{2}(E' - E'^t) = \frac{1}{2}(AEA^t - (AEA^t)^t) = \frac{1}{2}(AEA^t - AE^tA^t) = A\frac{1}{2}(E - E^t)A^t \\ = ABA^t \qquad [8.340]$$

The mass-shell condition [8.306] then becomes simply

$$\begin{aligned} \alpha' m'^2 &= n(A^t)^{-1}G^{-1}A^{-1}n + \alpha' w(AGA^t - ABA^t(A^t)^{-1}G^{-1}A^{-1}ABA^t)w \\ &\quad + 2wABA^t(A^t)^{-1}G^{-1}A^{-1}n \\ &= n(A^t)^{-1}G^{-1}A^{-1}n + \alpha' wA(G - BG^{-1}B)A^t w + 2wABG^{-1}A^{-1}n \\ &= \left(A^{-1}n\right)^t G^{-1}(A^{-1}n) + \alpha'(A^t w)(G - BG^{-1}B)(A^t w) \\ &\quad + 2(A^t w)BG^{-1}(A^{-1}n) \end{aligned} \qquad [8.341]$$

From this we immediately see that if we define $n' = A^{-1}n$ and $w' = A^t w$ then we can write this as

$$\alpha' m'^2 = n' G^{-1} n' + w'(G - BG^{-1}B)w' + 2w' BG^{-1} n' \qquad [8.342]$$

with n' and w' ranging over all integers, by virtue of the fact that $A \in GL(k, \mathbb{Z})$ and that A^{-1} exists. Hence, this is also a symmetry of the theory. We see from the form of n' and w' that this symmetry actually orresponds to a change of of basis for the compactification lattice.

3. A third symmetry is given by the matrix

$$G_{D_i} = \begin{pmatrix} \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \\ \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \end{pmatrix} \qquad [8.343]$$

Here $\mathbb{1}$ is the k-dimensional unit matrix and $\mathbb{e}_i$ is a k-dimensional matrix, with all zero entries, except for the ii component, which is one; i.e. its elements are $(\mathbb{e}_i)_{jk} = \delta_{ij}\delta_{ik}$. Note that $(\mathbb{e}_i)^2 = \mathbb{e}_i$ (no sum) and the $\mathbb{e}^{-1} = \mathbb{e}_i^t = \mathbb{e}_i$.

Let us proceed as for the previous cases

$$\begin{aligned} G_{D_i}^t J G_{D_i} &= \begin{pmatrix} \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \\ \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \\ \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \\ \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \end{pmatrix} \begin{pmatrix} \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \\ \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \end{pmatrix} \\ &= \begin{pmatrix} (\mathbb{1} - \mathbb{e}_i)\mathbb{e}_i + \mathbb{e}_i(\mathbb{1} - \mathbb{e}_i) & (\mathbb{1} - \mathbb{e}_i)^2 + \mathbb{e}_i^2 \\ \mathbb{e}_i^2 + (\mathbb{1} - \mathbb{e}_i)^2 & \mathbb{e}_i(\mathbb{1} - \mathbb{e}_i) + (\mathbb{1} - \mathbb{e}_i)\mathbb{e}_i \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{e}_i - \mathbb{e}_i^2 + \mathbb{e}_i - \mathbb{e}_i^2 & \mathbb{1} + \mathbb{e}_i^2 - 2\mathbb{e}_i + \mathbb{e}_i^2 \\ \mathbb{e}_i^2 + \mathbb{1} + \mathbb{e}_i^2 - 2\mathbb{e}_i & \mathbb{e}_i - \mathbb{e}_i^2 + \mathbb{e}_i - \mathbb{e}_i^2 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = J \end{aligned} \qquad [8.344]$$

so G_{D_i} is an appropriate transformation.

The background fields then transform as

$$E \longrightarrow E' = \left[(\mathbb{1} - \mathbb{e}_i)E + \mathbb{e}_i\right](\mathbb{e}_i E + \mathbb{1} - \mathbb{e}_i)^{-1} \qquad [8.345]$$

with G and B following from the symmetric and antisymmetric part respectively. These are in general messy formula. To understand what these transformations means, let us look at the case $k = 2$. Taking $i = 2$ one finds after some algebra that

$$G'_{11} = \frac{B_{12}^2 + G}{G_{22}}; \quad G'_{12} = \frac{B_{12}}{G_{22}}; \quad G'_{22} = \frac{1}{G_{22}}; \quad B'_{12} = \frac{G_{12}}{G_{22}} \qquad [8.346]$$

where, as usual $G = \det G_{ij}$. Note also that $\det G' = \det G/G_{22}^2$. To see what this means, let us take the example where the two dimensions are compactified in two circles that factorise completely. This means that $G_{12} = 0$ and that also the anti-symmetric field vanishes, $B_{12} = 0$. In that case, the above transformation reduces to

$$G'_{11} = G_{11}; \quad G'_{12} = 0; \quad G'_{22} = \frac{1}{G_{22}}; \quad B'_{12} = 0 \qquad [8.347]$$

What happens with the mass spectrum? With this background we have

$$\alpha' m^2 = \frac{n_1^2}{G_{11}} + \frac{n_2^2}{G_{22}} + G_{11} w_1^2 + G_{22} w_2^2 \qquad [8.348]$$

After the transformation with $i = 2$ we thus find

$$\begin{aligned} \alpha' m'^2 &= \frac{n_1^2}{G'_{11}} + \frac{n_2^2}{G'_{22}} + G'_{11} w_1^2 + G'_{22} w_2^2 \\ &= \frac{n_1^2}{G_{11}} + G_{22} n_2^2 + G_{11} w_1^2 + \frac{w_2^2}{G_{22}} \end{aligned} \qquad [8.349]$$

If we interchange n_2 with w_2, then we get exactly $\alpha' m^2$. We thus see that this case corresponds exactly to the T-duality transformation of the second compactified dimension. Similarly, of course $i = 1$ corresponds to the T-duality transformation of the first compactified dimension. We can thus see that the transformation for a general background corresponds to generalisations of the T-duality.

At this point, Giveon et al. write that "It can be shown straightforwardly that this transformation leaves the partition function invariant as well". As so often happens, straightforward usually means anything but that. If it were that straightforward why do they not even give a hint? At other points in the text they sometimes go to excriucating detail to show very simple facts. Alas, I have not found this straightforward at all and shall just have to accept the result. Any help on this is more than welcome.

4. A last symmetry comes from the worldsheet parity $\sigma \longrightarrow -\sigma$. It corresponds to a change of sign of the antisymmetric field $B \longrightarrow -B$. Contrary to the three previous symmetries, this is not an $O(k, k, \mathbb{Z})$ symmetry. In fact it interchanges p_L with p_R.

The first three symmetries of the theory are $O(k, k, \mathbb{Z})$ symmetries. Indeed in all three cases the matrix has integer entries. Contrary to the first three symmetries, the last symmetry is not an $O(k, k, \mathbb{Z})$ symmetry. In fact it interchanges p_L with p_R. For convenience it is in the discussion treated as part of the $O(k, k, \mathbb{Z})$ symmetry group.

An important caveat is on order. We have only that the above transformations are a symmetry of of the mass-shell spectrum, without the oscillator terms. In order to show that they are a symmetry of the full interacting theory, we need to show that the oscillator terms and the correlation functions are invariant as well. We refer to the Giveon et al article for details on this.

From here on we will revert to the original unhatted fields and compactification radius.

8.77 p 255: Eq. (8.4.36)-(8.4.37) The Complex Moduli τ and ρ

There are four moduli from the spacetime action: three from the symmetric spacetime metric G_{mn}, i.e. $G_{24,24}, G_{25,25}, G_{24,25}$ and one from the antisymmetric tensor B_{mn}, i.e. $B_{24,25}$. We rewrite these four moduli in terms of two complex fields τ and ρ and find the transformation law between them.

Just under (8.4.12) b_{mn} is defined as $b_{mn} = R^2 B_{mn}/\alpha'$, which explains the first term of (8.4.36). For the second term, recall that $G^{1/2} = (\det G_{mn})^{1/2}$ gives the unit volume of the two-dimensional surface described by the metric G_{mn}. The volume of the two torus of the compactified dimension is $V = \int dX^{24} dX^{25}\, G^{1/2} = (2\pi R)^2 G^{1/2}$. We thus have

$$i\frac{R^2}{\alpha'}G^{1/2} = \frac{i}{4\pi^2\alpha'}(2\pi R)^2 G^{1/2} = \frac{i}{4\pi^2\alpha'}(2\pi R)^2 \frac{V}{(2\pi R)^2} = \frac{iV}{4\pi^2\alpha'} \qquad [8.350]$$

which gives the second term of (8.4.36). We thus already have

$$\begin{aligned} \rho_1 &= \frac{R^2}{\alpha'} B_{24,25} \\ \rho_2 &= \frac{R^2}{\alpha'} \sqrt{G} \end{aligned} \qquad [8.351]$$

Consider now (8.4.37). Recall that G_{mn} defines the spacetime metric, i.e.

$$ds^2 = G_{mn} dX^m dX^n = G_{24,24} dX^{24} dX^{24} + G_{25,25} dX^{25} dX^{25} + 2G_{24,25} dX^{24} dX^{25} \qquad [8.352]$$

From (8.4.37) we also have

$$\begin{aligned} ds^2 &= \frac{\alpha' \rho_2}{R^2 \tau_2}(dX^{24} + \tau dX^{25})(dX^{24} + \bar{\tau} dX^{25}) \\ &= \frac{\alpha' \rho_2}{R^2 \tau_2}(dX^{24} dX^{24} + |\tau|^2 dX^{25} dX^{25} + 2\tau_1 dX^{24} dX^{25}) \end{aligned} \qquad [8.353]$$

and thus

$$\begin{aligned} G_{24,24} &= \frac{\alpha' \rho_2}{R^2 \tau_2} \\ G_{25,25} &= \frac{\alpha' \rho_2 |\tau|^2}{R^2 \tau_2} \\ G_{24,25} &= \frac{\alpha' \rho_2 \tau_1}{R^2 \tau_2} \\ B_{24,25} &= \frac{\alpha' \rho_1}{R^2} \end{aligned} \qquad [8.354]$$

We have also added the expression for the antisymmetric tensor for completeness.

For later use, it turns out to be convenient to write this as

$$G_{mn} = \frac{\alpha' \rho_2}{R^2} \mathfrak{G}_{mn}(\tau) \qquad [8.355]$$

with

$$\mathfrak{G} = \mathfrak{G}_{mn}(\tau) = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \qquad [8.356]$$

We first note that

$$\det \mathfrak{G}_{mn} = \frac{1}{\tau_2^2}(|\tau|^2 - \tau_1^2) = \frac{1}{\tau_2^2}(\tau_1^2 + \tau_2^2 - \tau_1^2) = 1 \qquad [8.357]$$

We thus have

$$\mathfrak{G}^{-1} = \mathfrak{G}^{mn}(\tau) = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix} \qquad [8.358]$$

We also have

$$G^{mn} = \frac{R^2}{\alpha' \rho_2} \left(\mathfrak{G}^{-1}\right)^{mn} \qquad [8.359]$$

Indeed, one see immediately that $G_{mn} G^{mp} = \delta_m^p$ as it should. Finally, using [8.357] we also have

$$\begin{aligned} \sqrt{\det G_{mn}} &= \frac{\alpha' \rho_2}{R^2} \\ \sqrt{\det G^{mn}} &= \frac{R^2}{\alpha' \rho_2} \end{aligned} \qquad [8.360]$$

Repeating [8.351]t and inverting [8.354] we find

$$\begin{aligned}\rho_1 &= \frac{R^2}{\alpha'}B_{24,25} = b_{24,25}\\ \rho_2 &= \frac{R^2}{\alpha'}\sqrt{G_{24,24}G_{25,25} - G_{24,25}^2} = \frac{R^2}{\alpha'}\sqrt{G}\\ \tau_1 &= \frac{G_{24,25}}{G_{24,24}}\\ \tau_2 &= \frac{\sqrt{G_{24,24}G_{25,25} - G_{24,25}^2}}{G_{24,24}} = \frac{\sqrt{G}}{G_{24,24}}\end{aligned} \qquad [8.361]$$

as can be checked by direct calculation.

8.78 p 255: Eq. (8.4.38) The Full T-Duality Group for Two Compactified Dimensions

[1] The symmetry transformation (8.4.32) for $k = 2$ i.e. $x^m \longrightarrow L^m{}_n x^n$ with $L \in SL(2,\mathbb{Z})$ does not affect ρ as can be seen from (8.4.36). Indeed $b_{24,25}$ is unchanged by this transformation and and so is the volume of the two-torus V. So these transformations can only affect the τ moduli. There is moreover a symmetry $X^m \longrightarrow -X^m$ and so the symmetry group is $PSL(2,\mathbb{Z})$ and it acts on the modulus τ of the space-time two-torus, leaving ρ invariant.

Let us now repeat this using the framework of our section 8.76. We simplify the dimension by renaming $(24,25) \equiv (1,2)$. We have an $O(2,2,\mathbb{Z})$ transformation with

$$g_L = \begin{pmatrix} L & 0 \\ 0 & (L^t)^{-1}\end{pmatrix} \qquad [8.362]$$

with L a matrix of integers with $\det L = 1$. The moduli transform according to [8.339] and [8.340]. Working this out we find

$$\begin{aligned}G'_{11} &= L_{11}^2 G_{11} + 2L_{11}L_{12}G_{12} + L_{12}^2 G_{22}\\ G'_{12} &= L_{11}L_{21}G_{11} + (L_{11}L_{22} + L_{12}L_{21})G_{12} + L_{12}L_{22}G_{22}\\ G'_{22} &= L_{21}^2 G_{11} + 2L_{21}L_{22}G_{12} + L_{22}^2 G_{22}\\ B'_{12} &= B_{12}\end{aligned} \qquad [8.363]$$

We also find that $\det G' = \det G$. In these expressions, we have used $\det L = 1$. From this and the relation with the complex moduli ρ in [8.361] we find that

$$\begin{aligned}\rho'_1 &= B'_{12} = B_{12} = \rho_1\\ \rho'_2 &= \sqrt{G'} = \sqrt{G} = \rho_2\end{aligned} \qquad [8.364]$$

and so ρ does indeed not change under this transformation. For τ we find from [8.361]

$$\tau_1' = \frac{G_{12}'}{G_{11}'} = \frac{L_{11}L_{21}G_{11} + G_{12}(L_{11}L_{22} + L_{12}L_{21}) + L_{12}L_{22}G_{22}}{L_{11}^2 G_{11} + 2L_{11}L_{12}G_{12} + L_{12}^2 G_{22}}$$
$$\tau_2' = \frac{\sqrt{G'}}{G_{11}'} = \frac{\sqrt{G}}{L_{11}^2 G_{11} + 2L_{11}L_{12}G_{12} + L_{12}^2 G_{22}} \quad [8.365]$$

We now express the moduli G in terms of τ and ρ using [8.354] and find, after some algebra

$$\tau_1' = \frac{L_{12}L_{22}(\tau_1^2+\tau_2^2) + (L_{11}L_{22}+L_{12}L_{21})\tau_1 + L_{11}L_{21}}{L_{12}^2(\tau_1^2+\tau_2^2) + 2L_{11}L_{12}\tau_1 + L_{11}^2}$$
$$\tau_2' = -\frac{\tau_2}{L_{12}^2(\tau_1^2+\tau_2^2) + 2L_{11}L_{12}\tau_1 + L_{11}^2} \quad [8.366]$$

We now wish to show that this corresponds to an $SL(2,\mathbb{Z})$ transformation of τ, i.e. for $a, b, c, d \in \mathbb{Z}$

$$\begin{aligned}\tau' = \tau_1' + i\tau_2' &= \frac{a\tau+b}{c\tau+d} = \frac{(a\tau+b)(c\bar{\tau}+d)}{(c\tau+d)(c\bar{\tau}+d)} \\ &= \frac{ac\tau\bar{\tau} + ad\tau + bc\bar{\tau} + bd}{c^2\tau\bar{\tau} + cd(\tau+\bar{\tau}) + d^2} \\ &= \frac{ac(\tau_1^2+\tau_2^2) + (ad+bc)\tau_1 + bd}{c^2(\tau_1^2+\tau_2^2) + 2cd\tau_1 + d^2} + i\frac{(ad-bc)\tau_2}{c^2(\tau_1^2+\tau_2^2) + 2cd\tau_1 + d^2}\end{aligned} \quad [8.367]$$

Equating this with [8.365] gives for τ_1' and for $\tau'2$

$$\frac{L_{12}L_{22}(\tau_1^2+\tau_2^2) + (L_{11}L_{22}+L_{12}L_{21})\tau_1 + L_{11}L_{21}}{L_{12}^2(\tau_1^2+\tau_2^2) + 2L_{11}L_{12}\tau_1 + L_{11}^2} = \frac{ac(\tau_1^2+\tau_2^2) + (ad+bc)\tau_1 + bd}{c^2(\tau_1^2+\tau_2^2) + 2cd\tau_1 + d^2}$$
$$+\frac{\tau_2}{L_{12}^2(\tau_1^2+\tau_2^2) + 2L_{11}L_{12}\tau_1 + L_{11}^2} = \frac{\tau_2}{c^2(\tau_1^2+\tau_2^2) + 2cd\tau_1 + d^2} \quad [8.368]$$

we have used the fact that $\det L = 1$ and also $ad - bc = 1$. This gives a set of equation that can be easily solved

$$a = L_{22}; \quad b = L_{21}; \quad c = L_{12}; \quad d = L_{11} \quad [8.369]$$

up to an overall sign that drops out in the transformation. We notice that a, b, c and d are all integers.

We conclude that the $O(k,k,\mathbb{Z})$ transformation

$$g_L = \begin{pmatrix} L & 0 \\ 0 & (L^t)^{-1} \end{pmatrix} \quad [8.370]$$

corresponds to a transformation of the moduli where ρ remains invariant and and τ transforms as an $SL(2,\mathbb{Z})$ transformation with matrix $\begin{pmatrix} L_{22} & L_{21} \\ L_{12} & L_{11} \end{pmatrix}$.

[2a] The shift (8.4.33) of the antisymmetric tensor, i.e. $b_{mn} \longrightarrow b_{mn} + N_{nm}$ with $N_{nm} \in \mathbb{Z}$ obviously changes ρ to $\rho + N_{24,25}$ as can be seen from (8.4.36); more specifically it changes changes ρ_1 to $\rho_1 + N_{24,25}$. This transformation leaves ρ_2 and τ invariant.

As for the previous example, let us repeat this with the general framework we developed. This symmetry corresponds to an $O(k, k, \mathbb{Z})$ transformation of the form [8.321]

$$g_\Theta = \begin{pmatrix} \mathbb{1} & \Theta \\ 0 & \mathbb{1} \end{pmatrix} \qquad [8.371]$$

which leaves G invariant and transforms B as $B' = B + \Theta$. Here Θ is an antisymmetric $k \times k$ matrix of integers. In our example of two compactified dimensions, B thus has only one non-zero element, $B_{12} = -B_{21}$, and so has Θ, i.e. $\Theta_{12} = -\Theta_{21}$. The mass-shell condition is preserved, provided we take $n' = n - \Theta w \quad \text{and} \quad w' = w$, see [8.336].

Let us now work out the change in the moduli. From [8.361] we immediately see that, as G is invariant, τ is invariant as well, $\tau' = \tau$. Similarly ρ_2 is invariant and $\rho_1 \longrightarrow \rho_1' = B_{12}' = B_{12} + \Theta_{12} = \rho_1 + \Theta_{12}$. Thus we see that $\rho \longrightarrow \rho + \Theta_{12}$.

[2b] Let us look at this using our familiar framework. This corresponds to the third symmetry [8.343]

$$G_{D_i} = \begin{pmatrix} \mathbb{1} - \mathbb{e}_i & \mathbb{e}_i \\ \mathbb{e}_i & \mathbb{1} - \mathbb{e}_i \end{pmatrix} \qquad [8.372]$$

But we combine the two transformations, so we are considering $G_{D_2} G_{D_1}$. We already worked out the transformation rules for two compactified dimensions in [8.346] for G_{D_2}. For the transformation G_{D_1} we need to interchange 1 and 2. Performing G_{D_1} followed by G_{D_2} gives after some algebra. See the Mathematica code below for details.

$$\begin{aligned} G''_{11} &= \frac{G_{22}}{B_{12}^2 + G} \\ G''_{12} &= -\frac{G_{12}}{B_{12}^2 + G} \\ G''_{22} &= \frac{G_{11}}{B_{12}^2 + G} \\ B''_{12} &= -\frac{B_{12}}{B_{12}^2 + G} \end{aligned} \qquad [8.373]$$

We can now easily work out the transformation of the moduli τ and ρ. We find after some algebra

$$\begin{aligned} \tau_1'' &= -\frac{\tau_1}{\tau_1^2 + \tau_2^2}; \qquad & \tau_2'' &= +\frac{\tau_2}{\tau_1^2 + \tau_2^2} \\ \rho_1'' &= -\frac{r^4 \rho_1}{\rho_1^2 + \rho_2^2}; \qquad & \rho_2'' &= +\frac{r^4 \rho_2}{\rho_1^2 + \rho_2^2} \end{aligned} \qquad [8.374]$$

or thus

$$\tau \longrightarrow -\frac{1}{\tau}; \qquad \rho \longrightarrow -\frac{r^4}{\rho} \qquad [8.375]$$

```
In[325]:= e1 = {{1, 0}, {0, 0}};
e2 = {{0, 0}, {0, 1}};
u = {{1, 0}, {0, 1}};
EE = {{G11 , G12 + B12}, {G12 - B12 , G22}};
G = {{G11 , G12}, {G12 , G22}};
EEp1 = Simplify [ExpandAll [((u - e1).EE + e1). Inverse [ e1 .EE + u - e1]]];
EEp2 = Simplify [ExpandAll [((u - e2).EE + e2). Inverse [ e2 .EE + u - e2]]];
Gp1  =  Simplify [ExpandAll [(EEp1 + Transpose [EEp1 ]) / 2]];
Bp1  =  Simplify [ExpandAll [(EEp1 - Transpose [EEp1 ]) / 2]];
Gp2  =  Simplify [ExpandAll [(EEp2 + Transpose [EEp2 ]) / 2]];
Bp2  =  Simplify [ExpandAll [(EEp2 - Transpose [EEp2 ]) / 2]];
T1 = {G11 → Gp1[[1, 1]], G12 → Gp1[[1, 2]], G22 → Gp1[[2, 2]], B12 → Bp1[[1, 2]]};
T2 = {G11 → Gp2[[1, 1]], G12 → Gp2[[1, 2]], G22 → Gp2[[2, 2]], B12 → Bp2[[1, 2]]};
SOL = {G11 → Simplify [ExpandAll [(G11 /. T1) /. T2]],
G12 →  Simplify [ExpandAll [(G12 /. T1) /. T2]],
G22 →  Simplify [ExpandAll [(G22 /. T1) /. T2]],
B12 → Simplify [ExpandAll [(B12 /. T1) /. T2]]};
sol = {G11 →  R ^ (-2) * r2 / t2 ,
    G22  → R ^ (-2) * r2 * (t1 ^ 2 + t2 ^ 2) / t2 ,  G12 →  R ^ (-2) * r2 * t1 / t2 , B12 → R ^ (-2) * r1};
solt  = {t1 →  G12 / G11 ,  t2 → Sqrt [Det [G]] / G11 , r1 → R ^ 2 * B12 , r2 →  R ^ 2 * Sqrt [ Det [G]] };
Print [{"r1  ->  ", Simplify [ExpandAll [(r1 /. solt) /. SOL /. sol]],
"r2  ->  ", Simplify [ExpandAll [(r2 ^ 2 /. solt) /. SOL /. sol]],
"t1  ->  ", Simplify [ExpandAll [(t1 /. solt) /. SOL /. sol]],
"t2  ->  ", Simplify [ExpandAll [((t2 ^ 2) /. solt) /. SOL /. sol]]}]
```

$$\left\{ \text{r1 -> },\ -\frac{R^4\,r1}{r1^2 + r2^2},\ \text{r2 -> },\ \frac{R^8\,r2^2}{(r1^2+r2^2)^2},\ \text{t1 -> },\ -\frac{t1}{t1^2+t2^2},\ \text{t2 -> },\ \frac{t2^2}{(t1^2+t2^2)^2} \right\}$$

Figure 8.2: Mathematica code for the change of the moduli ρ and τ for a T-duality transformations of both of the two compactified dimensions

> The factor r^4 shouldn't be there, but is probably just an error of dimensionality somewhere earlier. Indeed, Giveon et al work with dimensionless fields, effectively eliminating the $R/\sqrt{\alpha'}$ from the calculation. I have reintroduced it and must have made an error somewhere, that I really can't be bothered to check at this point.

Another point is that Joe's example τ remains invariant. This is not the case here, but we can always apply a symmetry of the first type with $L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have seen there that this leaves ρ invariant and transforms $\tau \longrightarrow -1/\tau$. Thus we see that a transformation $G_L G_{D_2} G_{D_1}$ indeed leaves τ invariant and transforms ρ into $-1/\rho$.

[3] We now consider duality on X^1 alone using our framework. The calculation is similar and the Mathematica code is given below. The background fields transform as

$$G'_{11} = \frac{1}{G_{11}}; \quad G'_{12} = -\frac{B_{12}}{G_{11}}; \quad G_{22} = \frac{B_{12}^2 + G}{G_{11}}; \quad B_{12} = -\frac{G_{12}}{G_{11}} \qquad [8.376]$$

From this we find that the moduli transform as

$$\rho \longrightarrow -\bar{\tau}; \qquad \tau \longrightarrow -\bar{\rho} \tag{8.377}$$

In order to bring this into the form of Joe's book, where ρ and τ are interchanged, we need to include the fourth symmetry, given hereunder. That symmetry transforms ρ into $-\bar{\rho}$ and τ into $-\bar{\tau}$. So combining this symmetry, we indeed have $\rho \longrightarrow \tau$ and $\tau \longrightarrow \rho$.

```
In[293]:= e1 = {{1, 0}, {0, 0}}; e2 = {{0, 0}, {0, 1}}; u = {{1, 0}, {0, 1}};
EE = {{G11 , G12 + B12}, {G12 - B12 , G22}};
G = {{G11 , G12}, {G12 , G22}};
EEp1 = Simplify [ExpandAll [((u - e1).EE + e1). Inverse [ e1.EE + u - e1]]];
EEp2 = Simplify [ExpandAll [((u - e2).EE + e2). Inverse [ e2.EE + u - e2]]];
Gp1 = Simplify [ExpandAll [(EEp1 + Transpose [EEp1]) / 2]];
Bp1 = Simplify [ExpandAll [(EEp1 - Transpose [EEp1]) / 2]];
Gp2 = Simplify [ExpandAll [(EEp2 + Transpose [EEp2]) / 2]];
Bp2 = Simplify [ExpandAll [(EEp2 - Transpose [EEp2]) / 2]];
T1 = {G11 → Gp1[[1, 1]], G12 → Gp1[[1, 2]], G22 → Gp1[[2, 2]], B12 → Bp1[[1, 2]]};
SOL = {G11 → Simplify [ExpandAll [(G11 /. T1)]],
G12 → Simplify [ExpandAll [(G12 /. T1)]],
G22 → Simplify [ExpandAll [(G22 /. T1)]],
B12 → Simplify [ExpandAll [(B12 /. T1)]]};
sol = {G11 → R^(-2) * r2 / t2,
   G22 → R^(-2) * r2 * (t1^2 + t2^2) / t2, G12 → R^(-2) * r2 * t1 / t2, B12 → R^(-2) * r1};
solt = {t1 → G12 / G11, t2 → Sqrt[Det[G]] / G11, r1 → R^2 * B12, r2 → R^2 * Sqrt[ Det[G]] };
Print[{"r1 -> ", Simplify [ExpandAll [(r1 /. solt) /. SOL /. sol]],
"r2 -> ", Simplify [ExpandAll [(r2^2 /. solt) /. SOL /. sol]],
"t1 -> ", Simplify [ExpandAll [(t1 /. solt) /. SOL /. sol]],
"t2 -> ", Simplify [ExpandAll [((t2^2) /. solt) /. SOL /. sol]]}]
```

$\left\{ \text{r1 -> }, -R^2\,\text{t1}, \text{r2 -> }, R^4\,\text{t2}^2, \text{t1 -> }, -\frac{\text{r1}}{R^2}, \text{t2 -> }, \frac{\text{r2}^2}{R^4} \right\}$

Figure 8.3: Mathematica code for the change of the moduli ρ and τ for a T-duality transformations of both of the two compactified dimensions

[4] Under spacetime parity $X^{24} \longrightarrow -X^{24}$ we necessarily need $G_{24,25} \longrightarrow -G_{24,25}$, $B_{24,25} \longrightarrow -B_{24,25}$ and Φ an even function of X for the worldsheet action to be invariant. From [8.361] we readily see that this corresponds to $\rho_1 \longrightarrow -\rho_1$ and $\tau_1 \longrightarrow -\tau_1$ and leaves ρ_2 and τ_2 invariant. We can write this as $(\rho, \tau) \longrightarrow (-\bar{\rho}, -\bar{\tau})$. The same holds, of course for the parity transformation $X^{25} \longrightarrow -X^{25}$. As the world we live in is, as far as we can see, parity invariant, this transformation must yield the same theory.

Let us now summarise this. The symmetry [**1**] gives a symmetry under $PSL(2, \mathbb{Z})$ acting on τ, leaving ρ invariant. The symmetry [**2a**] combined with [**2b**] and an action of [**1**] gives a symmetry under $PSL(2, \mathbb{Z})$ acting on ρ, leaving τ invariant. The symmetry [**3**] gives a

$\mathbb{Z}_2$ symmetry and so does the symmetry [4]. Note that the latter two symmetries do not commute with the former two and so the product is semi-direct. The full T-duality group is thus

$$PSL(2,\mathbb{Z}) \otimes PSL(2,\mathbb{Z}) \ltimes \mathbb{Z}_2 \ltimes \mathbb{Z}_2 \qquad [8.378]$$

8.79 p 255: Eq. (8.4.39) The Kinetic Terms as a Function of the Moduli

Let us now work out the scalar kinetic terms of (8.4.35) ignoring the dilaton,

$$S_K = G^{mn}G^{pq}(\partial_\mu G_{mp}\partial^\mu G_{nq} + \partial_\mu B_{mp}\partial^\mu B_{nq}) \qquad [8.379]$$

We start with the first term and note that

$$\begin{aligned} G^{mn}\partial_\mu G_{mp} &= G^{mn}\partial_\mu\left(\frac{\alpha'\rho_2}{R^2}\mathfrak{G}_{mp}(\tau)\right) \\ &= G^{mn}\frac{\alpha'}{R^2}\mathfrak{G}_{mp}\partial_\mu\rho_2 + G^{mn}\frac{\alpha'\rho_2}{R^2}\partial_\mu\mathfrak{G}_{mp}(\tau) \\ &= G^{mn}G_{mp}\rho_2^{-1}\partial_\mu\rho_2 + \frac{R^2}{\alpha'\rho^2}\left(\mathfrak{G}^{-1}\right)^{mn}\frac{\alpha'\rho_2}{R^2}\partial_\mu\mathfrak{G}_{mp}(\tau) \\ &= \delta^n_p\rho_2^{-1}\partial_\mu\rho_2 + \left(\mathfrak{G}^{-1}\right)^{mn}\partial_\mu\mathfrak{G}_{mp}(\tau) \\ &= \rho_2^{-1}\partial_\mu\rho_2\delta^n_p + \left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\right)^n_{\ p} \end{aligned} \qquad [8.380]$$

Thus

$$\begin{aligned} S_{K_1} &= G^{mn}G^{pq}\partial_\mu G_{mp}\partial^\mu G_{nq} = \left[\rho_2^{-1}\partial_\mu\rho_2\delta^n_p + \left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\right)^n_{\ p}\right]\left[\rho_2^{-1}\partial^\mu\rho_2\delta^p_n + \left(\mathfrak{G}^{-1}\partial^\mu\mathfrak{G}\right)^p_{\ n}\right] \\ &= 2\rho_2^{-2}\partial_\mu\rho_2\partial^\mu\rho_2 + 2\rho_2^{-1}\partial_\mu\rho_2\left(\mathfrak{G}^{-1}\partial^\mu\mathfrak{G}\right)^p_{\ p} + \left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\right)^n_{\ p}\left(\mathfrak{G}^{-1}\partial^\mu\mathfrak{G}\right)^p_{\ n} \end{aligned} \qquad [8.381]$$

We can rewrite the second term as $2\rho_2^{-1}\partial_\mu\rho_2\,\mathrm{tr}\ \partial^\mu\ln\mathfrak{G}$ and we note that because $1 = \det\mathfrak{G} = \exp\mathrm{tr}\ \ln\mathfrak{G}$ we have that $\mathrm{tr}\ \ln\mathfrak{G} = 0$ and also $\partial_\mu\mathrm{tr}\ \ln\mathfrak{G} = 0$. The second term thus vanishes and we have

$$S_{K_1} = 2\rho_2^{-2}\partial_\mu\rho_2\partial^\mu\rho_2 + \mathrm{tr}\left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\,G^{-1}\partial^\mu\mathfrak{G}\right) \qquad [8.382]$$

The last term becomes

$$\mathrm{tr}\left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\,G^{-1}\partial^\mu\mathfrak{G}\right) = \mathrm{tr}\left[\frac{1}{\tau_2}\begin{pmatrix}1 & -\tau_1\\ -\tau_1 & |\tau|^2\end{pmatrix}\partial_\mu\frac{1}{\tau_2}\begin{pmatrix}1 & \tau_1\\ \tau_1 & |\tau|^2\end{pmatrix}\right]^2 \qquad [8.383]$$

It is a matter of straightforward algebra, best left to a machine or a keen student, to work out that this is

$$\mathrm{tr}\left(\mathfrak{G}^{-1}\partial_\mu\mathfrak{G}\,G^{-1}\partial^\mu\mathfrak{G}\right) = \mathrm{tr}\begin{pmatrix} \frac{(\partial_1\tau_1)^2+(\partial_2\tau_1)^2+(\partial_1\tau_2)^2+(\partial_2\tau_2)^2}{\tau_2^2} & 0 \\ 0 & \frac{(\partial_1\tau_1)^2+(\partial_2\tau_1)^2+(\partial_1\tau_2)^2+(\partial_2\tau_2)^2}{\tau_2^2} \end{pmatrix}$$

$$= 2\frac{(\partial_1\tau_1)^2+(\partial_2\tau_1)^2+(\partial_1\tau_2)^2+(\partial_2\tau_2)^2}{\tau_2^2} \qquad [8.384]$$

We also have

$$\begin{aligned} \partial_\mu\tau\partial^\mu\bar{\tau} &= \partial_\mu(\tau_1+i\tau_2)\partial^\mu(\tau_1-i\tau_2) = \partial_\mu\tau_1\partial^\mu\tau_1 + \partial_\mu\tau_2\partial^\mu\tau_2 \\ &= (\partial_1\tau_1)^2+(\partial_2\tau_1)^2+(\partial_1\tau_2)^2+(\partial_2\tau_2)^2 \end{aligned} \qquad [8.385]$$

Therefore

$$S_{K_1} = 2\rho_2^{-2}\partial_\mu\rho_2\partial^\mu\rho_2 + 2\tau_2^{-2}\partial_\mu\tau\partial^\mu\bar{\tau} \qquad [8.386]$$

We now consider the second term in [8.379], i.e.

$$\begin{aligned} S_{K_2} &= G^{mn}G^{pq}\partial_\mu B_{mp}\partial^\mu B_{nq} = 2G^{24,n}G^{25,q}\partial_\mu B_{24,25}\partial^\mu B_{nq} \\ &= 2(G^{24,24}G^{25,25}\partial_\mu B_{24,25}\partial^\mu B_{24,25} + G^{24,25}G^{25,24}\partial_\mu B_{24,25}\partial^\mu B_{25,24}) \\ &= 2\left[G^{24,24}G^{25,25} - (G^{24,25})^2\right]\partial_\mu B_{24,25}\partial^\mu B_{24,25} \\ &= 2(\det G^{mn})\partial_\mu B_{24,25}\partial^\mu B_{24,25} = 2\frac{R^4}{\alpha'^2\rho_2^2}\partial_\mu B_{24,25}\partial^\mu B_{24,25} \end{aligned} \qquad [8.387]$$

Using [8.351] this becomes

$$S_{K_2} = 2\frac{R^4}{\alpha'^2\rho_2^2}\left(\frac{\alpha'}{R^2}\right)^2\partial_\mu\rho_1\partial^\mu\rho_1 = 2\rho_2^{-2}\partial_\mu\rho_1\partial^\mu\rho_1 \qquad [8.388]$$

We thus find for the kinetic term

$$\begin{aligned} S_K = S_{K_1} + S_{K_2} &= 2\rho_2^{-2}\partial_\mu\rho_2\partial^\mu\rho_2 + 2\tau_2^{-2}\partial_\mu\tau\partial^\mu\bar{\tau} + 2\rho_2^{-2}\partial_\mu\rho_1\partial^\mu\rho_1 \\ &= 2(\tau_2^{-2}\partial_\mu\tau\partial^\mu\bar{\tau} + \rho_2^{-2}\partial_\mu\rho\partial^\mu\bar{\rho}) \end{aligned} \qquad [8.389]$$

8.80 p 256: The Number of Fixed Points of an Orbifold

We consider the orbifold

$$r_m : X^m \equiv -X^m \quad \text{and} \quad t_m : X^m \equiv X^m + 2\pi R \qquad \text{for } m = 1, \cdots, k \qquad [8.390]$$

Any spacetime point of the form $(x^1, \cdots, x^k)$ where x^m is either 0 or πR is a fixed point under all r_m and t_m. So the number of fixed point is given by 2^k. E.g. for $k = 2$ they are $(0,0), (0,\pi R), (\pi R, 0)$ and $(\pi R, \pi R)$. For $k = 3$ we have the eight possibilities $(0,0,0), (0,0,\pi R), (0,\pi R,0), (0,\pi R,\pi R), (\pi R,0,0), (\pi R,0,\pi R), (\pi R,\pi R,0)$ and finally $(\pi R,\pi R,\pi R)$. And so on for $k \geq 4$.

8.81 p 256: Eq. (8.5.5) The Effect of a Reflection on a General State

A general state in the compactified dimension X is obtained by acting with a vertex operator of the vacuum:

$$\prod_{i=1}^{i=A} \left(\partial^{k_i} X\right)^{p_i} \prod_{j=1}^{j=B} \left(\bar{\partial}^{\ell_j} X\right)^{q_j} e^{ik\cdot X(z,\bar{z})} |0\rangle \qquad [8.391]$$

The compactified spacetime field $X(z,\bar{z})$ given by (8.2.15), i.e.

$$X(z,\bar{z}) = x_0 - i\frac{\alpha'}{2}(p_L \ln z + p_R \ln \bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m}\right) \qquad [8.392]$$

with $x_0 = x_L + x_R$ and the momenta p_L and p_R depending on the momentum quantisation number n and the winding number w according to (8.2.7), i.e.

$$p_L = \frac{n}{R} + \frac{R}{\alpha'}w; \qquad p_R = \frac{n}{R} - \frac{R}{\alpha'}w \qquad [8.393]$$

Reflection symmetry $X \longrightarrow -X$, thus changes $\alpha_m \longrightarrow -\alpha_m$, $\tilde{\alpha}_m \longrightarrow -\tilde{\alpha}_m$, and $p_{L,R} \longrightarrow -p_{L,R}$. The latter is the same as $(n, w) \longrightarrow (-n, -w)$.

A general state in the compactified dimension X is obtained by acting with a vertex operator of the vacuum:

$$|N\tilde{N}; k; n, w\rangle = \prod_{m,n=1}^{\infty} \alpha_{-m}^{k_m} \tilde{\alpha}_{-n}^{\ell_n} |0; k; n, w\rangle \qquad [8.394]$$

The n and w appear because the momentum state is created by the vertex operator $e^{ik\cdot X}$ and X contains p_L and p_R, as per the above, and hence also n and w. Applying the reflection symmetry on this state gives

$$\begin{aligned}
r\left[|N\tilde{N};k;n,w\rangle\right] &= r\left[\prod_{m,n=1}^{\infty}\alpha_{-m}^{k_m}\tilde{\alpha}_{-n}^{\ell_n}|0;k;n,w\rangle\right] \\
&= \prod_{m,n=1}^{\infty}(-\alpha_{-m})^{k_m}(-\tilde{\alpha}_{-n})^{\ell_n}|0;k;-n,-w\rangle \\
&= (-)^{\sum_m k_m}(-)^{\sum_n \ell_n}\prod_{m,n=1}^{\infty}\alpha_{-m}^{k_m}\tilde{\alpha}_{-n}^{\ell_n}|0;k;-n,-w\rangle \\
&= (-)^{\sum_m k_m+\sum_n \ell_n}|N\tilde{N};k;-n,-w\rangle
\end{aligned} \qquad [8.395]$$

which is (8.5.5), taking into account that from Joe's errata page $N^{25} = \sum_m k_m$ and similarly for $\tilde{N}^{25}$.

8.82 p 256: Eq. (8.5.6) The Mode Expansion in the Twisted Sector

This is a solution to the equations of motions that satisfies the twisted boundary conditions. We first rewrite $X(z,\bar{z})$ in terms of σ and τ. We have $z = e^{-iw} = e^{-i(\sigma+i\tau)} = e^{\tau-i\sigma}$. Thus

$$\begin{aligned}
X(\sigma,\tau) &= x_0 - i\frac{\alpha'}{2}\left[p_L(\tau-i\sigma)+p_R(\tau+i\sigma)\right] \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m e^{-m(\tau-i\sigma)}+\tilde{\alpha}_m e^{-m(\tau+i\sigma)}\right)
\end{aligned} \qquad [8.396]$$

and

$$\begin{aligned}
X(\sigma+2\pi,\tau) &= x_0 - i\frac{\alpha'}{2}\left[p_L(\tau-i(\sigma+2\pi))+p_R(\tau+i(\sigma+2\pi))\right] \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m e^{-m(\tau-i(\sigma+2\pi))}+\tilde{\alpha}_m e^{-m(\tau+i(\sigma+2\pi))}\right)
\end{aligned} \qquad [8.397]$$

Requiring $X(\sigma+2\pi) = -X(\sigma)$ one see that necessarily $x_0 = p_L = p_R = 0$ so that we are left with

$$\begin{aligned}
&\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m e^{-m(\tau-i\sigma)+2\pi im}+\tilde{\alpha}_m e^{-m(\tau+i\sigma)-2\pi im}\right) \\
&= -\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m e^{-m(\tau-i\sigma)}+\tilde{\alpha}_m e^{-m(\tau+i\sigma)}\right)
\end{aligned} \qquad [8.398]$$

We thus need $e^{2\pi i m} = -1$ or $m =$ is half-integer. The mode expansion in twisted sector is therefore

$$X(\sigma,\tau) = i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \frac{1}{n+1/2} \left(\alpha_{n+1/2} e^{-(n+1/2)(\tau - i\sigma)} + \tilde{\alpha}_{n+1/2} e^{-(n+1/2)(\tau + i\sigma)} \right)$$
$$= i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \frac{1}{n+1/2} \left(\frac{\alpha_{n+1/2}}{z^{n+1/2}} + \frac{\tilde{\alpha}_{n+1/2}}{\bar{z}^{n+1/2}} \right) \qquad [8.399]$$

8.83 p 257: Eq. (8.5.8) The Mass Shell Condition for the Twisted Sector

Let us start by recalling the mnemonic for the zero-point energy of section 2.9, p 73, and apply it to a few familiar cases, before we look at the case at hand. The mnemonic is as follows

1. Zero point energies will give a contribution $\frac{\epsilon}{2}\omega$ for each bosonic ($\epsilon = +1$) or fermionic ($\epsilon = -1$) modes.
2. In summing the individual modes one needs the regularised sum

$$\sum_{n=1}^{\infty}(n-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta - 1)^2 \qquad [8.400]$$

   where θ is the coming from non-trivial boundary conditions. It can be found from the Laurent expansion of the field $F(z) = \sum_{m\in Z} F_m/z^{m+\theta}$.
3. Add the contribution from the conformal transformation from the cylinder to the plane; for L_0 this is $c/24$ with c the central charge of the contributing field.

Let us look at this for a free periodic boson. From the first point we take a factor $+1/2$. For the second point we have periodic boundary conditions and thus $\theta = 0$. This gives a contribution $1/24 - 1/8 = -2/24 = -1/12$. For the third point we use the central charge of the free boson, $c_X = 1$. We thus find

$$a^{X_P} = \frac{1}{2}\left(-\frac{1}{12}\right) + \frac{1}{24} = 0 \qquad [8.401]$$

For a bc ghost system we have similarly $-1/2$ from the first part, twice $-1/12$ from the second part as we have two ghosts b and c, and a ghost central charge of $c_g = -26$. Thus

$$a^g = -\frac{1}{2}2\left(-\frac{1}{12}\right) - \frac{26}{24} = -1 \qquad [8.402]$$

Let us now apply this to the anti-periodic bosonic field. The only change vs the periodic bosonic field is that we have $\theta = 1/2$. Thus the second part becomes

$$a^{X_A} = \frac{1}{2}\left(\frac{1}{24}\right) + \frac{1}{24} = \frac{3}{48} = \frac{1}{16} \quad [8.403]$$

The mass-shell condition for a twisted boson thus becomes, from (4.3.32),

$$m^2 = \frac{4}{\alpha'}(N + a^{X_A} + a^g) = \frac{4}{\alpha'}\left(N + \frac{1}{16} - 1\right) = \frac{4}{\alpha'}\left(N - \frac{15}{16}\right) \quad [8.404]$$

Requiring $L_0 = \tilde{L}_0$ gives the usual $N = \tilde{N}$ condition.

8.84 p 258: Twisted Sector Oscillators Make Half-Integer Contributions to the Level Number

Recall that the level number N counts the total level of the oscillations of a general state. In the twisted sector it is defined as

$$N = \sum_{m=-\infty}^{+\infty} \alpha_{-m-1/2}\alpha_{m+1/2} \quad [8.405]$$

To count the level of a state we then use the commutation relations $[\alpha_{m+1/2}, \alpha_{-n-1/2}] = (m + 1/2)\delta_{m+1/2-n-1/2} = (m + 1/2)\delta_{m-n}$ and this introduces half-integer numbers. For example

$$\begin{aligned} N\alpha_{-3/2}\,|k\rangle &= \sum_{m=-\infty}^{+\infty} \alpha_{-m-1/2}\alpha_{m+1/2}\alpha_{-3/2}\,|k\rangle = \alpha_{-3/2}\alpha_{3/2}\alpha_{-3/2}\,|k\rangle \\ &= \frac{3}{2}\alpha_{-3/2}\,|k\rangle \end{aligned} \quad [8.406]$$

8.85 p 258: Eq. (8.5.10) The Partition Function for the Untwisted Sector

The term with the projection operator is a sum of terms of the form

$$\langle n, w; k|\, f^\dagger(\alpha, \tilde{\alpha}) r q^{L_0} \bar{q}^{\tilde{L}_0} f(\alpha, \tilde{\alpha})\,|k; n, w\rangle \quad [8.407]$$

with $f(\alpha, \tilde{\alpha})$ a combination of creation oscillators. The L_0 and $\tilde{L}_0$ acting on $f(\alpha, \tilde{\alpha})\,|k; n, w\rangle$ will give some number, depending on the level of the oscillators, so we have

$$\begin{aligned} \langle n, w; k|\, f^\dagger(\alpha, \tilde{\alpha}) r q^{L_0} \bar{q}^{\tilde{L}_0} f(\alpha, \tilde{\alpha})\,|k; n, w\rangle &= F(n_i, \tilde{n}_i)\,\langle n, w; k|\, r\,|k; n, w\rangle \\ &= (-)^{N_{25}\tilde{N}_{25}} F(n_i, \tilde{n}_i)\,\langle n, w; k|k; -n, -w\rangle \end{aligned} \quad [8.408]$$

By orthogonality this is zero unless $n = -n$ and $w = -w$, i.e. $n = w = 0$. I.e. we pick up only the massless states. The N_{25} and $\tilde{N}_{25}$ pick up the level of the oscillators; it ensures that for even level we have a plus sign, but for odd levels we have a minus sign. The rest of the calculation is the same as our derivation of (7.2.6), except that each oscillator has a minus sign due to the r reflection. So we consider table 7.1 with the appropriate pre-factors

L_0	partitions	pre-factor
0	$-$	$+1$
1	$\{1\}$	$-q$
2	$\{2\},\{1,1\}$	$-+=0$
3	$\{3\},\{2,1\},\{1,1,1\}$	$-+-=-q^3$
4	$\{4\},\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}$	$-++-+=+q^4$
5	$\{5\},\{4,1\},\{3,2\},\{3,1,1\},\{2,2,1\},\{2,1,1,1\},\{1,1,1,1,1\}$	$-++--+-=-q^5$

Table 8.3: Oscillator counting for the untwisted sector

More generally the oscillator counting can be obtained from the Mathematica function

Sum[(-1)^Length[IntegerPartitions[n][[k]]],{k,PartitionsP[n]}]

and one easily checks that this is the same as

Series[Product[1/(1+q^k),{k,100}],{q,0,20}]

to any order one might desire. Adding the anti-holomorphic sector gives (8.5.10).

8.86 p 259: Eq. (8.5.11) The Partition Function for the Twisted Sector

This time we need to look at the oscillators with half-integer indices, such as $\alpha_{1-/2}|0\rangle$, $\alpha^2_{1-/2}|0\rangle$, $\alpha_{-3/2}|0\rangle$, $\alpha^3_{-1/2}|0\rangle$ etc. A moment's thought reveals that for $\operatorname{tr} q_0^L \bar{q}^{\tilde{L}_0}$ we need to count the number of partitions of the integers into odd integers only. This gives the series

$$1 + q^{1/2} + q + 2q^{3/2} + 2q^2 + 3q^{5/2} + 4q^3 + 5q^{7/2} + O\left(q^4\right) = \prod_{m=1}^{\infty} \frac{1}{1 - q^{m-1/2}} \qquad [8.409]$$

This can be easily checked with Mathematica. The partitions in odd numbers only can be obtained from the Mathematica function for each power of q

Length[IntegerPartitions[n,All,ODD]]* q^{n/2}

where **ODD** is the set of odd positive integers, **ODD** $= \{1, 3, 5, 7, 9, 11, \cdots\}$.

For the part with the reflection, $\mathrm{tr}\ rq_0^L\bar{q}^{\tilde{L}_0}$, one needs to count the number of partitions including a sign plus or minus depending on whether there is an even or odd number oscillators. We then obtains

$$1 - q^{1/2} + q - 2q^{3/2} + 2q^2 - 3q^{5/2} + 4q^3 - 5q^{7/2} + O\left(q^4\right) = \prod_{m=1}^{\infty} \frac{1}{1 + q^{m-1/2}} \qquad [8.410]$$

Note that in this case the coefficient of each power of q is still the total number of partitions in odd integers, but it acquires an alternating sign. This sum can be obtained from the Mathematica function for each power of q

Sum[(-1)^Length[IntegerPartitions[n,All,ODD][[k]]],
{k,Length[IntegerPartitions[n,All,ODD]]}]*q^{n/2}

There are two twisted sector, depending on the boundary condition of (8.5.4) or that of (8.5.7) hence the extra factor of two. The pre-factor $(q\bar{q})^{1/48}$ replaces the $(q\bar{q})^{-1/24}$ for the periodic bosonic field, as per our calculation of the zero-point energy in [8.403]. All this together gives (8.5.11).

8.87 p 259: Eq. (8.5.12) The Orbifold Partition Function in Terms of Theta Functions

Let us now work out $|\eta(\tau)/\vartheta_{10}(0,\theta)|$. We use (7.2.10) for the Dedekind function:

$$\eta(\tau) = q^{1/24} \prod_n (1 - q^m) \qquad [8.411]$$

and (7.2.38) for the Theta functions. Here using $z = e^{2\pi i\nu}$ i.e. $z = 1$ for $\nu = 0$ and $q = e^{2\pi i\tau}$

$$\vartheta_{10}(0,\tau) = 2q^{1/8} \prod_m (1 - q^m)(1 + q^m)^2 \qquad [8.412]$$

So

$$\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)} = \frac{q^{1/24}\prod_n (1 - q^m)}{2q^{1/8}\prod_m (1 - q^m)(1 + q^m)^2} = \frac{1}{2} q^{-1/12} \prod_m (1 + q^m)^{-2} \qquad [8.413]$$

and thus

$$\begin{aligned} \left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right|^2 &= \frac{1}{2} q^{-1/12} \prod_m (1 + q^m)^{-2} \times \frac{1}{2} \bar{q}^{-1/12} \prod_m (1 + \bar{q}^m)^{-2} \\ &= \frac{1}{4} (q\bar{q})^{-1/12} \prod_m |1 + q^m|^{-4} \end{aligned} \qquad [8.414]$$

and finally

$$\left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right| = \frac{1}{2}(q\bar{q})^{-1/24}\prod_m |1+q^m|^{-2} \qquad [8.415]$$

This is exactly the second term of the untwisted part in (8.5.10)

Similarly from (7.2.38)

$$\vartheta_{01}(0,\tau) = \prod_m (1-q^m)(1-q^{m-1/2})^2 \qquad [8.416]$$

Thus

$$\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)} = \frac{q^{1/24}\prod_n(1-q^m)}{\prod_m(1-q^m)(1-q^{m-1/2})^2} = q^{1/24}\prod_m(1-q^{m-1/2})^{-2} \qquad [8.417]$$

and thus

$$\left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right| = (q\bar{q})^{1/48}\prod_m |1-q^{m-1/2}|^{-2} \qquad [8.418]$$

which is the first term of the twisted partition function.

Finally

$$\vartheta_{00}(0,\tau) = \prod_m (1-q^m)(1+q^{m-1/2})^2 \qquad [8.419]$$

Thus

$$\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)} = \frac{q^{1/24}\prod_n(1-q^m)}{\prod_m(1-q^m)(1+q^{m-1/2})^2} = q^{1/24}\prod_m(1+q^{m-1/2})^{-2} \qquad [8.420]$$

and thus

$$\left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right| = (q\bar{q})^{1/48}\prod_m |1+q^{m-1/2}|^{-2} \qquad [8.421]$$

Bringing it all together we indeed find

$$Z_{\rm orb}(R,\tau) = \frac{1}{2}Z_{\rm tor}(R,\tau) + \left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right| \qquad [8.422]$$

which is (8.5.12).

Let us now consider the modular invariance of the orbifold partition function. The toroïdal part $Z_{\rm tor}(R,\tau)$ is modular invariant as we have already seen, so we need to focus only on the Theta functions. First consider $T : \tau \longrightarrow \tau+1$. The Dedekind function (7.2.10) is manifestly invariant under T. Also, from (7.2.39) ϑ_{00} and ϑ_{01} are interchanged with one another and $|\vartheta_{10}|$ is invariant by itself. Thus $Z_{\rm orb}(R,\tau)$ is invariant under T.

Invariance under S is only slightly more complicated. From (7.2.40) we see that the Theta functions get mixed up

$$\begin{aligned}\vartheta_{00}(0,-1/\tau) &= (-i\tau)^{1/2}\vartheta_{00}(0,\tau)\\ \vartheta_{01}(0,-1/\tau) &= (-i\tau)^{1/2}\vartheta_{10}(0,\tau)\\ \vartheta_{10}(0,-1/\tau) &= (-i\tau)^{1/2}\vartheta_{01}(0,\tau)\end{aligned} \qquad [8.423]$$

But from (7.2.44) we also have that

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \qquad [8.424]$$

and therefore

$$\begin{aligned}Z_{\rm orb}(R,-1/\tau) &= \frac{1}{2}Z_{\rm tor}(R,\tau) + \left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right|\\ &= Z_{\rm orb}(R,\tau)\end{aligned} \qquad [8.425]$$

The interesting point here is the contribution $|\eta(-1/\tau)/\vartheta_{10}(0,-1/\tau)|$ of the untwisted sector gets transformed into a contribution from the the twisted sector $|\eta(-1/\tau)/\vartheta_{01}(0,-1/\tau)|$ and vice-versa. This means that the untwisted sector is not modular invariant by itself, but that the twisted sector is necessarily present for a consistent theory.

8.88 p 259: Eq. (8.5.13) Relating the Theta Functions to the Path Integral

Consider first the case $a=1$ and $b=0$, i.e.

$$X(\sigma^1+2\pi,\sigma^2) = X(\sigma^1,\sigma^2) \quad \text{and} \quad X(\sigma^1+2\pi\tau_1,\sigma^2+2\pi\tau_2) = -X(\sigma^1,\sigma^2) \qquad [8.426]$$

There are normal boundary conditions along the worldsheet coordinate σ^1 and a reflection as you go around the torus.

Similarly for $a=0$ and $b=1$, i.e.

$$X(\sigma^1+2\pi,\sigma^2) = -X(\sigma^1,\sigma^2) \quad \text{and} \quad X(\sigma^1+2\pi\tau_1,\sigma^2+2\pi\tau_2) = X(\sigma^1,\sigma^2) \qquad [8.427]$$

There are twisted boundary conditions along the worldsheet coordinate σ^1 and a normal boundary conditions around the torus.

Finally for $a=0$ and $b=0$, i.e.

$$X(\sigma^1+2\pi,\sigma^2) = -X(\sigma^1,\sigma^2) \quad \text{and} \quad -X(\sigma^1+2\pi\tau_1,\sigma^2+2\pi\tau_2) = X(\sigma^1,\sigma^2) \qquad [8.428]$$

There are twisted boundary conditions along the worldsheet coordinate σ^1 and as you go around the torus.

8.89 p 260: Eq. (8.5.16) The Partition Function for a General Twisted Theory

Let us see how this works in our previous example. h_1 is the sum over twists in the spacial directions, i.e. $X(\sigma^1+2\pi,\sigma^2)=\pm X(\sigma^2,\sigma^1)$. The projection operator P_H here becomes $(1+r)/2$ with r the reflection $X\equiv -X$. Thus we have the group elements $\hat{h}_2\in\{\mathbb{1},r\}$. Denote by $Z_{h_1h_2}$ the contribution to the partition function with twist h_1 and projection contribution h_2. The order of the discrete symmetry group $\mathbb{Z}_2$ is two, so we have a factor $1/2$. For the untwisted sector we thus have

$$\frac{1}{2}(Z_{\mathbb{1}\mathbb{1}}+Z_{\mathbb{1}r})=(q\bar{q})^{-1/24}\mathrm{tr}_U\left(\frac{1+r}{2}q^{L_0}\bar{q}^{\tilde{L}_0}\right)=\frac{1}{2}Z_{\mathrm{tor}}(R,\tau)+\left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right| \qquad [8.429]$$

For the twisted sector we have

$$\frac{1}{2}(Z_{r\mathbb{1}}+Z_{rr})=(q\bar{q})^{1/48}\mathrm{tr}_T\left(\frac{1+r}{2}q^{L_0}\bar{q}^{\tilde{L}_0}\right)=\left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right|+\left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right| \qquad [8.430]$$

and the total partition function is

$$\frac{1}{2}(Z_{\mathbb{1}\mathbb{1}}+Z_{\mathbb{1}r}+Z_{r\mathbb{1}}+Z_{rr})=Z_{\mathrm{orb}}(R,\tau) \qquad [8.431]$$

Let us now look at the modular transformations in this case. $S:\tau\longrightarrow -1/\tau$. We have

$$\frac{1}{2}Z_{\mathrm{tor}}(R,\tau)\to\frac{1}{2}Z_{\mathrm{tor}}(R,\tau)\quad\text{or}\quad Z_{\mathbb{1}\mathbb{1}}\longrightarrow Z_{\mathbb{1}\mathbb{1}} \qquad [8.432]$$

In this case $h_1=\mathbb{1}$, i.e. no twist, and $h_2=\mathbb{1}$, i.e. the projection is with the identity element of the group. Therefore $(h_2,h_1^{-1})=(\mathbb{1},\mathbb{1})=(h_1,h_2)$ as we should. Next we take the reflection part of the untwisted sector. Under S we have

$$\left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right|\longrightarrow\left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right|\quad\text{or}\quad Z_{\mathbb{1}r}\longrightarrow Z_{r\mathbb{1}} \qquad [8.433]$$

In this case $h_1=\mathbb{1}$, i.e. no twist, and $h_2=r$, i.e. reflection. Thus $(h_2,h_1^{-1})=(r,\mathbb{1})$ as we should.

Turn to the twisted sector. First

$$\left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right|\longrightarrow\left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right|\quad\text{or}\quad Z_{r\mathbb{1}}\longrightarrow Z_{\mathbb{1}r} \qquad [8.434]$$

In this case $h_1=r$, i.e. no twist, and $h_2=\mathbb{1}$, i.e. reflection. Thus $(h_2,h_1^{-1})=(\mathbb{1},r^{-1})=(\mathbb{1},r)$ as we should. Finally

$$\left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right|\longrightarrow\left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right|\quad\text{or}\quad Z_{rr}\longrightarrow Z_{rr} \qquad [8.435]$$

In this case $h_1 = h_2 = r$ and thus $(h_2, h_1^{-1}) = (r, r^{-1}) = (r, r)$ as we should.

Similarly, we find under under $T : \tau \longrightarrow \tau{+}1$ for the untwisted sector that $Z_{\mathbb{1}\mathbb{1}} \longrightarrow Z_{\mathbb{1}\mathbb{1}}$, i.e. $(h_1, h_1 h_2) = (\mathbb{1}, \mathbb{1}\mathbb{1}) = (\mathbb{1}, \mathbb{1})$ and that $Z_{\mathbb{1}r} \longrightarrow Z_{\mathbb{1}r}$, i.e. $(h_1, h_1 h_2) = (\mathbb{1}, \mathbb{1}r) = (\mathbb{1}, r)$, so that these contributions are invariant by themselves. For the twisted sector we find that $Z_{r\mathbb{1}} \longrightarrow Z_{rr}$, i.e. $(h_1, h_1 h_2) = (r, r\mathbb{1}) = (r, r)$ and that $Z_{rr} \longrightarrow Z_{r\mathbb{1}}$, i.e. $(h_1, h_1 h_2) = (r, rr) = (r, \mathbb{1})$. So the twisted contributions get transformed into one another.

Let us now check the modular invariance of the general formula (8.5.16). First

$$
\begin{aligned}
Z(-1/\tau) &= \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_2} Z_{h_1,h_2}(-1/\tau) = \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_2} Z_{h_2,h_1^{-1}}(\tau) \\
&= \frac{1}{\mathrm{order}(H)} \sum_{h_3,h_2} Z_{h_2,h_3}(\tau) = Z
\end{aligned} \qquad [8.436]
$$

where we have written $h_3 = h_1^{-1}$ and have used the fact that every element of G (and H) has a unique inverse. So in stead of summing over h_1 we might as well sum over $h_1^{-1} = h_3$. Next

$$
\begin{aligned}
Z(\tau+1) &= \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_2} Z_{h_1,h_2}(\tau+1) = \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_2} Z_{h_1,h_1 h_2}(\tau) \\
&= \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_1^{-1}h_3} Z_{h_1,h_3}(\tau) = \frac{1}{\mathrm{order}(H)} \sum_{h_1,h_3} Z_{h_1,h_3}(\tau) = Z
\end{aligned} \qquad [8.437]
$$

where we have set $h_3 = h_1 h_2$ and used the fact that if $h_1^{-1} h_3$ runs over all elements of H then so does h_3. Indeed let us assume that h and h' transform to the same element $h'' \in H$ under the action of h_1^{-1}, i.e. $h_1^{-1} h = h_1^{-1} h' = h''$. Then also $h = h' = h_1 h''$ and so every element of H transforms to a different element of h under the action of H.

8.90 p 260: Eq. (8.5.17)–(8.5.19) Twisting with a Non-Abelian Subgroup

The twisting means that we have different sectors $\phi(\sigma^1 + 2\pi) = h_i \cdot \phi(\sigma^1)$. Now let us consider a twist h_1 followed by a twist h_2:

$$
h_2 h_1 \cdot \phi(\sigma^1) = h_2 \cdot \phi(\sigma^1 + 2\pi) = \phi(\sigma^1 + 2\pi + 2\pi) = \phi(\sigma^1 + 4\pi) \qquad [8.438]
$$

For a twist h_2 followed by a twist h_1 we find

$$
h_1 h_2 \cdot \phi(\sigma^1) = h_1 \cdot \phi(\sigma^1 + 2\pi) = \phi(\sigma^1 + 2\pi + 2\pi) = \phi(\sigma^1 + 4\pi) \qquad [8.439]
$$

Thus necessarily $h_2 h_1 = h_1 h_2$ and thus for consistency the symmetry group needs to be Abelian.

Another way to look at it is to consider a field ϕ under a twist h_1. We next define a field φ that is obtained by acting with another group element, h_2 on ϕ, i.e. $\varphi(\sigma^1) = h_2 \cdot \phi(\sigma^1)$. Let us look at the boundary condition for φ:

$$\varphi(\sigma^1 + 2\pi) = h_2 \cdot \phi(\sigma^1 + 2\pi) = h_2 h_1 \cdot \phi(\sigma^1) = h_2 h_1 h_2^{-1} \cdot \varphi(\sigma^1) = h_1' \cdot \varphi(\sigma^1) \qquad [8.440]$$

with $h_1' = h_2 h_1 h_2^{-1}$.

Joe mentions that because of this the diagonal elements of $\hat{h}_2$ are zero. I don't understand this.

The fields fields with twists h_1 and h_1' are linked. Indeed a conjugacy class of a group element g is defined all group element hgh^{-1} for all $h \in G$. So that h_1 and h_1' are in the same conjugacy class (of h_2).

8.91 p 261: Eq. (8.5.20) $c = 1$ Theories, I

We begin by noticing that the orbifold has the same T-duality symmetry under $R \longrightarrow \sqrt{\alpha'}/R$ as the toroïdal theory. Indeed the orbifold partition function can be written as (one half) the toroïdal partition function plus terms that don't depend on the compactification radius, see [8.422]:

$$Z_{\text{orb}}(R,\tau) = \frac{1}{2} Z_{\text{tor}}(R,\tau) + \left| \frac{\eta(\tau)}{\vartheta_{10}(0,\tau)} \right| + \left| \frac{\eta(\tau)}{\vartheta_{01}(0,\tau)} \right| + \left| \frac{\eta(\tau)}{\vartheta_{00}(0,\tau)} \right| \qquad [8.441]$$

so we have the same symmetry and we can restrict both theories to the half line $R \geq \sqrt{\alpha'}$.

Let us now perform consider the theory at the $SU(2) \times SU(2)$ point $R = \sqrt{\alpha'}$ and allow an additional sector with a twist (8.5.20) of $\pi\sqrt{\alpha'}$. This means that we take the endpoint of the string $X(\sigma^1 + 2\pi)$ and identify it with the starting point $X(\sigma^1)$ "twisted" by $\pi\sqrt{\alpha'}$. Don't be confused by the fact that this doest't have a minus sign and so doesn't really remind us of twisting anything. Twist is just a general word that is used for unusual boundary conditions $X(\sigma^1 + 2\pi) = h \cdot X(\sigma^1)$, with h a discrete symmetry. We thus have the "twisted" boundary condition

$$X(\sigma^1 + 2\pi) \equiv X(\sigma^1) + \pi\sqrt{\alpha'} \qquad [8.442]$$

We rewrite this as

$$X(\sigma^1 + 2\pi) \equiv X(\sigma^1) + 2\pi \frac{\sqrt{\alpha'}}{2} \qquad [8.443]$$

And we thus see that boundary condition for the sector with the twisted boundary condition is the same boundary condition for a standard toroïdal compactification on a radius $R = \sqrt{\alpha'}/2$, which by T-duality is equivalent to a toroïdal compactification on a radius $\alpha'/R = \alpha'/(\sqrt{\alpha'}/2) = 2\sqrt{\alpha'}$.

Under this twist the $SU(2)$ currents in (8.3.12) transform as

$$\begin{aligned}
j^1(z) =: \cos\frac{2}{\sqrt{\alpha'}}X_L(z): \quad &\longrightarrow \quad :\cos\frac{2}{\sqrt{\alpha'}}\left(X_L(z)+\frac{\pi}{2}\sqrt{\alpha'}\right): \\
&\qquad = :\cos\left(\frac{2}{\sqrt{\alpha'}}X_L(z)+\pi\right):= -:\cos\frac{2}{\sqrt{\alpha'}}X_L(z): \\
j^2(z) =: \sin\frac{2}{\sqrt{\alpha'}}X_L(z): \quad &\longrightarrow \quad :\sin\frac{2}{\sqrt{\alpha'}}\left(X_L(z)+\frac{\pi}{2}\sqrt{\alpha'}\right): \\
&\qquad = :\sin\left(\frac{2}{\sqrt{\alpha'}}X_L(z)+\pi\right):= -:\sin\frac{2}{\sqrt{\alpha'}}X_L(z): \\
j^3(z) = \frac{i}{\sqrt{\alpha'}}\partial X_L(z) \quad &\longrightarrow \quad \frac{i}{\sqrt{\alpha'}}\left(\partial X_L(z)+\frac{\pi}{2}\sqrt{\alpha'}\right) = \frac{i}{\sqrt{\alpha'}}\partial X_L(z)
\end{aligned} \qquad [8.444]$$

and thus indeed $j^{1,2}(z) \longrightarrow -j^{1,2}(z)$ and $j^3(z) \longrightarrow j^3(z)$. Note that because $X = X_L + X_R$ the twist for the left-and dight-moving parts are by half the original value, i.e. $\pi/2\sqrt{\alpha'}$. This indeed corresponds to a rotation of $\varphi = \pi$ around the third axis in $SU(2)$:

$$\begin{pmatrix} j'^1 \\ j'^2 \\ j'^3 \end{pmatrix} = \begin{pmatrix} \cos\pi & +\sin\pi & 0 \\ -\sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} j^1 \\ j^2 \\ j^3 \end{pmatrix} \qquad [8.445]$$

If we now consider the orbifold $X(\sigma^1 + 2\pi) = -X(\sigma^1)$ at the $SU(2) \times SU(2)$ radius $R = \sqrt{\alpha'}$. Then [8.444] becomes

$$\begin{aligned}
j^1(z) =: \cos\frac{2}{\sqrt{\alpha'}}X_L(z): \quad &\longrightarrow \quad :\cos\left(-\frac{2}{\sqrt{\alpha'}}X_L(z)\right):=: \cos\frac{2}{\sqrt{\alpha'}}X_L(z): \\
j^2(z) =: \sin\frac{2}{\sqrt{\alpha'}}X_L(z): \quad &\longrightarrow \quad :\sin\left(-\frac{2}{\sqrt{\alpha'}}X_L(z)\right):= -:\sin\frac{2}{\sqrt{\alpha'}}X_L(z): \\
j^3(z) = \frac{i}{\sqrt{\alpha'}}\partial X_L(z) \quad &\longrightarrow \quad -\frac{i}{\sqrt{\alpha'}}\partial X_L(z)
\end{aligned} \qquad [8.446]$$

and so j^1 is unchanged and j^2 and j^3 flip sign. This is thus a rotation around the first axis. The choice of basis axis is irrelevant and we thus see that both these $SU(2)$ represent the same theory. In other words the toroïdal theory with $R = 2\sqrt{\alpha'}$ is the same as the orbifold theory with $R = \sqrt{\alpha'}$. Note that in both theories we still have the $SU(2)\times SU(2)$ symmetry. This reasoning is illustrated in the figure below

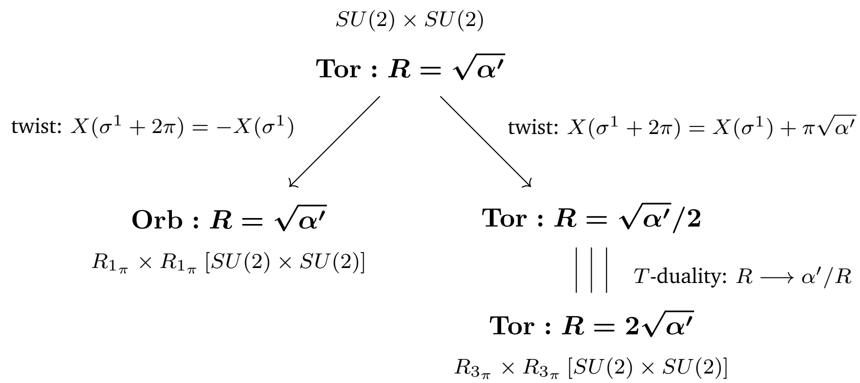


Figure 8.4: Equivalence of toroïdal theory at $\boldsymbol{R = 2\sqrt{\alpha'}}$ and orbifold theory at $\boldsymbol{R = \sqrt{\alpha'}}$

Joe then mentions that this equivalence can be verified via theta function identities. This may be so, but I fear this is not obvious and I will not try to do so.

8.92 p 262: The Low Energy Physics near the Crossing Point of the Toroïdal and the Orbifold Theory

Under the reflection $X(\sigma^1 + 2\pi) = X(\sigma^1) + \pi\sqrt{\alpha'}$, we have $j^{1,2} \longrightarrow -j^{1,2}$ and $j^3 \longrightarrow j^3$. The following $SU(2) \times SU(2)$ combinations thus remain invariant under this twist:

$$j^1\tilde{j}^1, \quad j^1\tilde{j}^2, \quad j^2\tilde{j}^1, \quad j^2\tilde{j}^2 \quad \text{and} \quad j^3\tilde{j}^3 \tag{8.447}$$

All the other combinations of the generators pick up a minus sign. The generic form of the potential at the $SU(2) \times SU(2)$ point was discussed on p 246. If M_{ij} are the nine massless scalars then the masslesness implies that the first possible term in a potential is cubic and the only cubic invariant is $\det M$, see (8.3.22). The solution of the equations of motion is similar as in the previous case, $M_{11}M_{22}M_{33} = M_{11}M_{22} = M_{11}M_{33} = M_{22}M_{33} = 0$, which implies three types of solutions, $M_{11} \neq 0, M_{22} = M_{33} = 0$ and its two permutations.

8.93 p 262: Extra Massless States on the Toroïdal Branch

We go back to the mass-shell condition (8.3.1), i.e. $m^2 = k_L^2 + 4(N-1)/\alpha' = k_R^2 + 4(\tilde{N} - 1)/\alpha'$ with $k_{L,R} = n/R \pm wR/\alpha'$ as per (8.2.7). Let us set the compactification radius as $R = k\sqrt{\alpha'}$ for some $k \in \mathbb{N}$. We find extra massless states at $(N, \tilde{N}; n, w) = (0, 0; \pm 2k, 0)$. Indeed, we then have

$$k_{L,R} = \frac{1}{\sqrt{\alpha'}}\left(\frac{n}{k} \pm kw\right) \tag{8.448}$$

and thus

$$m^2 = \frac{1}{\alpha'}\left(\frac{n}{k} \pm kw\right)^2 + \frac{4}{\alpha'}(N-1) = \frac{1}{\alpha'}\left[(2\pm 0)^2 - 4\right] = 0 \qquad [8.449]$$

8.94 p 263: Eq. (8.5.22) Twisting the $SU(2)\times SU(2)$ theory by $\mathbb{Z}_k$

We write this twist as

$$X(\sigma^1 + 2\pi) = X(\sigma^1) + 2\pi\frac{\sqrt{\alpha'}}{k} \qquad [8.450]$$

and so this is the same as a toroïdal theory with compactification radius $R = \sqrt{\alpha'}/k$. By T-duality this is equivalent to a toroïdal theory with compactification radius $R = k\sqrt{\alpha'}$.

8.95 p 263: Eq. (8.5.23) The Massless Scalars of the $\mathbb{Z}_k$ Twisted Theory

Let us define $j^\pm = j^1 \pm ij^2$. We thus have

$$j^\pm =: \cos\frac{2}{\sqrt{\alpha'}}X_L(z) : \pm i : \sin\frac{2}{\sqrt{\alpha'}}X_L(z) :=: e^{\pm i\frac{2}{\sqrt{\alpha'}}X_L(z)} : \qquad [8.451]$$

To check the transformation of $j^\pm$ under a twist recall that if X transforms as $X(\sigma^1+2\pi) = X(\sigma^1) + 2\pi\sqrt{\alpha'}/k$ then $X_{L,R}$ transform as $X_{L,R}(\sigma^1+2\pi) = X_{L,R}(\sigma^1) + \pi\sqrt{\alpha'}/k$. Thus

$$j^\pm(\sigma^1+2\pi) =: e^{\pm i\frac{2}{\sqrt{\alpha'}}\left[X_L(z) + \frac{\pi\sqrt{\alpha'}}{k}\right]} := e^{\frac{\pm 2\pi i}{k}} j^\pm(\sigma^1) \qquad [8.452]$$

and similar for the right-moving parts $\tilde{\jmath}^\pm$. We thus see that the combinations $j^+\tilde{\jmath}^-$ and $j^-\tilde{\jmath}^+$, as well as the combination $j^3\tilde{\jmath}^3$ remain invariant under the twist. We combine the first two as

$$\begin{aligned} j^+\tilde{\jmath}^- + j^-\tilde{\jmath}^+ &= (j^1+ij^2)(\tilde{\jmath}^1 - i\tilde{\jmath}^2) + (j^1 - ij^2)(\tilde{\jmath}^1 + i\tilde{\jmath}^2) = 2(j^1\tilde{\jmath}^1 + j^2\tilde{\jmath}^2) \\ j^+\tilde{\jmath}^- - j^-\tilde{\jmath}^+ &= (j^1+ij^2)(\tilde{\jmath}^1 - i\tilde{\jmath}^2) - (j^1 - ij^2)(\tilde{\jmath}^1 + i\tilde{\jmath}^2) = 2i(j^2\tilde{\jmath}^1 - j^1\tilde{\jmath}^2) \end{aligned} \qquad [8.453]$$

We thus have three massless invariants

$$j^3\tilde{\jmath}^3, \quad j^1\tilde{\jmath}^1 + j^2\tilde{\jmath}^2 \quad \text{and} \quad j^1\tilde{\jmath}^2 - j^2\tilde{\jmath}^1 \qquad [8.454]$$

Recall from p 246 the definition of a flat direction as a continuous family of static background solutions. Recall also that a solution for direction is that, in a diagonalised matrix M_{ij}, we cannot have two of the diagonal elements non-vanishing, or their product, say $M_{11}M_{22} \neq 0$, so that the flat condition $0 = \partial U(M)/\partial M_{33} = \partial \det M/\partial M_{33} = \partial(M_{11}M_{22}M_{33})/\partial M_{33}$ is not satisfied. Thus $j^3\tilde{\jmath}^3$ can remain a flat direction, but in $j^1\tilde{\jmath}^1 +$

$j^2\tilde{\jmath}^2$ not both $M_{11} = j^1\tilde{\jmath}^1$ and $M_{22} = j^2\tilde{\jmath}^2$ can be zero and so this this not a flat direction. I am not sure about the scond condition, $M_{12} = -M_{21}$

A complete analysis of the $c = 1$ theories can be found in [18]. We reproduce here his figure 14 of $c = 1$ theories and refer to that excellent review for details.

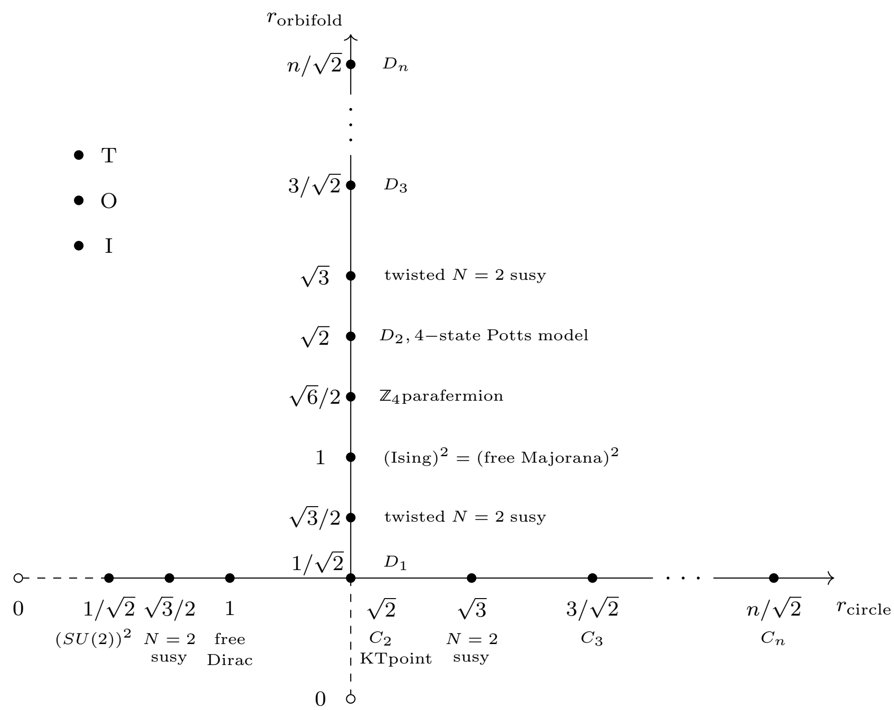


Figure 8.5: $c = 1$ CFTs. Axes not to scale.

8.96 p 263: Eq. (8.6.1) The $U(1)$ Constant Background Gauge Field

Equation (8.6.1) should be obviously correct. The statement that locally it is pure gauge means that locally we can set the field to zero by a $U(1)$ gauge transformation $A \longrightarrow A' = A + \partial\Omega = 0$. This implies that $-\theta/2\pi R + \partial\Omega = 0$ which is solved by a gauge transformation $\Omega = (\theta/2\pi R)x + c^{\text{te}}$.

The field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and obviously vanishes as A is constant. Under a space-time periodicity $x \longrightarrow x + 2\pi R$ we have

$$\Lambda \longrightarrow \exp\left[-\frac{i\theta}{2\pi R}(x + 2\pi R)\right] = e^{-i\theta}\Lambda \qquad [8.455]$$

and so Λ does not have the periodicity. Of course, $\Lambda^{-1}\partial\Lambda$ does have the required periodicity, so I must admit that I am a bit lost about the the subtlety of that remark. Indeed , the only thing you are doing is writing $1 = e^{-x}\partial e^{x}$ and you can do that at will. Hopefully the reason for this will become clear later!

8.97 p 263: Eq. (8.6.2) The Wilson Line

All physical quantities need to be gauge invariant, or they would depend on the gauge choice. As such the gauge field A is not a physical quantity and that is why we have the field tensor, which is gauge invariant. However there is another gauge invariant quantity we can build from the gauge field, i.e. the Wilson line as defined in (8.6.2).

Straightforward use of the definition of A gives

$$W_q = e^{iq\oint dx\,A} = e^{-iq\theta/(2\pi R)\oint dx} = e^{-iq\theta/(2\pi R)2\pi R} = e^{-iq\theta} \qquad [8.456]$$

This is indeed gauge invariant. Indeed performing a gauge transformation would give an additional contribution

$$e^{iq\oint dx\,\partial\Omega(x)} = e^{iq\Omega(x)}\Big|_{x=0}^{x=2\pi} = 0 \qquad [8.457]$$

as $\Omega(x)$ must be periodic with period 2π. For more details on the gauge invariant Wilson loops see e.g. the chapter on Yang-Mills in my notes on QFT [11] for details.

8.98 p 264: Eq. (8.6.3) The Action for a Point Particle with Charge q

The action (8.6.3) is the equivalent of the non-linear sigma model for a point particle. The field X^M is coupled to an external gauge field A_M. The coupling needs to happen via the derivative of the field, $\dot{X}^M$, in order to preserve worldline diffeomorphism invariance (recall that X^M and A_M are worldline scalars and that we take A_M to be a constant background field). Note that this action has Euclidean worldsheet time, compare to the action for Minkowski time that has a minus sign in the mass term, see (1.2.5).

The statement that the gauge action is simply $-iq\int dx^M\,A_M$ surely refers to the analogous statement for the string, where we found a low-energy spacetime action whose equations of motion are the conditions for Weyl invariance at the quantum level, i.e. the vanishing of the β functions of the worldsheet theory. From (8.6.2) we see that paths winding around the compact dimension indeed pick up a phase $e^{iq\theta}$ in the spacetime path integral.

8.99 p 264: Eq. (8.6.4) The Momentum of Point Particle with Charge q in a Compactified Dimension

We consider one compactified dimension X^d. We first rewrite the Euclidean action using $\dot{X}^d = -iv^d$:

$$S_M = \int d\tau \left(-\frac{1}{2}v^d v_d + \frac{1}{2}m^2 - qA_d v^d\right) = \int d\tau \left[-\frac{1}{2}(v^d)^2 + \frac{1}{2}m^2 - qA_d v^d\right] \qquad [8.458]$$

where we have used the fact that $v^d = v_d$ for our spacetime metric, Minkowski or otherwise. The canonical momentum in Minkowski space is defined as

$$p = \frac{\partial \mathcal{L}}{\partial(\partial_{t_M} X)} \qquad [8.459]$$

We now go to Euclidean time $t_E = it_M$ and use $\partial_{t_M} = i\partial_{t_E}$ to find

$$p = \frac{\partial \mathcal{L}}{\partial(i\partial_{t_E} X)} = -\frac{\partial \mathcal{L}}{\partial(i\dot{X})} = -\frac{\partial \mathcal{L}}{\partial v^d} \qquad [8.460]$$

Because we wish to integrate out the dynamics over the compactified field X, we replace the gauge field by its average contribution over the compactified dimension, i.e. we consider the case where the particle can wind around the compactified dimension and pick up a phase. Therefore

$$\begin{aligned} p_d &= -\frac{\partial}{\partial v^d}\left[-\frac{1}{2}(v^d)^2 + \frac{1}{2}m^2 - qA_d v^d\right] \\ &= v^d + \frac{q}{2\pi R}\int dx\, A = v^d - \frac{q\theta}{2\pi R} \end{aligned} \qquad [8.461]$$

8.100 p 264: Eq. (8.6.5) The Quantised Momentum of Point Particle with Charge q in a Compactified Dimension

The quantisation of the momentum p_d follows for the same reason as that that in (8.1.5). Therefore

$$p^d = \frac{n}{R} = v^d - \frac{q\theta}{2\pi R} \qquad [8.462]$$

From this we have that

$$v^d = \frac{n}{R} + \frac{q\theta}{2\pi R} = \frac{2\pi n + q\theta}{2\pi R} \qquad [8.463]$$

8.101 p 264: Eq. (8.6.6) The Hamiltonian Point Particle with Charge q in a Compactified Dimension

The Hamiltonian consists of a contribution from the non-compactified dimensions, which is simply $\frac{1}{2}(p_\mu p^\mu + m^2)$, plus that of the compactified dimension. For the compactified dimension we thus have, in Minkowski space,

$$\mathcal{H} = \frac{\partial X^d}{\partial t_M} p^d - \mathcal{L}_C = \frac{\partial X^d}{i\partial t_E} p^d - \mathcal{L}_C = -i\dot{X}^d p^d - \mathcal{L}_C = -v^d p^d - \mathcal{L}_C \tag{8.464}$$

Using [8.458] and [8.461] we find

$$\mathcal{H} = -\, v^d \left(v^d - \frac{q\theta}{2\pi R} \right) - \left[-\frac{1}{2}(v^d)^2 - q \left(-\frac{\theta}{2\pi R} \right) v^d \right] = -\frac{1}{2}(v^d)^2 \tag{8.465}$$

This has a different sign from Joe. I can't figure out what is incorrect.

8.102 p 264: Eq. (8.6.8) Diagonalising the Background Field with Chan-Paton Factors

Recall the discussion about Chan-Paton factors $\lambda^a, a = 1, \cdots, n$ starting on page 184. These are basically $n \times n$ Hermitian matrices and there are n of them. Each end-point of the open string has a Chan-Paton Factor associated to it. These factors always appear in amplitudes as traces of the form $\mathrm{tr}\ \lambda^{a_1}\lambda^{a_n}\cdots\lambda^{a_k}$ and thus we have a symmetry under Hermitian matrices $\lambda^a \longrightarrow U\lambda^a U^\dagger$. As explained in that section this symmetry results in a spacetime gauge symmetry.

As we have included Chan-Paton factors in our theory, the background is now an $n \times n$ matrix and this can be diagonalised using a Hermitian matrix U, i.e. $A^d \longrightarrow A_d^{\mathrm{diag}} = U^{-1}AU = U^\dagger AU$, i.e.

$$A_d = -\frac{1}{2\pi R}\mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_m) \tag{8.466}$$

where we have dropped the $^{\mathrm{diag}}$ superscript for convenience. Thus matrix is clearly a Hermitian matrix, as the θ_i are real, so it is an element of $U(N)$. It is moreover a matrix that commutes with all other elements of $U(N)$ and so is an element of the subgroup $U(1)^n$.[9]

[9]Each non-zero θ generates a $U(1)$, and all these $U(1)$ commute with one another, so a diagonal element of $U(n)$ sits automatically in $U(1)^n$.

8.103 p 264: Eq. (8.6.9) The Quantised Momentum with Chan-Paton Factors

It isn't clear to me why $|ij\rangle$ has charge $+1$ under $U(1)_i$ and change -1 under $U(1)_j$. The charge is linked to the phase the open string picks up in Wilson loop. The open string has an λ^i Chan-Paton factor on one end and a λ^j Chan Paton factor at the other end. Why do they pick up a different sign?

Assuming that the state $|ij\rangle$ has charge $+1$ under $U(1)_i$ and change -1 under $U(1)_j$ then the two ends are just like particles with charge $q = \pm 1$ from a Wilson line point of view. We can thus apply the same reasoning as for the point particle and we find indeed that we have the quantisation

$$v_d = \frac{2\pi\ell + \theta_i - \theta_j}{2\pi R} \qquad [8.467]$$

8.104 p 264: Eq. (8.6.10) The Mass Spectrum with Chan-Paton Factor

The mass spectrum is thus a combination of the non-compact and the compact dimensions. We have a contribution from the oscillator modes $\alpha'^{-1}(N-1)$ and from the Wilson line and quantised compact momentum v_d^2. Together this gives (8.6.10).

8.105 p 265: Open Strings with Neumann Boundary Conditions

Recall that the Neumann boundary conditions are $\partial^\sigma X^\mu = 0$ at the open string endpoints. These endpoints are thus free to move and can be unwound around any compactified dimension. Dirichlet boundary conditions, on the contrary, have the endpoint fixed and such strings cannot be unwound around a compactified dimension. But for Neumann boundary conditions there is thus no boundary condition $X(\sigma + 2\pi) = X(\sigma) + 2\pi R$. Ignoring the Chan-Paton factors, the mass spectrum (8.6.10) is thus given, in this case, by

$$m^2 = \frac{(2\pi\ell)^2}{4\pi^2 R^2} + \frac{N-1}{\alpha'} \qquad [8.468]$$

Strings with non-zero momentum, i.e. with $\ell \neq 0$, thus have infinite mass as $R \longrightarrow 0$. If we include the Chan-Paton factors we get for the numerator of the first term in the mass spectrum $(2\pi\ell - \theta_j + \theta_i)^2$. This can be zero for very specific combinations of θ_i and θ_j but this is not the case for generic such values. Because theres is no term w^2R^2/α'^2 that is present in the closed string torïdal compactification, there seems to be no duality and no infinite set of massless states. Because an open string with Neumann boundary conditions can just be unwound around the compactified dimension it seems like it lives in the the 25 dimensional spacetime, ignoring the compactified dimension. But as explained by Joe, this is not quite right; it is the endpoints only that live in 25 dimensional spacetime. The interior stuff of the string vibrates in the full 26 dimensions.

8.106 p 266: Eq. (8.6.15) The Boundary Conditions between a Theory and its Dual

We consider for simplicity a rectangular worldsheet

Figure 8.6: The boundary vectors on an open string worldsheet

We use the usual complex coordinates $z = e^{-iw} = e^{-i(\sigma+i\tau)} = e^{\tau-i\sigma}$ and $\bar{z} = e^{\tau+i\sigma}$. From this we have $\partial_\sigma = -i(\partial - \bar{\partial})$ and $\partial_\tau = \partial + \bar{\partial}$. Therefore

$$\partial_\sigma X(z,\bar{z}) = -i(\partial - \bar{\partial})\left[X_L(z) + X_R(\bar{z})\right] = -i\left[\partial X_L(z) - \bar{\partial} X_R(\bar{z})\right] \qquad [8.469]$$

and

$$\partial_\tau X'(z,\bar{z}) = (\partial + \bar{\partial})\left[X_L(z) - X_R(\bar{z})\right] = \partial X_L(z) - \bar{\partial} X_R(\bar{z}) \qquad [8.470]$$

and thus indeed

$$\partial_\sigma X(z,\bar{z}) = -i\partial_\tau X'(z,\bar{z}) \qquad [8.471]$$

The Neumann boundary condition on the X coordinate, i.e. $\partial_\sigma X(0) = \partial_\sigma X(2\pi) = 0$ implies that on the dual coordinate $\partial_\tau X'(0) = \partial_\tau X'(2\pi) = 0$ and so that X' is fixed at the end-points, i.e. Dirichlet boundary conditions for the dual coordinate X'.

8.107 p 266: Eq. (8.6.16) The Endpoints of the Compactified Open String Lie on one hyperplane

We drop the superscript ^{25} for convenience. The first line of (8.6.16) is obvious using (8.6.15). For the second line we use (8.2.16). But that is an equation for the closed string, so we first have to rewrite it for the open string. Using (2.7.26) we find

$$\begin{aligned} X_L(z) &= x_L - i\alpha' p_L \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m z^m} \\ X_R(\bar{z}) &= x_R - i\alpha' p_R \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m \bar{z}^m} \end{aligned} \qquad [8.472]$$

Recall that for the open string there is no $\tilde{\alpha}_m$; both left- and right-moving part have the same oscillators. Note also that we have $p_L = p_R = p$ in order to recover (2.7.26).

First we have

$$\partial_2 X_L(z) = \partial_2 \left[x_L - i\alpha' p_L \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m z^m} \right] \qquad [8.473]$$

With $z = e^{-iw} = e^{-i(\sigma^1 + i\sigma^2)} = e^{\sigma^2 - i\sigma^1}$ we find

$$\begin{aligned}\partial_2 X_L(z) =& \partial_2 \left[-i\alpha' p_L(\sigma^2 - i\sigma^1) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m} e^{m(\sigma^2 - i\sigma^1)} \right] \\ =& -i\alpha' p_L + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \alpha_m e^{m(\sigma^2 - i\sigma^1)} \end{aligned} \qquad [8.474]$$

Integrating, we find

$$\begin{aligned} -i\int_0^\pi d\sigma^1 \partial_2 X_L =& -i\int_0^\pi d\sigma^1 \left(-i\alpha' p_L + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \alpha_m e^{m(\sigma^2 - i\sigma^1)} \right) \\ =& -\pi\alpha' p_L + \sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \alpha_m \frac{e^{m(\sigma^2 - i\sigma^1)}}{-im} \Bigg|_{\sigma^1=0}^{\sigma^1=\pi} \\ =& -\pi\alpha' p_L + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m} \left(e^{m(\sigma^2 - i\pi)} - e^{m\sigma^2} \right) \\ =& -\pi\alpha' p_L + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m e^{m\sigma^2}}{m} \left((-1)^m - 1 \right) \end{aligned}$$

[8.475]

Similarly for the right-moving sector

$$\begin{aligned}\partial_2 X_R(z) =& \partial_2 \left[x_R - i\alpha' p_R \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m \bar{z}^m} \right] \\ \partial_2 X_L(z) =& \partial_2 \left[-i\alpha' p_R(\sigma^2 + i\sigma^1) + \sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m} e^{m(\sigma^2 + i\sigma^1)} \right] \\ =& -i\alpha' p_R + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \alpha_m e^{m(\sigma^2 + i\sigma^1)} \end{aligned} \qquad [8.476]$$

and integration yields

$$
\begin{aligned}
-i\int_0^\pi d\sigma^1 \partial_2 X_R =& -i\int_0^\pi d\sigma^1 \left(-i\alpha' p_R + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m e^{m(\sigma^2+i\sigma^1)}\right)\\
=& -\pi\alpha' p_R + \sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\alpha_m \frac{e^{m(\sigma^2+i\sigma^1)}}{im}\Bigg|_{\sigma^1=0}^{\sigma^1=\pi}\\
=& -\pi\alpha' p_R - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha_m}{m}\left(e^{m(\sigma^2+i\pi)} - e^{m\sigma^2}\right)\\
=& -\pi\alpha' p_R - i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha_m e^{m\sigma^2}}{m}\left((-1)^m - 1\right)
\end{aligned}
\tag{8.477}
$$

Therefore

$$
\begin{aligned}
-i\int_0^\pi d\sigma^1\, \partial_2 X =& -i\int_0^\pi d\sigma^1\, \partial_2 (X_L + X_R)\\
=& -\pi\alpha'(p_L + p_R) = -\pi\alpha' p = -2\pi\alpha' v
\end{aligned}
\tag{8.478}
$$

where we have used (8.6.4), $p_L + p_R = 2p = 2v$, with no Wilson line, i.e. $\theta = 0$. Using (8.6.9) with $\theta_i = 0$ we further have $v = \ell/R$ so that

$$
X'(\pi) - X'(0) = -\frac{2\pi\alpha'\ell}{R} = -2\pi\ell R' \tag{8.479}
$$

where in the last equality we have used compactification radius of the dual theory, $R' = \alpha'/R$.

8.108 p 266: The Endpoints of Two Interacting Open Strings Lie on the Same hyperplane

We can represent graphically the interaction of two open strings via a graviton using the second and fourth example of fig.3.1 that shows how an open string can evolve into a closed string and vice-versa. This gives

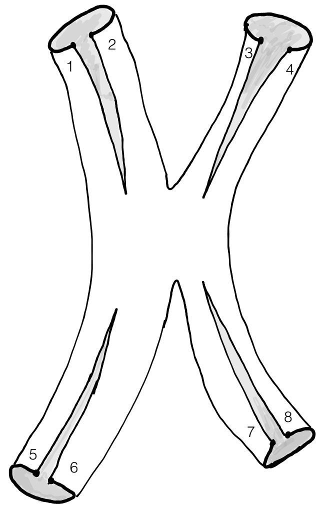


Figure 8.7: Two open strings interacting via a graviton

The diagram can be deformed in such a way that any endpoint may be connected to any other endpoint, i.e. there isn't a precise concept of which endpoint of a string corresponds to which other endpoint -remember crossing symmetry. As a result we can have the point **1** being an endpoint of an open string with other end-point $\mathbf{2}, \mathbf{3}, \cdots, \mathbf{8}$ and so by the argument of (8.6.16) the endpoint **1** must be on the same hyperplane as any of the endpoints $\mathbf{2}, \mathbf{3}, \cdots, \mathbf{8}$. This argument is, of course equally valid for the endpoint **2** etc, and so all the endpoints of these two interacting open strings must lie on the same hyperplane. But all open strings interact with one another (it is an interacting theory!). so all the endpoints of all the open strings must lie on the same hyperplane. Quite daunting to try to visualise if you ask me.

8.109 p 267: Eq. (8.6.17) The Endpoints of the Compactified Open String with Wilson Lines, I

We now have

$$\begin{aligned} X'(\pi) - X'(0) &= -2\pi\alpha' v = -2\pi\alpha' \frac{2\pi\ell - \theta_j + \theta_i}{2\pi R} = -\frac{\alpha'}{R}(2\pi\ell - \theta_j + \theta_i) \\ &= -(2\pi\ell - \theta_j + \theta_i)R' \end{aligned} \quad [8.480]$$

where we have again used the compactification radius of the dual theory, $R' = \alpha'/R$.

8.110 p 267: Eq. (8.6.18) The Endpoints of the Compactified Open String with Wilson Lines, II

The endpoint with $\sigma = 0$ contributes a Wilson line with θ_i and the endpoint with $\sigma = \pi$ contributes a Wilson line with θ_j. So if the endpoint with i sits on a hyperplane $X'(0) = \theta_i R' + k_i$ for some k_i and the endpoint with j sits on a hyperplane $X'(\pi) = \theta_i R' + k_j$ for some k_j, then

$$X'(\pi) - X'(0) = \theta_j R' + k_j - \theta_i R' - k_i = -(-\theta_j + \theta_i)R' + (k_j - k_i) \quad [8.481]$$

We recover (8.6.17) provided $k_j - k_i = 0$, up to the periodic boundary condition. In other words we have $X'(0) = \theta_i R' + k$ and $X'(\pi) = \theta_j R' + k$. So the endpoints are on the respective hyperplanes up to an arbitrary additive constant, k.

Using (8.6.8), i.e. the fact that the θ_i are the diagonal elements of gauge field, $\theta_i = -2\pi R A_{ii} = -(2\pi\alpha'/R')A_{ii}$ we get thus

$$X'(0) = -\frac{2\pi\alpha'}{R'} A_{ii} R' = -2\pi\alpha' A_{ii} \quad [8.482]$$

up to an additive arbitrary constant.

8.111 p 267: Eq. (8.6.19) The Mode Expansion of the Compactified Open String with Wilson Lines

Using the general expression for the left- and right-moving parts of the compactified dimension [8.472] we find that

$$\begin{aligned} X'(z,\bar{z}) &= X_L(z,\bar{z}) - X_R(\bar{z}) = x_L - x_R - i\alpha' p(\ln z - \ln \bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m}(z^m - \bar{z}^m) \\ &= x_L - x_R - i\alpha' p \ln z/\bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m}(z^m - \bar{z}^m) \end{aligned} \quad [8.483]$$

From (8.6.7) we have $p = v = (2\pi\ell - \theta_j + \theta_i)/2\pi R$. We thus have

$$X'(z,\bar{z}) = \theta_i R' - i\alpha' \frac{2\pi\ell - \theta_j + \theta_i}{2\pi R} \ln z/\bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m}(z^m - \bar{z}^m) \tag{8.484}$$

Here we have replaces $x_L - x_R$ by $\theta_i R'$. We will see shortly that this is necessary to ensure that both endpoints are fixed at the right hyperplane. Using $R' = \alpha/R$ we thus get

$$X'(z,\bar{z}) = \theta_i R' - \frac{iR'}{2\pi}(2\pi\ell - \theta_j + \theta_i) \ln z/\bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m}(z^m - \bar{z}^m) \tag{8.485}$$

which is the first line of (8.6.19).

Let us now check the second line of this. We have $z = e^{-iw} = e^{\sigma^2 - i\sigma^1}$. Thus

$$\begin{aligned}
X'(z,\bar{z}) &= \theta_i R' - \frac{iR'}{2\pi}(2\pi\ell - \theta_j + \theta_i) \ln \frac{e^{\sigma^2 - i\sigma^1}}{e^{\sigma^2 + i\sigma^1}} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m}\left(e^{-m\sigma^2 + im\sigma^1} - e^{-m\sigma^2 - im\sigma^1}\right) \\
&= \theta_i R' - \frac{iR'}{2\pi}(2\pi\ell - \theta_j + \theta_i) \ln e^{-2i\sigma^1} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m}{m} e^{-m\sigma^2}\left(e^{+im\sigma^1} - e^{-im\sigma^1}\right) \\
&= \theta_i R' - \frac{R'\sigma^1}{\pi}(2\pi\ell - \theta_j + \theta_i) - \sqrt{2\alpha'} \sum_{m\neq 0} \frac{\alpha_m}{m} e^{-m\sigma^2} \sin m\sigma^1 \\
&= \theta_i R' + \frac{\sigma^1}{\pi}\Delta X' - \sqrt{2\alpha'} \sum_{m\neq 0} \frac{\alpha_m}{m} e^{-m\sigma^2} \sin m\sigma^1
\end{aligned} \tag{8.486}$$

In the last line we have used (8.6.17). The last term has a different sign from (8.6.17), but that is an error found on Joe's errata page.

Finally, let us check that the endpoints are fixed. At $\sigma = 0$ we obviously have $X'(\sigma^1 = 0) - \theta_i R'$. At $\sigma^1 = \pi$ we find

$$X'(\sigma^1 = \pi) = \theta_i R' + \Delta X' = \theta_i R' - (2\pi\ell - \theta_j + \theta_i)R' = \theta_j R' - 2\pi\ell R' \equiv \theta_j R' \tag{8.487}$$

where we have used, once more, (8.6.17) and the periodicity of the compactified dimension. Both endpoints are thus fixed and satisfy (8.6.18) as they should.

8.112 p 268: Eq. (8.6.20) The Mass Spectrum of the Compactified Open String with Wilson Lines

From (8.6.10) and (8.6.17) we have

$$m^2 = \frac{(\Delta X'/R')^2}{4\pi^2 R^2} + \frac{N-1}{\alpha'} = \frac{(\Delta X' R/\alpha')^2}{4\pi^2 R^2} + \frac{N-1}{\alpha'} = \left(\frac{\Delta X'}{2\pi\alpha'}\right)^2 + \frac{N-1}{\alpha'} \tag{8.488}$$

8.113 p 268: Eq. (8.7.1) Vertex Operators for the Massless States

Massless states need $\Delta X' = 0$ and $N = 1$ according to (8.6.0). The first condition means, using (8.6.17) that

$$2\pi\ell - \theta_j + \theta_i = 0 \qquad [8.489]$$

For generic Wilson lines, this can only be the case if we take $i = j$ and $\ell = 0$. A massless state of a generic such theory is thus of the form $\alpha_{-1}^{M} |k; ii\rangle$. Recall from e.g. (3.6.26) that the open string vertex operator for the massless particle in the uncompactified theory is given by $\partial_\tau e^{ik\cdot X}$. In the compactified theory $\partial_\tau = \partial_t$, i.e. the derivative along the tangent of the hyperplane on which the end-points are fixed. Thus the vertex operator is of the form $\mathcal{V}e^{ik\cdot X}$ with $\mathcal{V} = i\partial_t X^M$. For a compactified dimension we can use (8.6.15) that relates the derivative along the tangent of the hyperplane on a compactified spacetime coordinate with the derivative along the normal of the hyperplane on the dual of the compactified spacetime coordinate, and we thus see that the state $\alpha_{-1}^{d} |k; ii\rangle$ can indeed be obtained from the vertex operator with $\mathcal{V} = i\partial_t X^d = \partial_n X^d$.

8.114 p 269: Eq. (8.7.1) The State with Perpendicular Polarisation is a Collective Coordinate for the hyperplane

This is my understanding of this statement. The hyperplane is fully determined by the normal coordinate and by the location of the endpoint at a given worldsheet time τ_0, see fig.8.8.

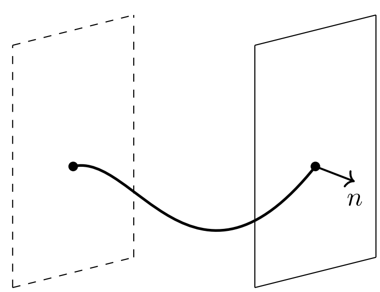


Figure 8.8: The hyperplane and its normal vector. Given the endpoint of the open string at a worldsheet time τ_0 and a normal, the hyperplane is fully determined.

Choosing a different constant background A^d will according to (8.6.18), i.e. $X' = -2\pi\alpha' A^d_{ii}$ will just translate the hyperplane in spacetime. It is in that sense that a constant gauge background corresponds to a uniform translation of the hyperplane. If the gauge field is now not constant, but depends on the (uncompactified) spacetime coordinates x^μ hen the translation becomes x^μ dependent , i.e. a curved hyperplane. The background field A^d_{ii} then determines the dynamics of the hyperplane.

8.115 p 270: Eq. (8.7.2) The D-Brane Action

I must admit that I find the discussion around (8.7.20) very hard to follow, especially the derivation of the Born-Infeld term $2\pi\alpha' F_{ab}$ in the D-brane action. I will therefore take a (few) step(s) back and follow the derivation of that term as presented in [22], albeit that their the superstring is considered, so that we will have to make the necessary adjustments. Also, when we actually calculate the path integral, I will use the original calculation [23] as I find this more clear. Unfortunately there will be quite a lot of repetition from what we have already explained. But if it that repetition doesn't benefit, it surely won't harm.

FIXED GAUGE BACKGROUND

Our aim is to describe the dynamics of a D-brane and to show that their collective coordinates can be described in terms of Wilson lines of a standard gauge theory. This will then naturally lead to the low-energy action of a D-brane including the Born-Infeld term.

Let us start by recalling what we call a Dp-brane: that is a $p+1$ dimensional hyperplane in a D dimensional spacetime onto which the endpoints of open strings can attach. These hyperplanes arise when the open string has ***Dirichlet*** boundary conditions, rather that ***Neumann*** boundary conditions. Remember that the Dirichlet boundary conditions arise by taking the T-duality of a compactified spacetime dimension with Neumann boundary conditions.

To be specific our string has Neumann boundary conditions in the directions along the hyperplane:

$$\partial_\sigma X^\mu\Big|_{\sigma=0,\pi} = 0 \qquad \text{for } \mu = 0, 1, \cdots, p \tag{8.490}$$

and Dirichlet boundary conditions in the transverse directions:

$$\partial_\tau X^\mu\Big|_{\sigma=0,\pi} = 0 \qquad \text{for } \mu = p+1, \cdots, D \tag{8.491}$$

The latter means that the end-points live on the hyperplane. We represent this graphically in fig.8.11.

The position of the D-brane is fixed at the boundary points of the space-time coordinates $X^{p+1}, \cdots, X^D$. Following (8.6.18) we write that the dual coordinates $X'^{p+1}, \cdots, X'^D$ have their endpoints fixed as

$$X'^k = -2\pi\alpha' A_{k,ii} \qquad \text{for } k = p+1, \cdots, D \tag{8.492}$$

where i is the Chan-Paton index. This links the end-points to a particular string theory background A_k.

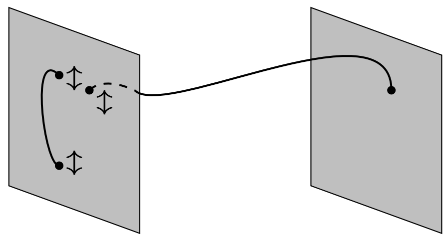


Figure 8.9: D-brane boundary conditions. The shaded (hyper)surfaces are two D-branes with open strings having their endpoints attached to them. One open string has both endpoints on the same D-brane; the other opens string has its end-points on two different D-branes. The open strings have Neumann boundary conditions in the directions along the D-brane – i.e. they can move in that hyperplane, but Dirichlet boundary conditions in the directions transverse to the D-branes – i.e. they are stuck to the hyperplane.

Let us now, for simplicity, first assume that only one dimension is compactified. We introduce $U(N)$ Chan-Paton factors and consider an open string vertex operator like in (8.7.1b) corresponding to a fixed background Abelian gauge field A_d. Due to the $U(N)$ Chan-Paton factors this is an $n \times N$ matrix. We consider the background

$$A_d = \frac{1}{2\pi R}\begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \theta_N \end{pmatrix} \quad [8.493]$$

with the θ_i constant. This background field can be written as

$$A_d = -i\Lambda^{-1}\partial_d\Lambda \quad [8.494]$$

with

$$\Lambda = \begin{pmatrix} e^{i\theta_1 X^d/2\pi R} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2 X^d/2\pi R} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & e^{i\theta_N X^d/2\pi R} \end{pmatrix} \quad [8.495]$$

and is pure gauge, i.e. it can be set to zero by a local gauge transformation $A \longrightarrow A' =$

$A + \partial\Omega = 0$ with

$$\Omega(X) = \begin{pmatrix} -\frac{\theta_1}{2\pi R} & 0 & \cdots & 0 \\ 0 & -\frac{\theta_2}{2\pi R} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & -\frac{\theta_N}{2\pi R} \end{pmatrix} \qquad [8.496]$$

We can thus gauge away A_d to zero by a local gauge transformation. But due to the compactification of the dimension we cannot gauge away this field globally. This is manifest from the fact that $\Lambda(X)$ is not periodic in X^d. Indeed

$$\begin{aligned} \Lambda(X^d + 2\pi R) &= \begin{pmatrix} e^{i\theta_1(X^d+2\pi R)/2\pi R} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2(X^d+2\pi R)/2\pi R} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & e^{i\theta_N(X^d+2\pi R)/2\pi R} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta_1}e^{i\theta_1 X^d/2\pi R} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2}e^{i\theta_2 X^d/2\pi R} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & e^{i\theta_N}e^{i\theta_N X^d/2\pi R} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & e^{i\theta_N} \end{pmatrix} \Lambda(X^d) = W \cdot \Lambda(X^d) \end{aligned} \qquad [8.497]$$

Gauge invariant states can thus pick up a phase after the periodic translation $X^d \longrightarrow X^d + 2\pi R$. But this phase is exactly the Wilson line (8.6.2)

$$W = e^{i \oint dx^d\, A_d} \qquad [8.498]$$

This is precisely is the gauge invariant observable that appears in the path integral of the Polyakov action for scattering amplitudes. We cannot set the gauge field globally to zero and hence we pick up non-trivial phases. This is responsible for the braking of the $U(N)$ symmetry into $\left[U(1)\right]^N$ as explained in the discussion around (8.6.8).

This symmetry breaking mechanism has a simple explanation when using T-duality. We know from (8.6.4) that the string momenta along the compactified dimensions are quantised, and so in the T-dual description the dual field has open string end-points that, in general, lie on different hyperplanes, see (8.6.17).

Consider now an open string state with Chan-Paton factors on its end-points, say $|k; ij\rangle$. Under a $U(1)$ gauge transformation [8.495] the corresponding wave function $\Psi(k; ij)$ will pick up a phase, in order to leave the corresponding Lagrangian invariant. For the end-point i this phase is $e^{-i\theta_i/2\pi R}$. For the end-point j this phase is $e^{-i\theta_j/2\pi R}$.[10] The open string wavefunction will therefore pick up phase $e^{-i(\theta_i-\theta_j)/2\pi R}$ an under a periodic translation $X^d \to X^2 + 2\pi R$ the state $|k;ij\rangle$ transforms as

$$|k;ij\rangle \longrightarrow e^{i(\theta_j-\theta_i)/2\pi R}\,|k;ij\rangle \qquad [8.499]$$

Following the derivation of (8.6.9) we then see that the momentum of that state is quantised as

$$p^D_{(ij)} = \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R} \qquad [8.500]$$

Continuing with Joe's derivation, this then leads to the fact that the open string end-points of the dual coordinates are fixed on hyperplanes, (8.6. 18)

$$X'^D(\sigma = 0; i) = 2\pi\alpha' A^D_{ii} \qquad \text{and} \qquad X'^D(\sigma = \pi; j) = 2\pi\alpha' A^D_{jj} \qquad [8.501]$$

The open string with $D-p$ coordinates X^m for $m = p+1, \cdots, D$ compactified thus has its end-points on N hyperplanes of dimension $p+1$. These are called ***Dp-branes***. Just to be sure there is no misunderstanding; the end-points of the open string of the original theory $X^\mu = X^\mu_L + X^\mu_R$ for $\mu = 0, \cdots, D$ are not fixed and still satisfy Neumann boundary conditions. It is the dual coordinates of the compactified dimensions $X'^m = X^D_L - X^D_R$ for $m = p+1, \cdots, D$ that satisfy Dirichlet boundary conditions and have end-points fixed on the Dp-branes. From this point of view we can consider the uncompactified string as having $p = D$. The end-points are then fixed on DD-branes of dimension $p+1 = D+1$ which is the entire space-time. The end-points of the uncompactified open string are thus free to move in space-time.

We can now consider applying a T-duality to space-time coordinates parallel and perpendicular to a D-brane. As a T-duality transformation interchanges Neumann and Dirichlet boundary conditions, if we apply it to a coordinate parallel to a Dp-brane, we change a Neumann boundary condition into a Dirichlet boundary condition and the result is a $D(p-1)$-brane. If we apply a T-duality transformation to a coordinate perpendicular to a Dp-brane, we change a Dirichlet boundary condition into a Neumann boundary condition and the result is a $D(p+1)$-brane. Duality transformations thus allow us to view a given theory in terms of different types of D-branes. This important remark allows one to establish dualities between different string theories, e.g.type IIA and type IIB superstrings.

10

Here again, it is not entirely clear to me why that is the case.

THE MASS SPECTRUM, THE MASSLESS VERTEX OPERATORS AND THE GAUGE SYMMETRY

We now wish to consider how the massless open string states contribute to the fluctuation of the D-brane. Recall from the mass spectrum (8.6.20) for one compactified dimension. This formula is easily generalised if more than one dimension is compactified:

$$m^2 = \sum_{k=p+1}^{D} \left(\frac{\Delta X'^k}{2\pi\alpha'} \right)^2 + \frac{N-1}{\alpha'} \tag{8.502}$$

Generic massless states are thus level one states with end-points on the same hyperplane, with zero winding, i.e. the states in (8.7.1).

These massless states are split into two:

- $p+1$ states $\alpha_{-1}^{\mu} |k;ii\rangle$ for $\mu = 0, \cdots, p$ with polarisation parallel to the hyperplane, making a gauge field A^μ on the hyperplane. These provide a $\left[U(1)\right]^{p+1}$ gauge symmetry. The vertex operator for that field is of the form $\partial_{\|} X^\mu e^{ik\cdot X}$.
- $D-p$ states with perpendicular polarisation $\alpha_{-1}^{k} |k;ii\rangle$ for $\mu = p+1, \cdots, D$. These are the gauge fields $A_{k,ii} = \Phi_k$, $k = p+1, \cdots, D$ in the compact directions. In the dual theory they are linked to the dual coordinates X'^k according to (8.6.18) and these dual coordinates determine the location of the hyperplanes, i.e. the D-branes. The vertex operator for that field is of the form $\partial_{\|} X^m e^{ik\cdot X} = \partial_{\perp} X'^m e^{ik\cdot X}$.

We have thus established the fact that the gauge fields in the compactified dimensions Φ_m, $m = p+1, \cdots, D$ are scalars that determine the location of the D-branes. Thus, a hyperplane has fluctuations described by massless open string states that correspond to gauge fields. This provides a description of D-branes in terms of gauge theory. Now all of this has been explained for constant gauge fields, but if these gauge fields are not constant then the hyperplanes would becomes curved and the gauge fields would therefore describe the dynamics of the D-branes. These facts provide the basis for the gauge/gravity correspondences in string theory such as the famous AdS/CFT correspondence.

So far we have considered the generic case where all θ_i are different. Let us now assume that k of them have the same value,

$$\theta_1 = \theta_2 = \cdots = \theta_k = \theta \tag{8.503}$$

In that case the mass spectrum

$$m^2 = \sum_{k=p+1}^{D} \left(\frac{\Delta X'^k}{2\pi\alpha'} \right)^2 + \frac{N-1}{\alpha'} = \sum_{k=p+1}^{D} \left(\frac{2\pi\ell^k - \theta_i + \theta_j}{2\pi\alpha'} \right)^2 + \frac{N-1}{\alpha'} \tag{8.504}$$

For $1 \leq i, j \leq k$ we have massless states for $\ell^k = 0$ and $N = 1$ even if the end-points are on different hyperplanes. The physical interpretation of this is that these different hyperplanes are actually located at the same place and so strings stretched between these hyperplanes can attain a vanishing length and so require no extra energy. There are k^2 such extra massless states and they transform under the adjoint representation of $U(k)$. It corresponds to the Wilson line in [8.497]

$$W = \begin{pmatrix} e^{i\theta}\mathbb{1}_{k\times k} & 0_{k\times 1} & \cdots & 0_{k\times 1} \\ 0_{1\times k} & e^{i\theta_2} & \cdots & 0 \\ & & \ddots & \\ 0_{1\times k} & \cdots & 0 & e^{i\theta_N} \end{pmatrix} \qquad [8.505]$$

The result is an unbroken $U(k)$ symmetry and we have now a gauge theory based on $U(k) \times \left[U(1)\right]^{p-k}$. The $U(k)$ symmetry means that the state $\alpha^{\mu}_{-1}\left|k;ij\right\rangle$ for $\mu = 0, \cdots, p$ and $1 \leq i, j \leq k$ corresponding to a gauge field $A^{\mu}_{ij}(\xi^a$ on the Dp-brane. The k^2 states $\alpha^{\mu}_{-1}\left|k;ij\right\rangle$ for each $m = p+1, \cdots, D$ and $1 \leq i, j \leq k$ correspond to massless scalar fields $\Phi^m_{ij}(\xi^a)$ describing the dynamics of the Dp-brane. Here ξ^a are the coordinates parametrising the Dp-brane. They are the analogues of worldsheet coordinates τ and σ of a string.

Something special has happened here. We have selected k end-points that have identical θ_i. These k end-points lie on k D-branes and from this emerged matrices $\Phi^m_{ij}(\xi^a)$ for m the index of the compactified dimensions and $1 \leq i, j \leq k$ that transform under the adjoint representation of $U(k)$. This suggests that as the compactification radius becomes small – smaller that the typical length of a string – the theory is best described by (non-commuting) matrix-valued fields. Just as the existence of the duality $R \longrightarrow \alpha'/R$, this too illustrates how string theory somehow implies that the structure of space-time itself is altered at very small distances.

The Born-Infeld Action for the Uncompactified String

We have just seen that gauge fields live on the worldvolume of a D-brane. What are their dynamics and what kind of gauge theory do they give? Let us finally derive the low-energy action that describes the dynamics of these D-branes.

We start with the coupling of a free open string to a constant background field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in the uncompactified theory. We have seen that we can also view this as a theory of open strings with end-points on a D-brane of dimension D, the entire space-time. Later on we will derive the result for compactified dimensions and generic D-branes using T-duality.

Our goal is to work out a tree-level scattering amplitude using perturbation theory. We are thus interested in a disc diagram with insertions at the boundary, or via a conformal transformation a unit circle with four insertions on the boundary. Let us, of course, not forget the Chan-Paton factors at the end-points. This is shown in fig.8.10.

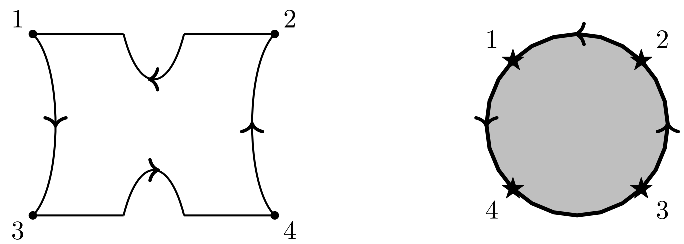


Figure 8.10: An open string four point function with Chan-Paton factors at the end-points.

We will work on the unit disc using polar coordinates $z = re^{i\theta}$ and will need the Green's function on the disc with Neumann boundary conditions, i.e. the solution to the PDE

$$\partial\bar{\partial} f(z,z') = \delta(z-z') \qquad \text{with} \qquad \partial_r f(z,z')\Big|_{r=1} = 0 \tag{8.506}$$

The solution to this is

$$f(z,z') = \frac{1}{2\pi}\left(\ln|z-z'| + \ln|z-\bar{z}'^{-1}|\right) \tag{8.507}$$

as can be checked by direct calculation. We will need to evaluate this at the boundary of the disk. On this boundary we have

$$z' = \bar{z}'^{-1} = e^{i\theta} \qquad \text{and} \qquad \bar{z}' = z'^{-1} = e^{-i\theta} \tag{8.508}$$

and plugging this in [8.507] we find that

$$\begin{aligned} f(\theta,\theta') &= \frac{1}{\pi}\ln|z-z'| = \frac{1}{2\pi}\ln\big(z-z'\big)(\bar{z}-\bar{z}') \\ &= \frac{1}{2\pi}\ln\left(e^{i\theta}-e^{i\theta'}\right)\left(e^{-i\theta}-e^{-i\theta'}\right) = \frac{1}{2\pi}\ln\left(2-e^{i(\theta-\theta')}-e^{-i(\theta-\theta')}\right) \\ &= \frac{1}{2\pi}\ln\left[2-2\cos\big(\theta-\theta'\big)\right] \end{aligned} \tag{8.509}$$

We can now use the mathematical formula

$$\ln\big(1+b^2-2b\cos x\big) = -2\sum_{m=1}^{\infty}\frac{b^m}{m}\cos mx \qquad \text{for } b\le 1 \tag{8.510}$$

which can be found in Gradshtein and Rizhik, *Tables of Integrals*. Setting $b=1$ we get

$$f(\theta,\theta') = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos n(\theta-\theta') \tag{8.511}$$

We can also simply prove this. On the boundary we have from the second line of [8.509]

$$\begin{aligned} f(\theta,\theta') &= \frac{1}{2\pi}\ln\left(e^{i\theta}-e^{i\theta'}\right)\left(e^{-i\theta}-e^{-i\theta'}\right) \\ &= \frac{1}{2\pi}\left[\ln\left(e^{i\theta}-e^{i\theta'}\right)+\ln\left(e^{-i\theta}-e^{-i\theta'}\right)\right] \\ &= \frac{1}{2\pi}\left[\ln e^{i\theta}+\ln\left(1-e^{i\theta'-\theta}\right)+\ln e^{-i\theta}+\ln\left(1-e^{-i\theta'-\theta}\right)\right] \\ &= -\frac{1}{2\pi}\left[\sum_{n=1}^{\infty}\frac{e^{+n(\theta-\theta')}}{n}+\sum_{n=1}^{\infty}\frac{e^{-n(\theta-\theta')}}{n}\right] = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left[n(\theta-\theta')\right]}{n} \end{aligned} \qquad [8.512]$$

Later on we will also need the inverse of this function. By inverse we mean the function $f^{-1}(\theta,\theta')$ such that $\int d\theta\, f(\theta,\theta')f^{-1}(\theta',\theta'') = \delta(\theta-\theta'') - 1/2\pi$. We claim that this is given by

$$f^{-1}(\theta,\theta') = -\frac{1}{\pi}\sum_{n=1}^{\infty} n\cos n(\theta-\theta') \qquad [8.513]$$

and can check this by direct calculation. Indeed

$$\begin{aligned} &\int_0^{2\pi} d\theta\, f(\theta,\theta')f^{-1}(\theta',\theta'') = \frac{1}{\pi^2}\int_0^{2\pi} d\theta' \sum_{m,n=1}^{\infty}\frac{m}{n}\cos[n(\theta-\theta')]\cos[n(\theta'-\theta'')] \\ &= \frac{1}{\pi^2}\sum_{m,n=1}^{\infty}\frac{m}{n}\int_0^{2\pi} d\theta'\,(\cos n\theta\cos n\theta' + \sin n\theta\sin n\theta')(\cos m\theta'\cos m\theta'' + \sin m\theta'\sin m\theta'') \\ &= \frac{1}{\pi^2}\sum_{m,n=1}^{\infty}\frac{m}{n}\int_0^{2\pi} d\theta'\,\Big[\cos n\theta(\cos n\theta'\cos m\theta'\cos m\theta'' + \cos n\theta'\sin m\theta'\sin m\theta'') \\ &\qquad + \sin n\theta(\sin n\theta'\cos m\theta'\cos m\theta'' + \sin n\theta'\sin m\theta'\sin m\theta'')\Big] \end{aligned} \qquad [8.514]$$

With the orthogonality relations

$$\int_0^{2\pi} d\theta'\,\cos m\theta'\cos n\theta' = \int_0^{2\pi} d\theta'\,\sin m\theta'\sin n\theta'\,\pi\delta_{m,n} \qquad [8.515]$$

and

$$\int_0^{2\pi} d\theta'\,\cos m\theta'\sin n\theta' = 0 \qquad [8.516]$$

we find

$$\int_0^{2\pi} d\theta\, f(\theta,\theta')f^{-1}(\theta',\theta'') = \frac{1}{\pi}\sum_{m,n=1}^{\infty}\frac{m}{n}\Big[\cos n\theta(\cos m\theta''+0)+\sin n\theta(0+\sin m\theta'')\Big]\delta_{m,n}$$
$$= \frac{1}{\pi}\sum_{m=1}^{\infty}(\cos m\theta\cos m\theta''+\sin m\theta\sin m\theta'')$$
$$= \frac{1}{\pi}\sum_{m=1}^{\infty}\cos m(\theta-\theta'') \qquad [8.517]$$

Next, we use the Fourier decomposition relation of the delta function

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\cos m(\theta-\theta'') = \delta(\theta-\theta') - \frac{1}{2\pi} \qquad [8.518]$$

and so we find indeed that

$$\int_0^{2\pi} d\theta\, f(\theta,\theta')f^{-1}(\theta',\theta'') = \delta(\theta-\theta') - \frac{1}{2\pi} \qquad [8.519]$$

This is, of course to be viewed in terms of distributions. The $-1/2\pi$ will then lead to a divergence that we can absorb in the normalisation of the path integral and we can ignore it without harming our derivation.

Let us now consider the bosonic string minimally coupled to a photon field. In the conformal gauge this is

$$S[X,A] = \frac{1}{4\pi\alpha'}\int d^2z\,\partial X^\mu\bar{\partial}X_\mu - i\oint d\theta\,\partial_\theta A_\mu\Big|_{r=1} \qquad [8.520]$$

We wish to evaluate the Euclidean path integral

$$Z[F] = \frac{1}{g_s}\int \mathcal{D}X^\mu e^{-S[X,A]} \qquad [8.521]$$

Note that the contribution of the minimally coupled photon field can be seen to correspond to the insertion of an open string vertex operator. As we integrate out the spacetime fields X^μ the resulting partition function is a functional of the gauge field only, and because of gauge invariance of the Wilson loop it must be a functional of the field strength $F_{\mu\nu}$.

We will compute this path integral using the background method. We expand the spacetime coordinates into its disc zero modes and the fluctuations

$$X^\mu = X_0^\mu + \xi^\mu \qquad [8.522]$$

For the external gauge field background we will chose the *radial gauge*

$$\xi^\mu A_\mu(X_0+\xi)=0 \qquad \text{and} \qquad A_\mu(X_0)=0 \tag{8.523}$$

We will also assume that the vector potential is slowly varying. This implies that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is constant, i.e. $\partial F = 0$.

The path integral measure $\mathcal{D}X^\mu$ can be split in spacetime coordinates taking values in the interior of the disc, paramatrised by $z, \bar{z}$ with $|z|<1$ and on the boundary of the disc parametrised by θ, i.e.

$$\mathcal{D}X^\mu = \prod_{|z|<1} \mathcal{D}X^\mu(z,\bar{z}) \prod_{\text{boundary}} \mathcal{D}\xi^\mu(\theta) \tag{8.524}$$

We thus have for the path integral

$$\begin{aligned} Z[F] = \frac{1}{g_s}\int \prod\nolimits_{|z|<1} \mathcal{D}X^\mu(z,\bar{z}) \prod_{\text{boundary}} \mathcal{D}\xi^\mu(\theta) \\ \times \exp - \left[\frac{1}{4\pi\alpha'}\int d^2z\, \partial X^\mu \bar{\partial} X_\mu - i \oint d\theta\, \dot{X}^\mu A_\mu\Big|_{r=1}\right] \end{aligned} \tag{8.525}$$

We can integrate this over all interior points ξ of the disc D and reduce this to a path integral over the boundary ∂D of the disc. To do this, we introduce a new field $\eta(\theta)$ where θ parametrises the boundary of the disc, i.e. $0 \le \theta \le 2\pi$. We insert in [8.525] the identity as $1 = \int D\eta^\mu \delta^{(D)}(\xi^\mu\big|_{\partial D} - \eta^\mu)$. We then represent the delta function as a Gaussian path integral over some auxiliary fields $\nu^\mu(\theta)$ and carry out the integrations, first over ξ^μ, then over ν^μ. In detail, and writing o.c. for the contribution that already only depend on the boundary and the purely bulk contributions, the kinetic term gives a contribution from the boundary of the form

$$\begin{aligned} Z[F] &= \int \mathcal{D}\xi^\mu\, e^{-\frac{1}{4\pi\alpha'}\int d\theta\, \partial\xi^\mu\bar{\partial}\xi_\mu} \int \mathcal{D}\eta^\mu \delta^{(D)}(\xi^\mu - \eta^\mu) \times \text{o.c.} \\ &= \int \mathcal{D}\xi^\mu\, \mathcal{D}\eta^\mu e^{-\frac{1}{4\pi\alpha'}\int d\theta\, \partial\xi^\mu\bar{\partial}\xi_\mu} \frac{1}{(2\pi)^D}\int \mathcal{D}\eta^\mu\, e^{i\int d\theta\, \nu^\mu(\xi_\mu-\eta_\mu)} \times \text{o.c.} \\ &= \frac{1}{(2\pi)^D}\int \mathcal{D}\eta^\mu\, \mathcal{D}\nu^\mu \int \mathcal{D}\xi^\mu\, e^{\int d\theta\left[-\frac{1}{4\pi\alpha'}\partial\xi^\mu\bar{\partial}\xi_\mu + i\nu^\mu(\xi_\mu-\eta_\mu)\right]} \times \text{o.c.} \end{aligned} \tag{8.526}$$

We now rescale ξ and η by $\sqrt{2\pi\alpha'}$ and ignore constant pre-factors

$$\begin{aligned}
Z[F] &\propto \int \mathcal{D}\eta^\mu\,\mathcal{D}\nu^\mu \int \mathcal{D}\xi^\mu\, e^{-\xi^\mu\left(-\frac{\partial\bar\partial}{2}\right)\xi_\mu + i\sqrt{2\pi\alpha'}\nu^\mu(\xi_\mu-\eta_\mu)} \times \text{o.c.} \\
&\propto \int \mathcal{D}\eta^\mu\,\mathcal{D}\nu^\mu \left(\frac{\pi}{\det\left(-\partial\bar\partial/2\right)}\right)^{\frac{D}{2}} e^{-\nu^\mu\frac{4\pi\alpha'}{\partial\bar\partial}\nu_\mu - i\sqrt{2\pi\alpha'}\nu^\mu\eta_\mu} \times \text{o.c.} \\
&\propto \int \mathcal{D}\eta^\mu \left(\frac{\pi}{\det\left(-\partial\bar\partial/2\right)}\right)^{\frac{D}{2}} \left(\frac{\pi}{\det\left(4\pi\alpha'/\partial\bar\partial\right)}\right)^{\frac{D}{2}} e^{-\frac{2\pi\alpha'\eta^\mu\eta_\mu}{4\pi\alpha'/\partial\bar\partial}} \times \text{o.c.} \\
&\propto \int \mathcal{D}\eta^\mu\, e^{-\eta^\mu\frac{\partial\bar\partial}{2}\eta_\mu} \times \text{o.c.}
\end{aligned} \qquad [8.527]$$

We have used $\int_{-\infty}^{+\infty} e^{-ax^2+bx}dx = \sqrt{\pi/a}\, e^{b^2/4a}$ for the Gaussian integrations. For simplicity we have not written the integration over the polar angle anymore. But this reminds us that by $\partial\bar\partial$ in this expression we actually mean the inverse Green's function with Neumann boundary conditions [8.506] valued on the boundary of the disc. We thus conclude that from the integration of the kinetic term we obtain a contribution of the form, renaming η^μ by ξ^μ

$$\int \mathcal{D}\xi^\mu\, e^{-\frac{1}{2}\xi^\mu f^{-1}\xi_\mu} \qquad [8.528]$$

Let us now consider the boundary term with the gauge field. By the delta function we can just replace the ξ^μ by η^μ, which we rescaled by $\sqrt{2\pi\alpha'}$ and renamed ξ^μ. The upshot of this is that because that contribution is quadratic in the original ξ^μ we get an extra factor $2\pi\alpha'$. We will include this by rescaling the field tensor as $F_{\mu\nu} \longrightarrow 2\pi\alpha' F_{\mu\nu}$ and will just have to be careful not to forget this at the end of the calculation. We then find for this term

$$i\int_0^{2\pi} d\theta\, F_{\mu\nu}\xi^\mu\dot\xi^\nu = \frac{i}{2}\int_0^{2\pi} d\theta\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\xi^\mu\dot\xi^\nu \qquad [8.529]$$

We interchange the dummy indices in the second term and perform a partial integration:

$$\begin{aligned}
i\int_0^{2\pi} d\theta\, F_{\mu\nu}\xi^\mu\dot\xi^\nu &= \frac{i}{2}\int_0^{2\pi} d\theta\,\left(\partial_\mu A_\nu\xi^\mu\dot\xi^\nu - \partial_\mu A_\nu\xi^\nu\dot\xi^\mu\right) \\
&= \frac{i}{2}\int_0^{2\pi} d\theta\,\left(\partial_\mu A_\nu\xi^\mu\dot\xi^\nu + \partial_\mu A_\nu\dot\xi^\nu\xi^\mu + \partial_\theta(\partial_\mu A_\nu)\xi^\nu\xi^\mu\right) \\
&= i\int_0^{2\pi} d\theta\,\partial_\mu A_\nu\xi^\mu\dot\xi^\nu + o(\partial F)
\end{aligned} \qquad [8.530]$$

The last term vanishes because it is of the order $\partial^2 A \propto \partial F$. On the other hand, using the gauge condition $A_\mu(X_0) = 0$ we find

$$i \oint d\theta\, \dot{X}^\mu A_\mu(X_0 + \xi) = i \int_0^{2\pi} d\theta\, \dot{\xi}^\mu \left[A_\mu(X_0) + \xi^\nu \partial_\nu A_\mu(X_0)\right] = i \int_0^{2\pi} d\theta\, \dot{\xi}^\mu \xi^\nu \partial_\nu A_\mu(X_0)$$
$$= i \int_0^{2\pi} d\theta\, \partial_\mu A_\nu \xi^\mu \dot{\xi}^\nu + o(\partial F) \qquad [8.531]$$

so both expressions for the gauge field insertion are indeed the same.

Combining [8.528] and [8.531] gives us the path integral for the integration over the boundary

$$Z[F] \propto \int \mathcal{D}\xi^\mu \exp\left(-\frac{1}{2}\xi^\mu f^{-1} \xi_\mu + \frac{i}{2} F_{\mu\nu} \int_0^{2\pi} d\theta\, \xi^\mu \dot{\xi}^\nu\right) \qquad [8.532]$$

where we have used the fact that we are assuming that $F_{\mu\nu}$ is constant.

Let us now look at the field tensor contribution to the boundary action. We can use space-time Lorentz invariance to simplify this. Unfortunately an antisymmetric matrix $F_{\mu\nu}$ cannot be brought to a diagonal form by an $SO(D)$, but it can be brought to a canonical Jordan normal form. This consists of the smallest block diagonals and for an antisymmetric matrix these are 2×2 blocks. We can thus write, using a Lorentz transformation, that, as a matrix,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -f_1 & & 0 & 0 \\ f_1 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & -f_{D/2} \\ 0 & 0 & & f_{D/2} & 0 \end{pmatrix} \qquad [8.533]$$

We then have

$$\xi^\mu F_{\mu\nu} \dot{\xi}^\nu = \begin{pmatrix} \xi^1 & \cdots & \xi^D \end{pmatrix} \begin{pmatrix} 0 & -f_1 & & 0 & 0 \\ f_1 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & -f_{D/2} \\ 0 & 0 & & f_{D/2} & 0 \end{pmatrix} \begin{pmatrix} \dot{\xi}^1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\xi}^D \end{pmatrix}$$
$$= -\xi^1 \dot{\xi}^2 f_1 + \xi^2 f_1 \dot{\xi}^1 - \xi^3 \dot{\xi}^4 f_2 + \xi^4 \dot{\xi}^5 f_2 + \cdots - \xi^{D-1} \dot{\xi}^D f_{D/2} + \xi^D \dot{\xi}^{D-1} f_{D/2}$$
$$= \sum_{k=1}^{D/2} \left(-\xi^{2k-1} \dot{\xi}^{2k} + \xi^{2k} \dot{\xi}^{2k-1}\right) f_k \qquad [8.534]$$

We conclude that, taking the path integral factorises into $D/2$ blocks of size 2×2 of the form and using partial integration on the last term

$$Z[F] \propto \prod_{k=1}^{D/2} \int \mathcal{D}\xi^{2k-1}\mathcal{D}\xi^{2k} e^{-\left(\frac{1}{2}\xi^{2k-1}f^{-1}\xi^{2k-1}+\frac{1}{2}\xi^{2k}f^{-1}\xi^{2k}+if_k\dot{\xi}^{2k-1}\xi^{2k}\right)} \tag{8.535}$$

We perform the integration over ξ^{2k} and rename ξ^{2k-1} simply by ξ

$$\begin{aligned} Z[F] &\propto \prod_{k=1}^{D/2} \int \mathcal{D}\xi \sqrt{\frac{2\pi}{f^{-1}}} e^{-\frac{1}{2}\xi f^{-1}\xi - \frac{f_k^2\dot{\xi}\dot{\xi}}{4f^{-1}/2}} \propto \prod_{k=1}^{D/2} \int \mathcal{D}\xi \sqrt{2\pi f} e^{-\frac{1}{2}\xi f^{-1}\xi - \frac{1}{2}f_k^2\dot{\xi}f\dot{\xi}} \\ &\propto \prod_{k=1}^{D/2} \int \mathcal{D}\xi \sqrt{2\pi f} e^{-\frac{1}{2}\xi(f^{-1}+f_k^2\ddot{f})\xi} \end{aligned} \tag{8.536}$$

In the last line we performed partial integration[11] and defined $\ddot{f}(\theta,\theta') = \partial_\theta\partial_{\theta'}f(\theta,\theta')$. We perform the final integration and get

$$Z[F] \propto \prod_{k=1}^{D/2} \sqrt{2\pi f} \sqrt{\frac{2\pi}{\det\left(f^{-1}+f_k^2\ddot{f}\right)}} \propto \prod_{k=1}^{D/2} \Big[\det\left(1+f_k^2\ddot{f}f\right)\Big]^{-1/2} \tag{8.537}$$

In the last line we have kept only those factors that have a dependence on the gauge field, as that is what interests us.

Let us now remind ourselves that we rescaled the field tensor by $2\pi\alpha'$. We thus have

$$Z[F] = Z[0]\prod_{k=1}^{D/2}(\det\Delta_k)^{-1/2} \qquad \text{with} \qquad \Delta_k = 1+(2\pi\alpha' f_k)^2\ddot{f}f \tag{8.538}$$

To proceed we need to compute $f\ddot{f}$. From [8.511] we have immediately that, recalling that $\ddot{f} = \partial_\theta\partial_{\theta'}f(\theta,\theta')$,

$$\dot{f} = \frac{1}{\pi}\sum_{n=1}^{\infty}\sin n(\theta-\theta') \qquad \text{and} \qquad \ddot{f} = -\frac{1}{\pi}\sum_{n=1}^{\infty} n\cos n(\theta-\theta') \tag{8.539}$$

From [8.513] we see that $\ddot{f} = f^{-1}$ and therefore $\ddot{f}f = f^{-1}f = \delta(\theta-\theta') - \frac{1}{2\pi} = \bar{\delta}(\theta-\theta')$, where we have used [8.519]. We therefore get the nice result that

$$\Delta_k = 1+(2\pi\alpha' f_k)^2 \tag{8.540}$$

[11] Recall that by $\dot{\xi}f\dot{\xi}$ we actually mean $\int d\theta\, d\theta'\, \partial_\theta\xi(\theta)f(\theta,\theta')\partial_{\theta'}\xi(\theta')$.

We are being slightly sloppy as we are again ignoring the $-1/2\pi$.

Let us now evaluate $\det \Delta_k = \det \left[1 + (2\pi\alpha' f_k)^2\right]$. To do this we take an orthonormal basis on the boundary of the disc. The basis elements are $E_m = \pi^{-1/2}\cos m\theta$ and $F_m = \pi^{-1/2}\sin m\theta$ for $m = 1, \cdots, \infty$. We have indeed $\int E_m E_n = \int F_m F_n = \delta_{m,n}$ and $\int E_m F_n = 0$ and we can expand a field on the disc boundary as

$$\xi(\theta) = \sum_{m=1}^{\infty}(a_m E_m + b_m F_m) = \frac{1}{\sqrt{\pi}}\sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta) \tag{8.541}$$

which is just a Fourier decomposition. In this basis Δ_k, considered as an operator is diagonal,

$$\int E_m \Delta_k E_n = \int F_m \Delta_k F_n = \Delta_k \delta_{m,n} \qquad \text{and} \qquad \int F_m \Delta_k E_n = 0 \tag{8.542}$$

and so the determinant is just the product of the Eigenvalues, which are Δ_k. We thus have

$$\det \Delta_k = \prod_{m=1}^{\infty} \Delta_k^2 \tag{8.543}$$

the square appearing because the Eigenvalues appear not only for each m but also for all E's and all F's. Plugging this in [8.538] we find that

$$Z[F] = Z[0]\prod_{k=1}^{D/2}(\det \Delta_k)^{-1/2} = Z[0]\prod_{k=1}^{D/2}\prod_{m=1}^{\infty}\left[1 + (2\pi\alpha' f_k)^2\right]^{-1} \tag{8.544}$$

The infinite product seems to give a divergence, but once we realise that this comes from the determinant of an operator, we can regularise this. We choose ζ-regularisation to do this. To see how this work consider the Gaussian path integral $\int \mathcal{D}\eta e^{-c\eta^2}$ on the boundary of the disk. To make a link with our case, think of c as $[1 + (2\pi\alpha' f_k)^2]^{-1}$. In our orthogonal basis we have

$$\int \mathcal{D}\eta e^{-c\eta^2} \propto \int \prod_{m=1}^{\infty} da_m\, db_m\, e^{-c(a_m^2 + b_m^2)} \propto \prod_{m=1}^{\infty} c^{-1} \tag{8.545}$$

We can now write

$$\prod_{m=1}^{\infty} c^{-1} = e^{\ln \prod_{m=1}^{\infty} c^{-1}} = e^{-\sum_{m=1}^{\infty} \ln c} = e^{-\zeta(0)\ln c} \tag{8.546}$$

with the $\zeta(s)$ the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{for} \qquad \mathrm{Re}(s) > 1 \tag{8.547}$$

By analytic continuation $\zeta(0) = -1/2$ and thus

$$\prod_{m=1}^{\infty} c^{-1} = e^{\frac{1}{2}\ln c} = c^{1/2} \qquad [8.548]$$

Using this regularisation procedure in [8.544] gives

$$Z[F] = Z[0] \prod_{k=1}^{D/2} \left[1 + (2\pi\alpha' f_k)^2\right]^{1/2} \qquad [8.549]$$

Our final task for the calculation of this partition function is to put this back in a manifestly Lorentz invariance form. The infinite product is a product of Eigenvalues and has the determinant of the corresponding operator, and thus becomes $\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})$. Indeed, if we to that for the field tensor in the selected basis [8.533]

$$\begin{aligned}\det\big(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\big) &= \begin{pmatrix} 1 & -2\pi\alpha' f_1 & 0 & 0 \\ 2\pi\alpha' f_1 & 1 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 1 & -2\pi\alpha' f_{D/2} \\ 0 & 0 & 2\pi\alpha' f_{D/2} & 1 \end{pmatrix} \\ &= \prod_{k=1}^{D/2} \left[1 + (2\pi\alpha' f_k)^2\right] \end{aligned} \qquad [8.550]$$

Adding in the normalisation constant that we ignored and recalling that we still have to integrate over the zero modes we find for the tree level partition function of the uncompactified open bosonic string, coupled to a $U(1)$ gauge field

$$\frac{1}{(4\pi^2\alpha')^{D/2} g_s} \int \mathcal{D}X_0^{\mu} \left[\det\big(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\big)\right]^{1/2} \qquad [8.551]$$

This "open string effective action" is similar to the Born-Infeld action for electromagnetism. It is a non-linear extension proposed to remove the electron's self-energy in classical electrodynamics.

Note that we evaluated the partition function without having to use perturbation theory. Within the approximations made, i.e. constant field strength and radial gauge, the result is hence exact. In particular, it is exact in α'. But this also means that this results contains the contribution of all modes of the string, both massless and massive and is hence a pure string result.

THE BORN-INFELD ACTION FOR THE COMPACTIFIED STRING

We now extend the result for the uncompactified string to the compactified string. This will be surprisingly easy. We note that, in the previous derivation of the Born-Infeld action, nowhere did we need to assume that the a spacetime dimension was compactified or not. So to see what changes when one or more spacetime dimensions are compactified, we can just use the T-duality relations for these dimensions.

So we assume a Dp-brane, i.e. $D-p$ dimensions are compactified. These $D-p$ dimensions have Dirichlet boundary conditions for open strings whose end-points sit on $p+1$-dimensional hyperplanes. We assume that the compactified dimension are so small that we can neglect any derivatives along them, i.e. $\partial_m X = 0$ for $m = p+1, \cdots, D$. The remaining uncompactified worldvolume is thus described by X^a for $a = 0, 1, \cdots, p$.

Consider now the Born-Infeld action for the uncompactified string [8.551]. We will split $\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}$ as follows

$$\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} = \left(\begin{array}{c|c} \delta_{ab} + 2\pi\alpha' F_{ab} & 2\pi\alpha' F_{ma} \\ \hline 2\pi\alpha' F_{am} & \delta_{mn} + 2\pi\alpha' F_{mn} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \delta_{ab} + 2\pi\alpha' F_{ab} & -2\pi\alpha' \partial_a A_m \\ \hline 2\pi\alpha' \partial_a A_m & \delta_{mn} \end{array}\right) \qquad [8.552]$$

where we have used the fact that $\partial_m X = 0$ for $m = p+1, \cdots, D$. We now introduce a $(p+1) \times (p+1)$ matrix N and a $(p+1) \times (D-p)$ matrix A with elements

$$N_{ab} = \eta_{ab} + 2\pi\alpha' F_a b \qquad A_{am} = 2\pi\alpha' \partial_a A_m \qquad [8.553]$$

We can thus write

$$\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} = \left(\begin{array}{c|c} N & -A^T \\ \hline A & \mathbb{1} \end{array}\right) \qquad [8.554]$$

with $\mathbb{1}$ the $(D-p) \times (D-p)$ unit matrix. We now use the matrix identity

$$\det \begin{pmatrix} N & -A^T \\ A & M \end{pmatrix} = \det\bigl(N + A^T M^{-1} A\bigr) = \det\bigl(M + A N^{-1} A^T\bigr) \qquad [8.555]$$

Therefore, taking the first equality,

$$\det\bigl(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\bigr) = \det\bigl(N + A^T \mathbb{1} A\bigr) = \det\bigl(\eta_{ab} + 2\pi\alpha' F_{ab} + 2\pi\alpha' \partial_a A^m \partial_b A_m\bigr) \quad [8.556]$$

For the compactified dimensions $m = p+1, \cdots, D$ we know that we can go to the T-dual coordinates and can link the gauge field to the location of the Dp-brane by [8.492]

$$X'^m = -2\pi\alpha' A_m \qquad [8.557]$$

We therefore find

$$\det\big(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\big) = \det\big(\eta_{ab} + 2\pi\alpha' F_{ab} + \partial_a X'^m \partial_b X' m\big) \qquad [8.558]$$

Using reparametrisations of the worldvolume and of space-time we can chose that the $p+1$ dimensions of the hyperplane are aligned with (the first) $p+1$ coordinates of spacetime. The remaining $D-p$ coordinates are therefore transverse to the D-brane . This means that the actual $p+1$ fluctuations of the D-brane ξ^a correspond to the spacetime coordinates x^a. We have assumed that there is no dependence on the compactified dimensions ($\partial_m - 0$ f0r $m = p+1, \cdots, D$) and so these can be integrated out. The Born-Infeld [8.551] action thus becomes, reverting to Minkowski space so that we need to include a minus sign for the determinant to be positive

$$S = -\frac{T_p}{g_s} \int d^{p+1}\xi \sqrt{-\det(\eta_{ab} + \partial_a X'^m \partial_b X' m + 2\pi\alpha' F_{ab})} \qquad [8.559]$$

with

$$T_p = \frac{1}{\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p} \qquad [8.560]$$

the tension of the Dp-brane. This action is known as the ***Dirac-Born-Infeld action***.

Note that if there is no gauge field then the action reduces to

$$\begin{aligned} S[A=0] &= -\frac{T_p}{g_s} \int d^{p+1}\xi \sqrt{-\det(\eta_{ab} + \partial_a X'^m \partial_b X'^m)} \\ &= -\frac{T_p}{g_s} \int d^{p+1}\xi \sqrt{-\det(-\eta_{\mu\nu} \partial_a X'^\mu \partial_b X'^\nu)} \end{aligned} \qquad [8.561]$$

as $\partial_a X^c = \partial_a \xi^c = \delta_a^c$. The combination $-\eta_{\mu\nu}\partial_a X'^\mu \partial_b X'^\nu$ is the induced metric on the worldvolume and its determinant is the infinitesimal volume element of the D-brane. As such this is the natural generalisation of the action for a particle or as the Nambu-Goto action for a string.

Non-Flat Backgrounds

So far we have considered the string to live in a flat background. One can easily generalise the situation to curved spacetimes. We thus consider a worldsheet action (3.76). In the conformal gauge

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ \Big[(\eta^{ab} G_{\mu\nu}(X) + 2\pi\alpha' \epsilon^{ab} B_{\mu\nu}(X) \Big] \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \right\} \qquad [8.562]$$

We have also rescaled the antisymmetric field $B_{\mu\nu}$ for convenience. We will ignore the dilaton contribution.

Let us first assume that $B_{\mu\mu}$ is constant. We have seen earlier that in this case this contribution is just a boundary term

$$\int d^2\sigma\, \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu = \int_0^{2\pi} d\theta \frac{1}{2} B_{\mu\nu} X^\nu \dot{X}^\mu \Big|_{r=1} \qquad [8.563]$$

If we include the coupling to the gauge field and work in the radial gauge, i.e. we compare with [8.532], then we see that it amounts to a shift of the field strength

$$F_{\mu\nu} \longrightarrow \mathcal{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} - B_{\mu\nu} \qquad [8.564]$$

This action is has the gauge invariance

$$\begin{aligned} G_{\mu\nu} &\longrightarrow G_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \\ B_{\mu\nu} &\longrightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\ A_\mu &\longrightarrow A_\mu + \frac{1}{2\pi\alpha'}\Lambda_\mu \end{aligned} \qquad [8.565]$$

The gauge invariant field tensor is thus, in this case, $\mathcal{F}_{\mu\nu}$ and not $F_{\mu\nu}$.

We can then repeat the calculation of the disc partition function. Leaving out the details that we will leave to the industrious reader, the outcome is the effective action

$$S = -T_p \int d^{p+1}\xi\, e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \qquad [8.566]$$

where G_{ab} and B_{ab} are the "pull-backs" of the space-time fields to the Dp-brane:

$$\begin{aligned} G_{ab}(\xi) =& G_{\mu\nu}\big(X(\xi)\big)\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} \\ B_{ab}(\xi) =& B_{\mu\nu}\big(X(\xi)\big)\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} \end{aligned} \qquad [8.567]$$

In particular G_{ab} is the induced world-volume metric from the (curved) spacetime metric. Expanding the spacetime metric around a flat background we find

$$G_{ab} = \eta_{ab} + \partial_a X^\mu \partial_b X_\mu + o\big[(\partial x)^4\big] \qquad [8.568]$$

and we recover [8.559]. We also see that the open string coupling constant g_s comes from the dilaton contribution $g_s = e^\Phi$.

8.116 p 271: Eq. (8.7.5) The Geometric Factor in the Action

We are considering that we have $25-p$ compactified dimensions and we take their T-dual. We then have a p dimensional hyperplane. The coordinates are split as follows

$$X^0 \text{ : spacetime time coordinate}$$
$$X^1, \cdots X^p \text{ : parametrise a hyperplane (Neumann boundary conditions)} \quad [8.569]$$
$$X^{p+1}, \cdots X^D \text{ : compactified dimensions (Dirichlet boundary conditions)} \quad [8.570]$$

Let us visualise this in three dimension. We compactify one dimension and so the endpoint of the open string live on a two-dimensional plane.

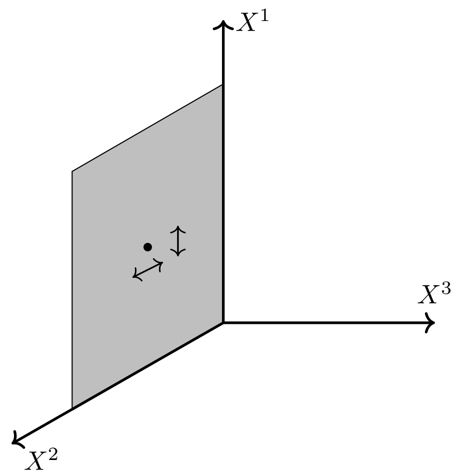


Figure 8.11: Visualisation of a D-brane in three dimensions. The third spacetime dimension is compactified, and so the endpoint of an open string is fixed on a (hyper)plane with $X^3 = c^{\text{te}}$. The endpoints can move freely in the X^1–X^2plane.

The end-points of the string are free to move in the X^1–X^2 plane; in these dimensions the end-points have Neumann boundary conditions. However we know that we can buils a T-dual theory of this and the dual coordinates then have Dirichlet boundary conditions.[12]

[12]I am actually confused by this. The coordinate X^2 is not compactified. We can treat this as the limit of a compactified spacetime coordinate with $R \longrightarrow \infty$. The T-dual coordinate X'^2 than has compactification radius $R \longrightarrow 0$. But do all our arguments about T-duality hold in this limit? I.e. is an uncompactified dimension really equivalent to its T-dual dimension with zero compactification radius?

So we take the T-dual of the X^2 coordinate. We have $X'2 = -2\pi\alpha' A^2$ by (8.6.18). Using the gauge condition $A_2 = X^1 F_{12}$ we find $X'2 = -2\pi\alpha' X^1 F_{12}$.

We have a theory with a p dimensional hyperplane, a Dp-brane. But we have taken the T-dual of one of the coordinates, X^2 and as on page 269 of Joe's book, this reduces the Dp-brane to a $D(p-1)$-brane.

So far so good, but how does (8.7.5) arise? In his Little Book of Strings [16] Joe refers to Pythagoras. Yes, it looks like an application of Pythagoras, but where does it come from? The original infinitesimal two-dimensional element $dX^1\,dX^2$ gets collapsed into a one-dimensional one. How does good-old Pythagoras come into play? This is a Greek mystery to me.

8.117 p 271: Eq. (8.7.10) The Field Tensor Invariant under the Transformations of the Gauge and the Antisymmetric Field

This was already discussed around [8.563].

8.118 p 272: Eq. (8.7.11) The Potential for Coinciding D-Branes

The field strength gives a contribution $-\frac{1}{4}\mathrm{tr}\ F_{\mu\nu}F^{\mu\nu}$ to the effective action. As $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$, this gives a term quartic in the A_m of the form $g^2\mathrm{tr}\left([A_m, A_n]\times [A^m, A^n]\right)$. Using (8.6.28) for the dual coordinates we find that this is a contribution to the effective action of the form

$$S = \cdots + (2\pi\alpha')^4 g^2 \mathrm{tr}\left([X_m, X_n]\times[X^m, X^n]\right) + \cdots \qquad [8.571]$$

This has the form of a potential with a quartic term. Calling this term V it is clear that

$$\left.\frac{\partial^2 V}{\partial X^m \partial X^n}\right|_{X^p=0} = 0 \qquad [8.572]$$

If we have n coinciding D-branes, this thus gives n^2 massless fields. If we, moreover have $25-p$ dimensions that we dualised, then we have a total of $(25-p)n^2$ massless fields.

8.119 p 273: Eq. (8.7.14)–(8.7.16) The D-Brane Tension Recursion Relation

The relation (8.3.31) between the dilation field Φ of an uncompactified D-dimensional theory and the dilaton field Φ' of the same theory with one dimension compactified is

$$e^{-\Phi} = \sqrt{\alpha'}R^{-1}e^{-\phi'} \qquad [8.573]$$

with R the compactification radius. We use this to rewrite (8.7.13)

$$T_p e^{-\Phi} \prod_{i=1}^{p}(2\pi R) = T_p\sqrt{\alpha'}R_p^{-1}e^{-\Phi'}(2\pi R_p)\prod_{i=1}^{p-1}(2\pi R) = 2\pi\sqrt{\alpha'}T_p e^{-\Phi'}\prod_{i=1}^{p-1}(2\pi R) \qquad [8.574]$$

We set this equal to the mass of a $D(p-1)$ torus wrapped around a $(p-1)$- brane

$$2\pi\sqrt{\alpha'}T_p e^{-\Phi'}\prod_{i=1}^{p-1}(2\pi R) = T_{p-1}e^{-\Phi'}\prod_{i=1}^{p-1}(2\pi R) \qquad [8.575]$$

from which it follows that

$$T_{p-1} = 2\pi\sqrt{\alpha'}T_p \qquad \Rightarrow \qquad T_p = \frac{1}{2\pi\sqrt{\alpha'}}T_{p-1} \qquad [8.576]$$

8.120 p 275: Eq. (8.7.17) The D-Brane Annulus Vacuum Amplitude, I

We need to consider the vacuum energy for a cylinder, but with $D-p$ coordinates fixed on a D-brane. We thus have only $p+1$ fluctuation fields. Eq. (7.4.1) is

$$Z_{C_2} = iV_D\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-D/2}\sum_{i\in\mathcal{H}_0^\perp} e^{-2\pi t(h_i-1)} \qquad [8.577]$$

Let us briefly recall, from its derivation, where the different factors originate from. We are calculating $Z_{C_2} = q^{-c/24}\mathrm{Tr}\, q^{L_0}$. For the matter sector this is [7.203]

$$\begin{aligned} Z_X(t) &= q^{-d/24}\mathrm{Tr}\, q^{\alpha' p^2 + \sum_{n=1}^{\infty} n\sum_{\mu=0}^{\infty}N_{\mu n}} \\ &= q^{-d/24}V_d\int\frac{d^dk}{(2\pi)^d}q^{\alpha' k^2}\prod_{n=1}^{\infty}\prod_{\mu=0}^{25}\sum_{N_{\mu n}} q^{nN_{\mu n}} \\ &= iV_d(8\pi^2 t\alpha')^{-d/2}\eta(it)^{-d} \end{aligned} \qquad [8.578]$$

The ghost sector give a contribution [7.205]

$$Z_g = \eta(it)^2 \qquad [8.579]$$

Combining these and integrating over the modulus we then get [7.207]

$$Z_{C_2} = in^2V_{26}\int_0^\infty\frac{dt}{2t}(8\pi^2 t\alpha')^{-D/2}\eta(it)^{-24} \qquad [8.580]$$

where we have also included the contribution n^2 from the Chan-Paton factors.

We now adapt this to the case at hand:

1. Replace D by $p+1$ as that is the number of degrees of freedom (the other ends are fixed on the hyperplane)

2. The weights h_i come from the L_0 part in the trace. L_0 is the Hamiltonian; it measures the energy of the string. The open string now is stretched between two hyperplanes at distance y and thus there is an extra contribution $y^2/4\pi^2\alpha'$. Why exactly that is the contribution is not clear to me.

3. There is no Chan-Paton factor per se, but a factor of two as explained in the book

All this means that we can write

$$\mathcal{A} = iV_{p+1}\int_0^\infty \frac{dt}{t}(8\pi^2 t\alpha')^{-(p+1)/2}e^{-2\pi t\frac{y^2}{4\pi^2\alpha'}}\eta(it)^{-24} \quad [8.581]$$

In order to derive the second line of (8.7.17), we follow the same procedure as for the derivation of (7.4.3). We use (7.4.2), change variables $t=\pi/s$ and use (7.4.4) to expand around $s=0$, i.e. around $t=\infty$, to see that

$$\begin{aligned}\eta(it)^{-24} &= \left[t^{-1/2}\eta(i/t)\right]^{-24} = t^{12}\eta(i\pi/s)^{-24} = t^{12}\left(e^{2s}+24+\cdots\right)\\ &= t^{12}\left(e^{2\pi/t}+24+\cdots\right)\end{aligned} \quad [8.582]$$

We therefore find that

$$\begin{aligned}\mathcal{A} &= iV_{p+1}\int_0^\infty \frac{dt}{t}(8\pi^2 t\alpha')^{-(p+1)/2}e^{-\frac{ty^2}{2\pi\alpha'}}t^{12}\left(e^{2\pi/t}+24+\cdots\right)\\ &= \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{-1-(p+1)/2+12}e^{-\frac{ty^2}{2\pi\alpha'}}\left(e^{2\pi/t}+24+\cdots\right)\\ &= \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(21-p)/2}e^{-\frac{ty^2}{2\pi\alpha'}}\left(e^{2\pi/t}+24+\cdots\right)\end{aligned} \quad [8.583]$$

8.121 p 275: Eq. (8.7.17) The *D*-Brane Annulus Vacuum Amplitude, II

Introduce the variable $u = ty^2/2\pi\alpha'$ and consider the second term in the expansion (8.7.17)

$$\begin{aligned}\mathcal{A} &= \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty \frac{2\pi\alpha' du}{y^2}\left(\frac{2\pi\alpha' u}{y^2}\right)^{(21-p)/2}e^{-u}\\ &= \frac{24iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}(2\pi\alpha')^{(23-p)/2}y^{p-23}\int_0^\infty du\, u^{(23-p)/2-1}e^{-u}\end{aligned} \quad [8.584]$$

Using the definition of the Gamma function:

$$\begin{aligned}\mathcal{A} =& 24iV_{p+1}2^{10-2p}\pi^{(21-3p)/2}\alpha'^{11-p}y^{p-23}\Gamma\left(\frac{23-p}{2}\right)\\ =& 24iV_{p+1}2^{10-2p-22+2p}\pi^{(21-3p)/2-22+2p}(4\pi^2\alpha')^{11-p}y^{p-23}\Gamma\left(\frac{23-p}{2}\right)\\ =& iV_{p+1}24\times 2^{-12}\pi^{(p-23)/2}(4\pi^2\alpha')^{11-p}|y|^{p-23}\Gamma\left(\frac{23-p}{2}\right)\end{aligned} \quad [8.585]$$

In the last line we have replaced y by $|y|$ as it is an absolute number anyway in view of it being the distance between the two D-branes. This the first line of (8.7.18). The second line follows with the definition

$$G_{25-p}(y) = \frac{\pi^{(p-23)/2}}{4}\Gamma\left(\frac{23-p}{2}\right)|y|^{p-23} \quad [8.586]$$

or, setting $25-p=d$,

$$G_d(y) = \frac{\pi^{(2-d)/2}}{4}\Gamma\left(\frac{d-2}{2}\right)|y|^{2-d} \quad [8.587]$$

We'll leave it as an exercise to the reader to show that $G_d(y)$ is indeed the Green's function for the massless scalar field in $d>2$ dimensions. One certainly immediately recognises the correct y behaviour.

8.122 p 275: Eq. (8.7.19) The Space-Time Action

As it has been some time, let us recall how the field theory approach is obtained. We consider the general non-linear sigma model on the worldsheet with action (3.7.6), i.e. of

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{g}\left[\left(g^{ab}G_{\mu\nu}(X)+i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu\partial_b X^\nu + \alpha' R\Phi(X)\right] \quad [8.588]$$

Just to remind ourselves, g^{ab} is the (Euclidean) worldsheet metric, $G_{\mu\nu}$ a symmetric tensor that we can view as being the spacetime metric, B_μ is an antisymmetric spacetime field, also known as the Kalb-Ramond field, and Φ is the dilation. Note that R here is the (two-dimensional) worldsheet curvature. This Lagrangian is the general form of a theory of interacting string. Indeed we saw that starting form the free string theory, and adding all possible vertex operators, i.e. asymptotic string states, we were naturally lead to this action.

We then required Weyl invariance of this theory after quantisation. This implied the vanishing of the β functions of the theory, (3.7.14-15), which give string corrections to the equations of general relativity. We then showed that it was possible to derive the vanishing of the β functions by an action principle from a field theory point of view. The corresponding action to leading order in α' was given by (3.7.20)

$$S = \frac{1}{2\kappa_0^2}\int d^D X\,\sqrt{-G}e^{-2\Phi}\left[\gamma + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + o(\alpha')\right] \qquad [8.589]$$

with $\gamma = -2(D-26)/3\alpha'$. Note that $\gamma = 0$ for the critical string. Here $\boldsymbol{R}$ is the spacetime curvature, $H_{\mu\nu\lambda}$ is the field tensor of the antisymmetric $B_{\mu\nu}$ field, defined in (3.7.8) and κ_0 is an undetermined normalisation constant. Note that there is no reference to the string or its worldsheet in this action. This is just a field theory of a coordinate X, hence the name field theory approach.

This action is, however, not convenient because the dilation couples with everything due to the $e^{-\Phi}$ factor. We showed that it was convenient to make a field redefinition $\tilde{G}_{\mu\nu}(x) = e^{2\omega(x)}G_{\mu\nu}(x)$ with $\omega = 2(\Phi-\Phi_0)/(D-2)$ and define $\tilde{\Phi} = \Phi - \Phi_0$, with Φ_0 being the (constant) expectation value of Φ. The spacetime action then becomes

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^D X\,\sqrt{-\tilde{G}}\Bigg[&\gamma e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}} - \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} \\ &- \frac{4}{D-2}\partial_\mu\tilde{\Phi}\tilde{\partial}^\mu\tilde{\Phi} + o(\alpha')\Bigg] \end{aligned} \qquad [8.590]$$

Here indices are raised with $\tilde{G}^{\mu\nu}$, as denoted by the tilde on $\tilde{\boldsymbol{R}}$, $\tilde{H}^{\mu\nu\lambda}$ and on $\tilde{\partial}^\mu$. Often $\tilde{G}_{\mu\nu}$ is referred to as the Einstein metric. Finally, $\kappa = \kappa_0 e^{\Phi_0}$.

Setting $D = 26$ and ignoring the Kalb-Ramond field (why can we do that?), we immediately recover (8.7.19).

8.123 p 275: Eq. (8.7.20) The D-Brane Action as a Function of the Spacetime Fields

As we are ignoring the gauge fields, (8.7.2) becomes

$$S_p = -T_p\int d^{p+1}\xi\, e^{-\Phi}\left[-\det G_{ab}\right]^{1/2} \qquad [8.591]$$

We wish to express this in terms of the tilde spacetime fields. We have, as G_{ab} is a $(p+1)\times(p+1)$ matrix

$$\begin{aligned} \det\tilde{G}_{ab} &= \det\left(e^{2\omega}\det G_{ab}\right) = e^{2(p+1)\omega}\det G_{ab} = e^{\frac{4(p+1)(\Phi_0-\Phi)}{D-2}}\det G_{ab} \\ &= e^{\frac{(p+1)(\Phi_0-\Phi)}{6}}\det G_{ab} \end{aligned} \qquad [8.592]$$

Therefore

$$\begin{aligned} S_p &= -T_p \int d^{p+1}\xi\, e^{-\tilde{\Phi}-\Phi_0} e^{-\frac{(p+1)(\Phi_0-\Phi)}{12}} \left[-\det \tilde{G}_{ab}\right]^{1/2} \\ &= -T_p e^{-\frac{p+13}{12}\Phi_0} \int d^{p+1}\xi\, e^{\frac{p-11}{12}\tilde{\Phi}} \left[-\det \tilde{G}_{ab}\right]^{1/2} \end{aligned} \qquad [8.593]$$

This is (8.7.20) but with a different definition of τ_p.

8.124 p 276: Eq. (8.7.20) The D-Brane Action as a Function of the Spacetime Fields

Including the gauge-fixing term the spacetime action becomes

$$S = \frac{1}{2\kappa^2} \int d^{26}D\, \sqrt{-\tilde{G}} \left[\tilde{\boldsymbol{R}} - \frac{1}{6}\left(\nabla_\mu \tilde{\Phi}\right)\left(\tilde{G}^{\mu\sigma}\nabla_\sigma \tilde{\Phi}\right) - \frac{1}{2}\eta^{\mu\nu} f_\mu f_\nu\right] \qquad [8.594]$$

This should be an entirely straightforward calculation, which we will not bother doing at this stage. For those of you impelled to do so by some higher force(s), recall that the inverse Einstein metric is given by $\tilde{G}^{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$ and that the dilaton is a spacetime scalar and we can replace its covariant derivative by an ordinary derivative.

8.125 p 276: Eq. (8.7.23) The Propagator for the Graviton and the Dilaton Field

Let us recall how we derive the propagator of a scalar field φ with Lagrangian $\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2$. Adding a source J, the path integrals is

$$Z[J] = \int \mathcal{D}\varphi e^{i\int d^d d\left[-\frac{1}{2}\varphi(\partial^2 - m^2)\varphi + J\varphi\right]} \qquad [8.595]$$

We can perform the Gaussian integral and up to a normalisation factor we find

$$Z[J] = e^{-\frac{i}{2}\int d^d x\, \int d^d y\, J(x) D(x-y) J(y)} \qquad [8.596]$$

where $D(x-y$ is the Green's function, a.k.a. propagator and satisfies

$$-(\partial^2 + m^2) = \delta^{(d)}(x-y) \qquad [8.597]$$

The use of the propagator becomes apparent in perturbation theory. E.g. if we add an interaction term $\mathcal{L}_I[\varphi]$ to the Lagrangian, then we can write the path integral of the interacting theory as

$$Z_I[J] = e^{i\int d^4 w\, \mathcal{L}_I[\delta/i\delta J(w)]} e^{-\frac{i}{2}\int d^d x\, \int d^d y\, J(x) D(x-y) J(y)} \qquad [8.598]$$

and thus once we know the propagator we can use it to work out the perturbative expansion. This is, of course, the basis of the Feynman diagrams. To derive the form of the propagator we write the delta function in its Fourier transform

$$\delta^{(d)}(x-y) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \qquad [8.599]$$

and one can check by direct computation that the propagator is then given by

$$D(x-y) = \int \frac{d^d k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+m^2+i\varepsilon} \qquad [8.600]$$

and in momentum space

$$D(k) = \frac{1}{k^2+m^2} \qquad [8.601]$$

All this should be well-known to 99.9% of you, with the 0.1% being those who believed that you can learn string theory without knowing any quantum field theory.

Let us now apply this to the case at hand. The dilaton is a spacetime scalar and has kinetic term $(-1/12\kappa^2)\partial_\mu\tilde{\Phi}\partial^{\hat{\mu}}\tilde{\Phi}$. The propagator in momentum space is hence

$$\langle\tilde{\Phi}(k)\tilde{\Phi}(-k)\rangle = -\frac{6i\kappa^2}{k^2} \qquad [8.602]$$

The kinetic term for the graviton is

$$\begin{aligned}
\mathcal{L}_{\text{grav}} &= -\frac{1}{8\kappa^2}\left(\partial_\mu h_{\nu\lambda}\partial^{\hat{\mu}}h^{\hat{\nu}\hat{\lambda}} - \frac{1}{2}\partial_\mu h_\nu^{\hat{\nu}}\partial^{\hat{\mu}}h_\lambda^{\hat{\lambda}}\right)\\
&= -\frac{1}{8\kappa^2}\left(-\eta^{\mu\rho}\eta^{\nu\kappa}\eta^{\lambda\tau}h_{\nu\lambda}\partial_\mu\partial_\rho h_{\kappa\tau} + \frac{1}{2}\eta^{\nu\lambda}\eta^{\mu\rho}\eta^{\kappa\tau}h_{\lambda\nu}\partial_\mu\partial_\rho h_{\tau\kappa}\right)\\
&= -\frac{1}{8\kappa^2}h_{\nu\lambda}\left(-\eta^{\mu\rho}\eta^{\nu\kappa}\eta^{\lambda\tau} + \frac{1}{2}\eta^{\nu\lambda}\eta^{\mu\rho}\eta^{\kappa\tau}\right)\partial_\mu\partial_\rho h_{\tau\kappa}\\
&= -\frac{1}{8\kappa^2}h_{\nu\lambda}\left(-\eta^{\nu\kappa}\eta^{\lambda\tau} + \frac{1}{2}\eta^{\nu\lambda}\eta^{\kappa\tau}\right)\partial^2 h_{\tau\kappa}\\
&= \frac{1}{16\kappa^2}h_{\nu\lambda}\left(\eta^{\nu\kappa}\eta^{\lambda\tau} + \eta^{\nu\tau}\eta^{\lambda\kappa} - \eta^{\nu\lambda}\eta^{\kappa\tau}\right)\partial^2 h_{\tau\kappa} = \frac{1}{16\kappa^2}h_{\nu\lambda}A^{\nu\lambda,\tau\kappa}\partial^2 h_{\tau\kappa} \qquad [8.603]
\end{aligned}$$

with

$$A^{\nu\lambda,\tau\kappa} = \eta^{\nu\kappa}\eta^{\lambda\tau} + \eta^{\nu\tau}\eta^{\lambda\kappa} - \eta^{\nu\lambda}\eta^{\kappa\tau} \qquad [8.604]$$

We now need the inverse of $A^{\nu\lambda,\tau\kappa}$. If we consider

$$\tilde{D}_{\mu\nu,\sigma\rho} = \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \alpha\eta_{\mu\nu}\eta_{\sigma\rho} \qquad [8.605]$$

It has the right symmetries and α can be determined. We have

$$\begin{aligned}\tilde{D}_{\mu\nu,\sigma\rho}A^{\sigma\kappa,\rho\tau} &= [\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \alpha\eta_{\mu\nu}\eta_{\sigma\rho}] \times (\eta^{\sigma\tau}\eta^{\kappa\rho} + \eta^{\sigma\rho}\eta^{\kappa\tau} - \eta^{\sigma\kappa}\eta^{\rho\tau}) \\ &= \delta^\tau_\mu\delta^\kappa_\nu + \eta_{\mu\nu}\eta^{\kappa\tau} - \delta^\kappa_\mu\delta^\tau_\nu + \delta^\kappa_\mu\delta^\tau_\nu + \eta_{\mu\nu}\eta^{\kappa\tau} - \delta^\tau_\mu\delta^\kappa_\nu \\ &\quad + \alpha\eta_{\mu\nu}(\eta^{\kappa\tau} + \eta^{\kappa\tau}\delta^\sigma_\sigma - \eta^{\kappa\tau}) \\ &= [2 + \alpha(1 + D - 1)]\,\eta_{\mu\nu}\eta^{\kappa\tau} = (2 + \alpha D)\eta_{\mu\nu}\eta^{\kappa\tau}\end{aligned} \tag{8.606}$$

But this does not have the right form; it should have a $\delta^\kappa_\mu\delta^\tau_\nu + \delta^\tau_\mu\delta^\kappa_\nu$.

If we assume that there is a typo in the spacetime action (8.7.22) and that the second term quadratic in the gravitons should come with a plus sign, then we find that

$$A^{\nu\lambda,\tau\kappa} = \eta^{\nu\kappa}\eta^{\lambda\tau} + \eta^{\nu\tau}\eta^{\lambda\kappa} + \eta^{\nu\lambda}\eta^{\kappa\tau} \tag{8.607}$$

and

$$\begin{aligned}\tilde{D}_{\mu\nu,\sigma\rho}A^{\sigma\kappa,\rho\tau} &= [\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \alpha\eta_{\mu\nu}\eta_{\sigma\rho}] \times (\eta^{\sigma\tau}\eta^{\kappa\rho} + \eta^{\sigma\rho}\eta^{\kappa\tau} + \eta^{\sigma\kappa}\eta^{\rho\tau}) \\ &= \delta^\tau_\mu\delta^\kappa_\nu + \eta_{\mu\nu}\eta^{\kappa\tau} + \delta^\kappa_\mu\delta^\tau_\nu + \delta^\kappa_\mu\delta^\tau_\nu + \eta_{\mu\nu}\eta^{\kappa\tau} + \delta^\tau_\mu\delta^\kappa_\nu \\ &\quad + \alpha\eta_{\mu\nu}(\eta^{\kappa\tau} + \eta^{\kappa\tau}\delta^\sigma_\sigma + \eta^{\kappa\tau}) \\ &= 2(\delta^\kappa_\mu\delta^\tau_\nu + \delta^\tau_\mu\delta^\kappa_\nu) + [2 + \alpha(1 + D + 1)]\,\eta_{\mu\nu}\eta^{\kappa\tau} \\ &= 2(\delta^\kappa_\mu\delta^\tau_\nu + \delta^\tau_\mu\delta^\kappa_\nu) + [2 + \alpha(D + 2)]\eta_{\mu\nu}\eta^{\kappa\tau}\end{aligned} \tag{8.608}$$

If we choose $\alpha = -2/(D+2)$ then we get

$$\tilde{D}_{\mu\nu,\sigma\rho}A^{\sigma\kappa,\rho\tau} = 2(\delta^\kappa_\mu\delta^\tau_\nu + \delta^\tau_\mu\delta^\kappa_\nu) \tag{8.609}$$

And so if we define $D_{\mu\nu,\sigma\rho} = \frac{1}{4}\tilde{D}_{\mu\nu,\sigma\rho}$ then

$$D_{\mu\nu,\sigma\rho}A^{\sigma\kappa,\rho\tau} = \frac{1}{2}(\delta^\kappa_\mu\delta^\tau_\nu + \delta^\tau_\mu\delta^\kappa_\nu) \tag{8.610}$$

and D is the inverse of A. We could therefore write the propagator as

$$\langle h_{\mu\nu}(k)h_{\sigma\rho}(-k)\rangle = \frac{16i\kappa^2}{k^2}\frac{1}{4}(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \alpha\eta_{\mu\nu}\eta_{\sigma\rho}) \tag{8.611}$$

But unfortunately, this is not (8.7.23).

It seems like the action has the right symmetry structure for the graviton, and so has the propagator, but still I cannot reproduce one from the other. Either there must be some strange typo somewhere, or I am doing something wrong, The latter is the more plausible explanation. In any case the form of the propagator is consistent with the rest of the derivation, so I assume there is no error in that formula.

8.126 p 276: Eq. (8.7.25) The Amplitude for a Propagating Graviton and Dilaton

We are evaluating the propagation of a graviton and a dilaton between two D-branes. This means that the D-brane is a source/sink for the graviton dilaton. We thus wish to evaluate the path integral

$$\tilde{\mathcal{A}} = \int \mathcal{D}h\, \mathcal{D}\tilde{\Phi} e^{iS} e^{iS_p} = \int \mathcal{D}h\, \mathcal{D}\tilde{\Phi} e^{iS} e^{-i\tau_p \int d^{p+1}\xi\, \left(\frac{p-11}{12}\tilde{\Phi} - \frac{1}{2}h_{aa}\right)} \qquad [8.612]$$

Here S is the spacetime action (87.7.22) and S_p is the D-brane action (8.7.24), both taken around a flat spacetime. As usual we expand the exponential and, as we are looking at the (lowest order) contribution for the propagation of a graviton and dilaton between the D-branes, we only keep the terms quadratic in $\tilde{\Phi}$ and the terms quadratic in h. This gives

$$\begin{aligned} \tilde{\mathcal{A}} &= \int \mathcal{D}h\, \mathcal{D}\tilde{\Phi} e^{iS} \left(-\frac{\tau_p^2}{2}\right) \left[\left(\frac{p-11}{12}\right)^2 \tilde{\Phi}\tilde{\Phi} + \frac{1}{4} h_{aa} h_{bb}\right] + \cdots \\ &= -\frac{\tau_p^2}{2} \left[\left(\frac{p-11}{12}\right)^2 \langle\tilde{\Phi}\tilde{\Phi}\rangle + \frac{1}{4}\langle h_{aa} h_{bb}\rangle\right] + \cdots \end{aligned} \qquad [8.613]$$

For $D = 26$ we have from (8.7.23)

$$\begin{aligned} \langle\tilde{\Phi}\tilde{\Phi}\rangle &= -\frac{6i\kappa^2}{k_\perp^2} \\ \langle h_{aa} h_{bb}\rangle &= -\frac{2i\kappa^2}{k_\perp^2}\left(\eta_{ab}\eta_{ab} + \eta_{ab}\eta_{ab} - \frac{1}{12}\eta_{aa}\eta_{bb}\right) \\ &= -\frac{2i\kappa^2}{k_\perp^2}\left[2(p+1) - \frac{1}{12}(p+1)^2\right] \end{aligned} \qquad [8.614]$$

We have use the fact that $\eta_{aa} = p+1$ as $a = 0, 1, \cdots, p$. Here $k_\perp$ is the momentum of the graviton/dilaton, i.e. the momentum perpendicular to the D-brane; it has $p+1$ components. Using this we find that

$$\begin{aligned} \tilde{\mathcal{A}} &= -\frac{\tau_p^2}{2}\left[\left(\frac{p-11}{12}\right)^2\left(-\frac{6i\kappa^2}{k_\perp^2}\right) + \frac{1}{4}\left\{-\frac{2i\kappa^2}{k_\perp^2}\left[2(p+1) - \frac{1}{12}(p+1)^2\right]\right\}\right] + \cdots \\ &= \frac{i\tau_p^2\kappa^2}{2k_\perp^2}\left\{6\left(\frac{p-11}{12}\right)^2 + \frac{1}{2}\left[2(p+1) - \frac{1}{12}(p+1)^2\right]\right\} \\ &= \frac{i\tau_p^2\kappa^2}{2k_\perp^2}\left(\frac{p^2}{24} + \frac{121}{24} - \frac{11p}{12} + p + 1 - \frac{p^2}{24} - \frac{p}{12} - \frac{1}{24}\right) = \frac{3i\tau_p^2\kappa^2}{k_\perp^2} \end{aligned} \qquad [8.615]$$

We have been quite sloppy about this calculation and it is now time to make some corrections. In [8.613] we forget the (double) integration over the worldvolume coordinates. We then used the propagators in momentum space, whilst we needed them in coordinate space. This correction leads to a delta function for the momentum conservation, which takes care of one of the integrals. The other integral will give a volume factor V_{p+1}. Moreover the graviton and dilaton can propagate from one D-brane to the other or vice versa. This gives a doubling of the contribution. Our final result is thus

$$\mathcal{A} = \frac{6i\tau_p^2\kappa^2}{k_\perp^2}V_{p+1} \tag{8.616}$$

8.127 p 276: Eq. (8.7.26) The Relation Between τ_p and κ

Before we can compare (8.7.18) with (8.7.15) we need to get the momentum representation of (8.7.18). The Fourier transform just gives the traditional k^2, as can also be seen from the fact that we are looking ate the inverse of $\nabla^2 = \partial^2$, thus yielding a k^{-2}. Comparing the two results thus gives

$$iV_{p+1}\frac{24\pi}{2^{10}}(4\pi^2\alpha')^{11-p}\frac{1}{k_\perp^2} = \frac{6i\tau_p^2\kappa^2}{k_\perp^2}V_{p+1} \tag{8.617}$$

or, cleaning up,

$$\frac{\pi}{2^8}(4\pi^2\alpha')^{11-p} = \tau_p^2\kappa^2 \qquad \Rightarrow \qquad \tau_p^2 = \frac{\pi}{256\kappa^2}(4\pi^2\alpha')^{11-p} \tag{8.618}$$

The recursion relation (8.7.16) is, using $T_p = e^{\Phi_0}\tau_p$

$$\tau_p = \frac{1}{(4\pi^2\alpha')^{1/2}}\tau_{p-1} \tag{8.619}$$

and (8.7.26) clearly satisfies this.

8.128 p 276: Eq. (8.7.27) The Gauge Field Action for the $D25$-Brane

The $D25$-brane has $p = 1$ and so the hyperplane has dimension $p+1 = 26$ and is the entire spacetime. I.e. this is the standard open string theory with all dimensions having Neumann boundary conditions. We now attache to each end-point n-values Chan-Paton factors. The $D25$-brane action (8.7.12) is then

$$S_{25} = -T_{25}\int d^{26}X \,\mathrm{tr}\,\left\{e^{-\Phi}\left[-\det\big(G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\big)\right]^{1/2} + \cdots\right\} \tag{8.620}$$

we can replace the indices a, b by $\mu\nu$ as they now run over all dimensions anyway. In order to find the quadratic terms in F we can set $B = 0$. We can't set $G_{\mu\nu} = 0$ as this is the spacetime metric necessary to raise and lower indices. We also use the dynamic dilaton field $\tilde{\phi} = \phi - \phi_0$. Then $e^{-\Phi} = e^{-\tilde{\Phi}-\Phi_0}$. We also have that $T_p = e^{\Phi_0}\tau_p$. The Φ_0 contribution then cancels and we can ignore the $\tilde{\Phi}$ contributions as these would give interactions with the dilaton and nor contribute to the gauge field kinetic term. We thus need

$$S_{25} = -\tau_{25} \int d^{26}X \, \mathrm{tr} \, \left[-\det(G_{\mu\nu} + f_{\mu\nu})\right]^{1/2} + \cdots \qquad [8.621]$$

where we have defined $f_{\mu\nu} = 2\pi\alpha' F_{\mu\nu}$.

Now recall that, as it should, this action is spacetime parametrisation invariant. This is obviously the case if we set $f_{\mu\nu} = 0$, but as this has the same tensorial structure as $G_{\mu\nu}$, we also have that $\int d^{26}X \sqrt{-\det(G+f)}$ is invariant. So, if there is a term quadratic in $f_{\mu\nu}$ it must be of the form $\sqrt{-G} f_{\mu\nu} f^{\mu\nu} = \sqrt{-G} f^{\mu\nu} G^{\mu\sigma} G^{\nu\rho} f_{\sigma\rho}$. This means that the only thing we really need to do is to find the coefficient in front of that term, i.e. find the α in

$$\left[-\det(G_{\mu\nu} + f_{\mu\nu})\right]^{1/2} = \alpha\sqrt{-G} f_{\mu\nu} f^{\mu\nu} = \alpha\sqrt{-G} f_{\mu\nu} G^{\mu\sigma} G^{\nu\rho} f_{\sigma\rho} + \cdots \qquad [8.622]$$

This in itself is messy, but fortunately we can focus on one special case. For example, let us take four dimensions and let us check the coefficient of the quadratic term in $f_{12} f_{23}$. If that coefficient is non-zero, it will be the same for all other (non-zero) combinations of indices and hence also equal to α. In order to find this coefficient we simply calculate

$$\tilde{\alpha} = \frac{\partial^2}{\partial f_{12} \partial f_{23}} \left[-\det(G_{\mu\nu} + f_{\mu\nu})\right]^{1/2} \Bigg|_{f_{\mu\nu}=0} \qquad [8.623]$$

This is as well a tedious calculation and best performed by Mathematica, see fig.8.12. The results turns out to be

$$\tilde{\alpha} = \frac{G_{14}G_{34} - G_{13}G_{44}}{\sqrt{-G}} \qquad [8.624]$$

In other words we have a term in the expansion of $\left[-\det(G_{\mu\nu} + f_{\mu\nu})\right]^{1/2}$ of the form

$$f_{12} \frac{G_{14}G_{34} - G_{13}G_{44}}{\sqrt{-G}} f_{23} \qquad [8.625]$$

```
In[63]:= ClearAll[dim, M, G, F, Mat, GMat, IGMat, DM, DG, alphatilde , beta0, beta1];
dim = 4;
M[m_, n_] := G[m, n] + F[m, n]
Do[F[m, m] = 0, {m, dim}]
Do[F[n, m] = -F[m, n], {n, dim}, {m, n}]
Do[G[n, m] = G[m, n], {n, dim}, {m, n}]
Mat = {{M[1, 1], M[1, 2], M[1, 3], M[1, 4]}, {M[2, 1], M[2, 2], M[2, 3], M[2, 4]},
{M[3, 1], M[3, 2], M[3, 3], M[3, 4]}, {M[4, 1], M[4, 2], M[4, 3], M[4, 4]}};
GMat = {{G[1, 1], G[1, 2], G[1, 3], G[1, 4]}, {G[2, 1], G[2, 2], G[2, 3], G[2, 4]},
{G[3, 1], G[3, 2], G[3, 3], G[3, 4]}, {G[4, 1], G[4, 2], G[4, 3], G[4, 4]}};
IGMat = Inverse[GMat];
DM = Expand[Det[Mat]];
DG = Expand[Det[GMat]];
F[m_, n_, p_, q_] := D[D[Sqrt[-DM], F[m, n]], F[p, q]]
alphatilde = Simplify[F[1, 2, 2, 3]];
Do[F[m, n] = 0, {m, dim}, {n, dim}] ;
alphatilde = Simplify[test1];
Simplify[alphatilde - (G[1, 4] * G[3, 4] - G[1, 3] * G[4, 4]) / Sqrt[-DG]]

Out[78]= 0

In[79]:= beta0 = Simplify[IGMat[[1, 2]] * IGMat[[2, 3]] - IGMat[[1, 3]] * IGMat[[2, 2]]];
beta1 = Simplify[(G[1, 4] * G[3, 4] - G[1, 3] * G[4, 4]) / (-DG)];
beta0 - beta1

Out[81]= 0
```

Figure 8.12: Mathematica code for the $D25$-brane kinetic field strength term. alphatilde corresponds to $\tilde{\alpha}$ in [8.624]; beta0 is $G^{12}G^{23} - G^{13}G^{22}$ in [8.629] and beta1 is $(G_{14}G_{34} - G_{13}G_{44})/(-G)$ in [8.629].

We now need to link this to a term in $\sqrt{-G} f_{\mu\nu} f^{\mu\nu}$. We consider $f_{12} f^{12}$ first. This is

$$\begin{aligned} f_{12} f^{12} &= f_{12} G^{1\mu} G^{2\nu} f_{\mu\nu} = f_{12} G^{12} G^{23} f_{23} + f_{12} G^{13} G^{22} f_{32} + \cdots \\ &= f_{12} (G^{12} G^{23} - G^{13} G^{22}) f_{23} + \cdots \end{aligned} \qquad [8.626]$$

where we have only shown the term in $f_{12} f_{23}$. We have a similar contribution from $f_{21} f^{21}$. But there is also a contribution from $f_{23} f^{23}$

$$\begin{aligned} f_{23} f^{23} &= f_{23} G^{2\mu} G^{3\nu} f_{\mu\nu} = f_{23} G^{21} G^{32} f_{12} + f_{23} G^{22} G^{31} f_{21} + \cdots \\ &= f_{12} (G^{12} G^{23} - G^{13} G^{22}) f_{23} + \cdots \end{aligned} \qquad [8.627]$$

and a contribution from $f_{32}f^{32}$. We thus have Therefore we thus conclude the

$$\alpha\sqrt{-G}f_{\mu\nu}f^{\mu\nu} = 4\alpha\sqrt{-G}\, f_{12}(G^{12}G^{23} - G^{13}G^{22})f_{23} \qquad [8.628]$$

Direct calculation, see fig.8.12, teaches us that

$$\beta = G^{12}G^{23} - G^{13}G^{22} = \frac{G_{14}G_{34} - G_{13}G_{44}}{-G} \qquad [8.629]$$

Therefore we thus conclude the

$$\begin{aligned}\alpha\sqrt{-G}f_{\mu\nu}f^{\mu\nu} &= 4\alpha\sqrt{-G}f_{12}\frac{G_{14}G_{34} - G_{13}G_{44}}{-G}f_{23} \\ &= 4\alpha f_{12}\frac{G_{14}G_{34} - G_{13}G_{44}}{\sqrt{-G}}f_{23}\end{aligned} \qquad [8.630]$$

Comparing [8.625] with [8.630] we find that $4\alpha = 1$ or hence $\alpha = 1/4$, i.e.

$$[-\det(G_{\mu\nu} + f_{\mu\nu})]^{1/2} = \frac{1}{4}\sqrt{-G}f_{\mu\nu}f^{\mu\nu} + \cdots \qquad [8.631]$$

We will leave it as en exercise to the industrious reader to show the same results for spacetime dimensions higher than four.

Using this in [8.621] we therefore find that

$$S_{25} = -\frac{\tau_{25}}{4}\int d^{26}X\,\sqrt{-G}f_{\mu\nu}f^{\mu\nu} + \cdots = -\frac{\tau_{25}}{4}\int d^{26}X\,\sqrt{-G}(2\pi\alpha')^2F_{\mu\nu}F^{\mu\nu} + \cdots \qquad [8.632]$$

which is what we set out to show (as usual the $\sqrt{-G}$ is necessary for spacetime parametrisation invariant and is understood as part of the measure of the action).

8.129 p 276: Eq. (8.7.28) The Relation Between the Coupling Constants

Let us first recall the different formula mentioned and what they mean. (6.5.14) Relates the coupling constant for an open string tachyon g_0 with that of an open string gauge boson g'_0

$$g'_0 = \frac{1}{\sqrt{2\alpha'}}g_0 \qquad [8.633]$$

The effective spacetime action for the open string, up to first order in momenta, that reproduces the open string amplitudes is given by (6.5.16)

$$S = \frac{1}{g_0'^2}\int d^{26}x\left(-\frac{1}{2}\mathrm{tr}\ D_\mu\varphi D^\mu\varphi + \frac{1}{2\alpha'}\mathrm{tr}\ \varphi^2 + \frac{2}{3\sqrt{2\alpha'}}\mathrm{tr}\ \varphi^3 - \frac{1}{4}\mathrm{tr}\ F_{\mu\nu}F^{\mu\nu}\right) \qquad [8.634]$$

In (6.6.18) we relate the closed string coupling constant g_c to the gravitational constant κ

$$\kappa = 2\pi g_c \qquad [8.635]$$

From [8.633] and [8.635] we find that

$$\frac{g_0^2}{g_c} = \frac{2\alpha' g_0'^2}{\kappa/2\pi} = \frac{4\pi\alpha' g_0'^2}{\kappa} \qquad [8.636]$$

Now comparing the coupling constants between [8.634] with [8.632] we see that

$$\frac{1}{g_0'^2} = \tau_{25}(2\pi\alpha')^2 = \frac{\sqrt{\pi}}{16\kappa}(4\pi^2\alpha')^{(11-p)/2}(2\pi\alpha')^2 \qquad [8.637]$$

where we have also used (8.7.26). Thus, setting $p = 25$,

$$\begin{aligned}\frac{g_0^2}{g_c} &= \frac{4\pi\alpha'}{\kappa}\frac{16\kappa}{\sqrt{\pi}}\frac{1}{(4\pi^2\alpha')^{-7}(2\pi\alpha')^2} = 2^{2+4+14-2}\pi^{1-1/2+14-2}\alpha'^{1+7-2}\\ &= 2^{18}\pi^{25/2}\alpha'^6\end{aligned} \qquad [8.638]$$

Note that we did not need (6.6.15), the equivalent of (6.5.14) but for the closed string, that relates the coupling constant for an closed string tachyon g_c with that of an closed string gauge boson g_c'

$$g_c' = \frac{2}{\alpha'}g_c \qquad [8.639]$$

8.130 p 277: Eq. (8.8.1) The Impact of the Worldsheet Parity on the Worldsheet Coordinates

Recall that the worldsheet parity Ω inverts the orientation of the coordinate coordinate σ_1, i.e. i.o. running from 0 to 2π it now runs from 2π to zero. What moved to the right is now moving to the left, and vice-versa. The worldsheet coordinates themselves don't change. Thus the action of Ω is simply

$$X_L^M(z) \longleftrightarrow X_R^M(z) \qquad \text{for} \qquad M = 1, \cdots, D \qquad [8.640]$$

Note that we have $X_R^M(z)$ and not $X_M^R(\bar{z})$ as the definition of the coordinates don't change.

On the non-dualised coordinates we have

$$\Omega\, X^\mu(z,\bar{z}) = \Omega\left(X_L^\mu(z) + X_R^\mu(z)\right) = X_R^\mu(z) + X_L^\mu(\bar{z}) = X^\mu(\bar{z}, z) \qquad [8.641]$$

On the dualised coordinates we get

$$\Omega\, X'^m(z,\bar{z}) = \Omega\left(X_L'^m(z) - X_R'^m(z)\right) = X_R'^m(z) - X_L'^m(\bar{z}) = -X'^m(\bar{z}, z) \qquad [8.642]$$

The latter is the combination fo a worldsheet parity $x \leftrightarrow \bar{z}$ and a spacetime reflection $X \leftrightarrow -X$.

8.131 p 277: Eq. (8.8.3) The Fields G_{MN} and B_{MN} of an Orientifold

We need to consider again the worldsheet action (3.7.6)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[\left(g^{ab} G_{MN}(X) + i\epsilon^{ab} B_{MN} \right) \partial_a X^M \partial_b X^N + \alpha' R\Phi(X) \right] \qquad [8.643]$$

The orientifold is a parity operation on the worldsheet coordinates of the spacetime fields. In addition, for the dualised coordinates, we also have a spacetime inversion. Consider first the $g^{ab}G_{MN}(X)\partial_a X^M \partial_b X^N$. In the gauge $g^{ab} = \delta^{ab}$ we have

$$g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = G_{\mu\nu}(X)(\partial_1 X^\mu \partial_1 X^\nu + \partial_2 X^\mu \partial_2 X^\nu) \qquad [8.644]$$

The worldsheet parity operator transforms ∂_1 into $-\partial_1$ and X into X'. Thus here Ω acts as

$$\begin{aligned} \Omega\, g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu &= G_{\mu\nu}(X')\left[(-\partial_1 X^\mu)(-\partial_1 X^\nu) + \partial_2 X^\mu \partial_2 X^\nu\right] \\ &= g^{ab}G_{\mu\nu}(X')\partial_a X^\mu \partial_b X^\nu \end{aligned} \qquad [8.645]$$

Requiring $\Omega = 1$ thus implies

$$G_{\mu\nu}(X') = G_{\mu\nu}(X) \qquad [8.646]$$

If one of the spacetime coordinates is dualised we obtain

$$\begin{aligned} \Omega\, g^{ab}G_{m\nu}(X)\partial_a X^m \partial_b X^\nu &= G_{\mu\nu}(X')\left[\big(-\partial_1(-X^m)\big)(-\partial_1 X^\nu) + \partial_2(-X^m)\partial_2 X^\nu\right] \\ &= -\, g^{ab}G_{m\nu}(X')\partial_a X^m \partial_b X^\nu \end{aligned} \qquad [8.647]$$

from which we deduce that

$$G_{m\nu}(X') = -G_{m\nu}(X) \qquad [8.648]$$

and similarly we find for two dualised spacetime coordinates that

$$G_{mn}(X') = -G_{mn}(X) \qquad [8.649]$$

Looking now at the antisymmetric tensor, we see that

$$\epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = 2B_{\mu\nu}(X)\partial_1 X^\mu \partial_2 X^\nu \qquad [8.650]$$

Therefore

$$\Omega\,\epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = 2B_{\mu\nu}(X')(-\partial_1 X^\mu)\partial_2 X^\nu) = -2\epsilon^{ab}B_{\mu\nu}(X')\partial_a X^\mu \partial_b X^\nu \qquad [8.651]$$

so that

$$B_{\mu\nu}(X') = -B_{\mu\nu}(X) \qquad [8.652]$$

Similarly we find, obviously, that

$$B_{m\nu}(X') = B_{m\nu}(X) \qquad [8.653]$$

and

$$B_{mn}(X') = -B_{mn}(X) \qquad [8.654]$$

8.132 p 278: Fig. 8.6 The Torus and the Klein Bottle

In order to make sure we understand this figure, let us start from the simplest case, the cylinder. We take a rectangle and draw arrows pointing in the same direction on opposite sides. We then "sew" together the opposite sides so that the arrows point in the same directions. The result is a cylinder, see fig.8.13. Our graphical representation is thus a rectangle with on two opposite sides arrows pointing in the same direction

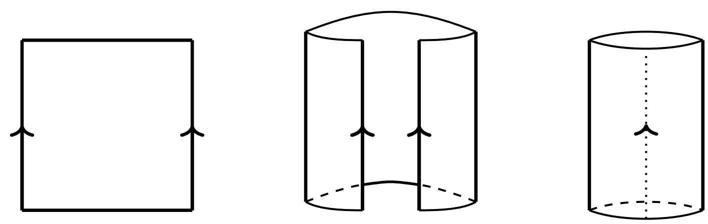


Figure 8.13: Representation of the cylinder. The two opposite sides are sewn together so that the two arrows align. The result is a cylinder.

If the arrows on the two opposite sides of the rectangle now point in opposite directions and we sew them together than we have to twist one side to achieve this. The result is the Möbius strip, see fig.8.15.

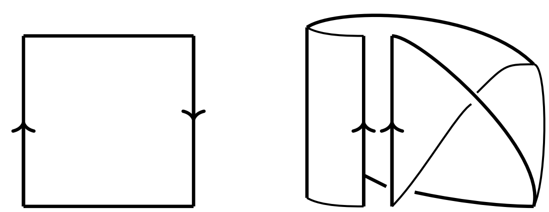


Figure 8.14: Representation of the Möbius strip. The two opposite sides are sewn together so that the two arrows align. This needs a twist of one side and the result is the Möbius strip

It should now be clear how we can represent the torus, viz. as a cylinder where also the two remaining opposite ends with arrows in the same direction are identified. The Klein bottle, finally is obtained from the cylinder with with the two remaining ends identified, but after a twist, hence with arrows on that end pointing to opposite directions.

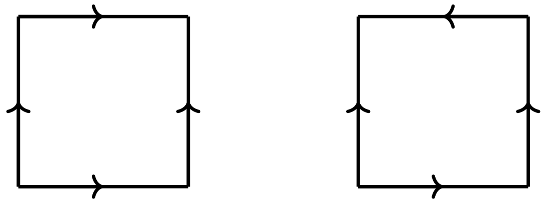


Figure 8.15: Representation of the torus and the Klein bottle. Both start from a cylinder with the open ends identified. For the Klein bottle there is an extra twist.