General Relativity

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Introduction

These notes provide my personal road to understanding general relativity. There are fortunately several excellent text books and lecture notes on the subjects, so I cannot not claim any originality to my approach. I have done nothing else than combine parts from these different sources, trying to avoid bringing in too many errors. Any errors in the reasoning or in the text are entirely due to me.

If you want to help improve these Notes, either by correcting errors, changing, adding material, or answering open questions please contact me on hepnotes@hotmail.com.

For these notes I have made more or less extensive uses of the resources given in the bibliography. One may notice the absence of some classics such as Hawking & Ellis and MTW. This is by no means a judgment on the quality of these books.

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Chapter 2

Special Relativity

2.1 Introduction

Familiarity with special relativity is assumed. This chapter is thus necessarily brief and serves more as a reminder of concepts and important formulas, and to establish notation.

2.2 Invariant Line Element, Light Cone, Proper Time

The **Invariant Line Element** of Minkowski spacetime in d dimensions is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + \sum_{i=1}^{d-1} (dx^d)^2 \tag{2.1}$$

In four dimensions this becomes in Cartesian coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.2}$$

At each point in spacetime we have a lightcone **Light Cone** defined by the line element

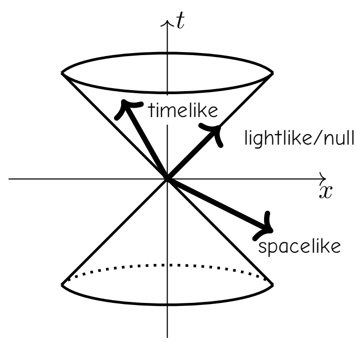


Figure 2.1: Lightcone and timelike, lightlike and spacelike paths

We have

$$\begin{aligned} ds^2 < 0 \quad &\Rightarrow \frac{dx^2}{dt^2} < 1 \quad : v < c \quad : \textbf{Timelike} \\ ds^2 = 0 \quad &\Rightarrow \frac{dx^2}{dt^2} = 1 \quad : v = c \quad : \textbf{Lightlike} \text{or } \textbf{Null} \\ ds^2 > 0 \quad &\Rightarrow \frac{dx^2}{dt^2} > 1 \quad : v > c \quad : \textbf{Spacelike} \end{aligned} \tag{2.3}$$

The worldline of a massive particle is in the light-cone with $v < c$. A massless particles moves on the boundary of the lightcone with $v = c$. No particles can move outside the lightcone.

The **Proper Time** is defined as

$$d\tau^2 = -ds^2 \tag{2.4}$$

and is thus positive for massive particles and zero for massless particles. The proper time measures the time elapsed between two events as seen by an observer moving on a straight path with constant velocity between the events. For timelike paths we can integrate this as

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \tag{2.5}$$

where λ parametrises the path.

In $3d$ space the shortest distance between two points is the straight line; in spacetime the straight trajectory has the longest proper time.

2.3 Lorentz Transformations and the Poincaré Group

The line element ds^2 is invariant under a Lorentz transformation transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with Λ satisfying

$$\Lambda^t \eta \Lambda = \eta \tag{2.6}$$

Thus $\Lambda \in O(1, d-1)$, called the **Lorentz Group**. In order to ensure that the Lorentz transformations are connected to the identity we require $\det\Lambda = 1$ which implies $\Lambda \in SO(1, d-1)$. But time reversal and parity transformations have unit determinant and are not connected to the identity. In order to avoid these, we also impose $\Lambda^0_0 \geq 1$ which then leads to the **Proper Orthochronous Lorentz Group** $\Lambda \in SO(1, d-1)^{\uparrow}$. We will often just call this the Lorentz group.

The dimension of the Lorentz group is $n(n-1)/2$ so in four dimensions the Lorentz group is six dimensional. In four dimensions a basis of the generators of the Lorentz group consists of **Spacetime Rotations** and **Lorentz Boosts**. The spacetime rotations are the usual $SO(3)$ rotations in $\mathbb{R}^3$. E.g. a rotation in the x–y plane is given by

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

A Lorentz boost is a transformation that rotates space and time. E.g. a boost in the x direction is given by

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.8}$$

The line element ds^2 is also invariant under spacetime translations $x^\mu \to x'^\mu = x^\mu + a^\mu$. These form four copies of the Abelian group T_4. The full symmetry group of Minkowski space is thus the semi-direct product $SO(1, d-1) \otimes T_4$ and is called the **Poincaré Group**.

2.4 Energy and Momentum

The velocity four vector of a massive particle is defined as

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{2.9}$$

where τ is the proper time, and satisfies

$$U^2 = \eta_{\mu\nu} U^\mu U^\mu = -1 \tag{2.10}$$

The momentum four vector is then

$$p^\mu = m U^\mu \tag{2.11}$$

and satisfies the dispersion relation

$$p^2 = -m^2 \tag{2.12}$$

The **energy-momentum Tensor $T^{\mu\nu}$** is the Noether current of the spacetime translations. It is the flux of the four momentum p^μ across a surface of constant x^μ and its component have the following interpretation

$$\begin{aligned} T^{00} &\quad : \text{energy density} \\ T^{0i} &\quad : \text{momentum density} \\ T^{ii} &\quad : \text{pressure} \\ T^{i\neq j} &\quad : \text{shear} \end{aligned} \tag{2.13}$$

In cosmology one often uses the example of an energy-momentum tensor describing a **Perfect Fluid**

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu} \tag{2.14}$$

where ρ is the energy density and p the pressure in the rest frame. Different equations of states, i.e. relations between ρ and p give different types of perfect fluids

$$\begin{aligned} p = 0 &\quad : \quad \textbf{Dust or Matter} \\ p = \frac{1}{3}\rho &\quad : \quad \textbf{Radiation or Isotropic Photon Gas} \\ p = -\rho &\quad : \quad \textbf{Vacuum Energy or Cosmological Constant} \end{aligned} \tag{2.15}$$

The dust just consists of collision-less non-relativistic particles. Radiation consists of photons or ultra relativistic particles with negligible contribution to their energy from their mass. The factor follows from the tracelessness of the energy-momentum tensor of the Maxwell field. The vacuum case just gives a non-vanishing cosmological constant as we will see when we discuss the Einstein field equations.

The conservation of energy-momentum $\partial_\mu T^{\mu\nu} = 0$ for a perfect fluid in the the non-relativistic limit gives the energy density continuity equation

$$\partial_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{2.16}$$

and the Euler equation

$$\rho\big[\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\big] = -\nabla p \tag{2.17}$$

Chapter 3

Generalities: the Equivalence Principle, the Metric and Causality

3.1 Equivalence Principles

The **Weak Equivalence Principle** has three different formulation

1. The inertial and gravitational mass of an object is the same. This leads to the famous statement that all objects fall with the same speed in vacuum.

2. There exists a preferred class of trajectories in spacetime, so-called **Inertial Trajectories**, on which unaccelerated particles travel. Here we define **Unaccelerated Particles** as particles subject to gravity only, and not to any other force.

3. In a small enough regions of spacetime the motion of freely falling particles in a gravitational field cannot be distinguished from particles undergoing constant acceleration .

The **Einstein Equivalence Principle** says that in small enough regions of spacetime the laws of physics reduce to the laws of special relativity. This means that it is impossible to detect the presence of a gravitational field by a local experiment.

The **Strong Equivalence Principle** encompasses both the weak and the Einstein equivalence principle.

From the weak equivalence principle we deduce that gravity is actually not really a force, at least not in the traditional way.

3.2 The Metric

Mathematically speaking the metric is a two-form, i.e. a $(0,2)$ tensor. Recall that a differential form is a linear function from the tangent space T_pM at a point p of a manifold M to the real numbers $\mathbb{R}$. A two-form thus depends on two vectors and gives a number:

$$g(V,W) = g_{\mu\nu}V^\mu W^\mu \tag{3.1}$$

Physically speaking the metric $g_{\mu\nu}$ is the generalisation of the Minkowski metric $\eta_{\mu\nu}$ to a general smooth manifold.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3.2}$$

Here the metric depends in general on the point of the manifold. The line element is invariant under general coordinate transformations, also called **Diffeomorphisms** $x^\mu \to x'^\mu$. The metric then transforms as a $(0,2)$ tensor

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) \tag{3.3}$$

One way to characterise metrics is by the sign of their Eigenvalues. If all Eigenvalues are positive then the metric and by extension the manifold is called **Euclidean** or **Riemannian**. If all Eigenvalues are positive except one that is negative then the metric and the manifold are called **Lorentzian** or **Pseudo-Riemannian**. If the metric has several positive and negative manifolds then we say the metric is indefinite. We do not consider the degenerate case of zero Eigenvalues.

For a general manifold the metric cannot be transformed into a flat metric by a coordinate transformation. However it is always possible to go to **Local Inertial Coordinates** in a **Local Lorentz Frame** where the metric is locally flat and its first derivatives disappear. I.e. at a point p we have

$$g'_{\mu\nu}(x')\Big|_p = \eta_{\mu\nu} \qquad \text{and} \qquad \frac{\partial}{\partial x'^\sigma}g'_{\mu\nu}(x')\Big|_p = 0 \tag{3.4}$$

It is not possible to set all second order and higher derivatives to zero. For example the second derivatives have, in four dimensions, 100 components, but only 80 of them can be fixed. The remaining 20 cannot be fixed and these are, not coincidentally, the number of components of the curvature tensor as we will see.

3.3 Causality

Here we consider the initial value problem: given initial conditions in some sub-manifold can we fully determine the state of the system at a later time. The key definition is that of a Cauchy surface, but before we get there we need to define a few other concepts

Causal Curve: this is a worldline taken by a particle that is either timelike or lightlike, i.e. the path taken by a massive or a massless particle.

Causal Future $J^+(S)$: Given a subset S of a manifold the causal future of S is the set of points that can be reached by the causal curves starting in S.

Chronological Curve: this is a worldline taken by a particle that is timelike, i.e. the past taken by a massive particle.

Chronological Future $I^+(S)$: Given a subset S of a manifold the causal future of S is the set of points that can be reached by the chronal curves starting in S.

There are similar definitions for causal past $J^-(S)$ and chronological past $I^-(S)$.

If in a subset S of a manifold M no two points are connected by a timelike curve, then S is called **Achronal**. As an example consider a two-dimensional Lorentzian manifold with a straight line $L : t = c^{\text{te}}$. The future lightcone of each point on the line L points towards the future and no point on L can be reached from any other point on the line. Thus, L is an achronal subset. More generally, any edgeless spacelike surface in Minkowski spacetime is achronal.

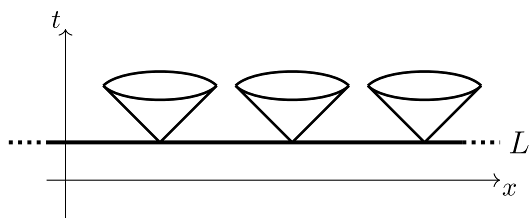


Figure 3.1: Achronal line in two dimensions. None of the points on the line L can be reached via a timelike path from any other point on L.

Given a closed achronal set S the **Future Domain of Dependence** of S, denoted by $D^+(S)$ is the set of all points p of the manifold such that every past-moving inextendible causal curve through p must intersect S. Inextendible means that the curve goes on forever, i.e. does not stop in a fixed point. By definition points in S are also in its future domain of dependence $D^+(S)$. We can similarly define a past domain of dependence $D^-(S)$ of an achronal set S.

The **Future Cauchy Horizon $H^+(S)$** is defined as the boundary of the future domain of dependence, $H^+(s) = \partial D^+(S)$, and likewise for the past Cauchy surface. The Cauchy surfaces are null surfaces, i.e. point on them are lightlike, and they divide the manifold into points that are in the domain of dependence of S and in points outside of it. Thus, if the manifold has a Cauchy horizon for S , then there are points in the manifold that cannot be reached via a timelike worldline from S.

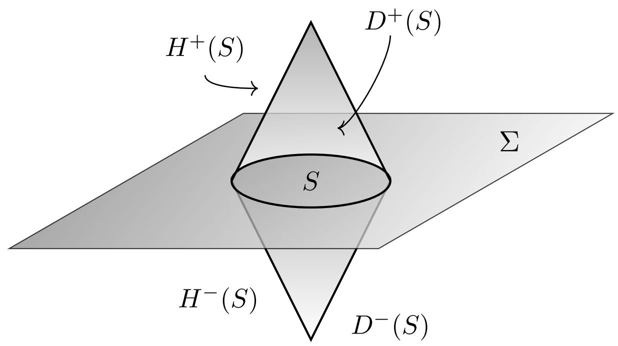


Figure 3.2: Causal structure. S is a subset of an achronal surface Σ. The future domain of dependence $D^+(S)$ comprises all future points that are linked by a timelike path to S and its boundary is the future Cauchy horizon $H^+(s) = \partial D^+(S)$.

These definitions just formalise our intuitive understanding of the initial value problem. An achronal surface is a slice of spacetime where all points are independent of one another, in the sense that the state of any one point of that surface is not determined by the state of a different point of that surface in the past. From such an achronal surface we can derive the part of spacetime that is determine from all its points, i.e. the domain of future dependence. What we now want is to define a surface so that we can use it as an initial condition for determining the future dependence of the entire manifold.

A **Cauchy Surface Σ** is a closed achronal surface whose domain of dependence $D(\Sigma)$ is the entire manifold.

Thus from the initial values on the Cauchy surface, we can determine what happens throughout the entire manifold. Note that as a Cauchy surface is achronal, i.e. no single point can be linked by a timelike path to another point on the surface, giving initial values on the Cauchy surface, will not lead to a contradiction in the sense that a spacetime point on the Cauchy surface as evolved from another point on that surface, could be different than its initial value given. There is however no guarantee that a manifold has a Cauchy surface, i.e. that there is a well defined initial value problem.

A spacetime that has a Cauchy surface is called **Globally Hyperbolic** .

Any set Σ that is closed, achronal and has no edge is called a **Partial Cauchy Surface**. Not ever partial Cauchy surface is a Cauchy surface. This can be due to a reason of its own – such as that not the entire manifold evolves from Σ – or from reasons due to specific structure of the manifold.

Note that in Newtonian physics times moves forward absolutely. In special relativity time is relative but it moves forward as well. Particles cannot move outside their lightcone and as Minkowski space is flat this means that a particle cannot return to its point. This is not the case in general relativity. It is possible that the worldline of a massive particles get bent so much that the particle returns to it spacetime original point. These are the **Closed Timelike Curves**.. A couple of examples will clarify all this.

- Consider the manifold $\mathbb{R} \times S^1$ with metric

$$ds^2 = -\cos\lambda\, dt^2 - 2\sin\lambda\, dx\, dt + \lambda\, dx^2, \qquad \lambda = \cot^{-1} t \tag{3.5}$$

The null lines $ds^2 = 0$ are given by

$$v^2 - (2/t)v - 1 = 0 \tag{3.6}$$

where $v = dx/dt$. Thus they are

$$v = (1 \pm \sqrt{t^2 + 1})/t \tag{3.7}$$

Hence for $t = \pm\infty$ these are $v = \pm 1$ but for $t = 0$ they are $v = 0$ and $v = +\infty$. So the lightcone at $t = 0$ is rotated $90°$ compared to far past and future times. With time evolves from the far past, the light cones get rotated and eventually allow a worldline at constant time. As x is periodic the worldline can return to itself, leading to a closed timelike curve. This is illustrated in figure 3.3.

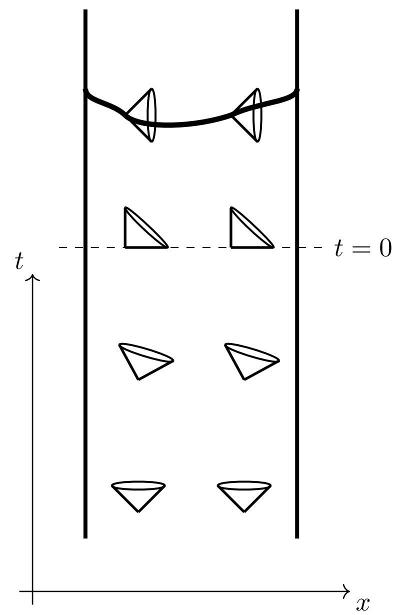


Figure 3.3: Closed timelike curves of (3.5). As t evolves, the light cones get rotated and eventually allow a worldline at constant time. As x is periodic the worldline can return to itself.

If we had specified a surface Σ to the past of that point, then none of the points in the region of the closed timelike curves can be obtained from starting in Σ and are hence in its domain of dependence $D^+(\Sigma)$ since the closed timelike curves don't intersect Σ. There is thus a Cauchy horizon at $t = 0$, and it thus seems that this spacetime does not have a well defined initial value problem.

- Another example is the occurrence of a **Singularity** in spacetime. Typically these are points with infinite curvature. They are at a finite distance from other points of the manifold, but are not part of it. Some points in the future of the singularity will have a worldline tracking back to the singularity. Hence these points cannot be in the domain of dependence of a hypersurface in the past of the singularity. This also leads to a Cauchy horizon.

Chapter 4

Connection and Curvature

4.1 Connection

The partial derivative of a tensor $\partial_\mu V^\mu$ does not transform as a tensor, i.e. it is coordinate dependent. We look for an alternative, the **Covariant Derivative $\nabla_\mu V^\nu$** that transforms as a tensor and is hence independent of the choice of coordinates. We also require the covariant derivative to be linear and satisfy the Leibniz rule. This can be obtained by adding a linear correction to the partial derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{4.1}$$

Requiring that hat both sides transform as a $(1,1)$ tensor gives the transformation law for the **Connection** $\Gamma^\nu_{\mu\lambda}$

$$\Gamma^\nu_{\mu\lambda} \to \left(\Gamma'\right)^\nu_{\mu\lambda} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\lambda} \frac{\partial x'^\nu}{\partial x^\kappa} \Gamma^\kappa_{\sigma\rho} + \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\lambda} \frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\rho} \tag{4.2}$$

which does not transform as a tensor, as it should, due to the second term.

Similarly for a one form we can write $\nabla_\mu \omega_\nu = \partial_\mu + \tilde{\Gamma}^\lambda_{\mu\nu}\omega_\lambda$. As scalars have no direction on the manifold (i.e. no components that depend on the coordinate system) it makes sense to require that the covariant derivate on scalars reduces to the partial derivative. Then, requiring this on the product of a vector and a one form , we find that $\tilde{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}$ and the covariant derivative on a one-form becomes

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \tag{4.3}$$

There is a certain freedom on choosing the connection on a manifold. However the difference between any two choices of connection is a tensor, as the guilty term in (4.2) does not depend on the choice of connection. This implies that any connection can be obtained from a "standard" connection plus a tensor.

From (4.2) we also see that if $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^\lambda_{\nu\mu}$ transform in the same way and are hence both connections. Their difference is thus a tensor, the **Torsion Tensor**

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \tag{4.4}$$

We now impose to additional conditions on the connection

1. The connection is **Torsion Free**, i.e. symmetric: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$
2. The connection is **Metric Compatible**: $\nabla_\sigma g_{\mu\nu} = 0$.

These requirements lead to a unique expression for the connection

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\sigma g_{\mu\nu} \right) \tag{4.5}$$

This connection is known as the **Christoffel** or **Levi-Civita Connection**, sometimes as the **Riemannian Connection**. This is the only connection we will work with, and hence we will just call it the connection as if there is no other one.

The metric compatibility condition is equivalent to the statement that the line element remains constant when we parallel transport it along any curve as we will see in (4.13).

As a reminder, one does not have to go to a curved manifold to have non-zero connections. Indeed curvilinear coordinates in flat space also results in non-zero connections. Take for example polar coordinates in $\mathbb{R}^2$ with metric $ds^2 = dr^2 + \sin^2\theta d\theta^2$. one easily finds that the following connections are not zero: $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$.

In local inertial coordinates where the metric and its first derivative is locally zero, i.e. at one point, the connection is zero

$$\Gamma^\sigma_{\mu\nu} = 0 \qquad \text{in a local Lorentz frame} \tag{4.6}$$

The contracted connection satisfies

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|} \tag{4.7}$$

from which it follows that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right) \tag{4.8}$$

This in turn implies a useful form of Stokes theorem

$$\int_\Sigma d^d x\,\sqrt{|g|}\,\nabla_\mu V^\mu = \int_{\partial\Sigma} d^{d-1}x\,\sqrt{|\gamma|}\,n_\mu V^\mu \tag{4.9}$$

where n_μ is the normal to the boundary $\partial\Sigma$ and γ_{ij} is the induced metric on $\partial\Sigma$.

4.2 Parallel Transport and Geodesics

Heuristically **Parallel Transport** is the moving of a vector from one point to another whilst keeping it constant. This is straightforward in flat space but on a curved manifold the parallel transport depends on the path between both points. Indeed, vectors live on the tangent space of a point and so it depends on how you reach that point.

This means that there is no unique way to compare vectors at different points. Thus the concept of relative velocity between two points does not make sense in GR. As an example the redshift of galaxies does strictly not mean that they are receding. What it means is that the metric between here and the galaxy has changed and the wavelength of the photon reaching us has shifted. It is of course tempting to think that the galaxies are receding.

Before we define parallel transport we need the concept of **Directional Covariant Derivative** along a path $x(\lambda)$

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu \tag{4.10}$$

We then define parallel transport of a tensor along a path $x(\lambda)$ as the requirement that

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = 0 \tag{4.11}$$

This is the **Equation of Parallel Transport**. For vectors it reduces to

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{\lambda}V^\rho = 0 \tag{4.12}$$

Because of metric compatibility, the metric is always parallel transported

$$\frac{D}{D\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0 \tag{4.13}$$

It follows that if two vectors are parallel transported, so is their inner product $W \cdot V = g_{\mu\nu} W^\mu V^\mu$ and thus also orthogonality of vectors. And vice versa: preserving the inner product during parallel transport along any curve implies metric compatibility

$$0 = \nabla_\sigma (g_{\mu\nu} v^\mu v^\nu) = (\nabla_\sigma g_{\mu\nu}) v^\mu v^\nu \tag{4.14}$$

as for parallel transport $\nabla_\sigma v^\mu = 0$.

4.3 The Geodesic Equation

A geodesic is the analog of a straight line in flat space to a curved manifold. There are two ways to generalise the straight line to a general manifold. A geodesic

1. is the path that parallel transports its own tangent vector, or
2. is the path that extremises its proper time.

Recall that in Minkowski space the proper time is maximal for a straight line, hence the second definition.

The first definition gives a straightforward equation for the geodesic. The tangent vector to a path $x^\mu(\lambda)$ is $dx^\mu/d\lambda$ so we require $D/D\lambda(dx^\mu/d\lambda) = 0$ or the **Geodesic Equation**

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \tag{4.15}$$

For the second definition, we have to maximise the proper time (2.5)

$$\Delta\tau = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda \tag{4.16}$$

The way to do this is to write this as $\delta\tau = \int \delta\big(\sqrt{-f}\big) d\lambda - = -\frac{1}{2} \int (-f)^{-1/2} \delta f d\lambda$ and then notice that if we choose for λ a specific parameter, the proper time itself, the $f = g_{\mu\nu} U^\mu U^\nu = -1$ and so we need to find the extremum of

$$I = \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\, d\tau \tag{4.17}$$

We now vary $x^\mu \to x^\mu + \delta x^\mu$ and $g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}) \delta x^\sigma$ and a bit of algebra then gives the equation

$$\frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} g^{\mu\lambda} \left(\partial_\sigma g_{\lambda\rho} + \partial_\rho g_{\lambda\sigma} - \partial_\lambda g_{\rho\sigma} \right) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \tag{4.18}$$

This is equivalent to the first definition provided the connection is the Christoffel connection, i.e. provided we have metric compatibility. This motivates that restriction.

Note that using the variational principle on (4.17) is a convenient way to calculate the connections as there is no point in having to work out in detail all the connections that are zero.

4.4 Properties of Geodesics

1. Geodesics describe the paths of unaccelerated **Test Particles**, i.e. particles that do not influence the geometry of spacetime. Whilst this is never true, this is often a very good approximation. One can also add forces, e.g. the Lorentz force for electromagnetism can be represented as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau} \tag{4.19}$$

2. The geodesic equation (4.15) contains the arbitrary parameter λ, whilst (4.18) used the proper time τ. A transformation of the proper time

$$\tau \to \lambda = a\tau + b \tag{4.20}$$

leaves (4.18) invariant. Any parameter λ related in such a way to the proper time τ is called an **Affine Parameter** and can be used as a proper time for the geodesic. The geodesic equation (4.15) seems to have a more general parameter λ, but that is not really the case. Requiring the tangent vector to be parallel transported constrains the parametrisation of the curve: the initial conditions of the parallel transport not only determine the geodesic, but also, up to an affine transformation, the parametrisation of the path. One can use a more general parametrisation, but then the geodesic equation will be more complicated.

3. For a timelike particle, we can rewrite the geodesic equation using the velocity $U^\mu = dx^\mu/d\tau$ or momentum $p^\mu = mU^\mu$ as

$$U^\mu\nabla_\mu U^\nu = p^\mu\nabla_\mu p^\nu = 0 \tag{4.21}$$

This shows that free falling test particles keep moving in the direction of their momentum.

4. For a null path the proper time does not exist, but we can always use a parameter λ, e.g. such that $p^\mu = dx^\mu/d\lambda$ is the momentum of the massless particle. An observer with velocity U^μ will then measure the energy of the particle to be

$$E = -p_\mu U^\mu \tag{4.22}$$

In order to show this, first go to the rest frame of the observer so that $U^\mu = (1,0,0,0)$ and then take locally inertial coordinates $g_{\mu\nu} = \eta_{\mu\nu}$. As E is a scalar under coordinate transformation, this results holds in any frame.

5. Because parallel transport preserves the inner product and ds^2 is an inner product, a timelike geodesic will always remain timelike and similarly for a lightlike or spacelike geodesic.

6. Let us explain the observation that timelike geodesics are maxima for the proper time. Any timelike geodesic can be approximated by "jagged" lightlike curves. Indeed at a given point of the geodesic, just take a lightlike curve, i.e. on the boundary of the lightcone. At the next spacetime point take again a lightlike curve that goes back to the original path. This is illustrated in fig. 4.1.

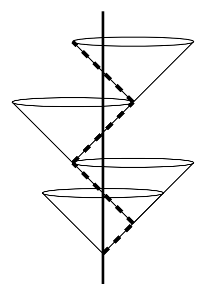


Figure 4.1: Approximating a timelike by a lightlike path

We can clearly approximate the timelike curve by a lightlike curve to any accuracy desired. So timelike curves cannot be curves of minimum proper time as they can be approximated by curves with less, in fact zero, proper time. Thus timelike geodesics are necessarily curves of maximum proper time.

7. The geodesic equation can be derived in heuristic way from classical mechanics using Einstein's Equivalence principle, that in a small enough region the laws of special relativity are valid. In order to make them valid we just have to write these equations in tensorial form. In classical mechanics a particle in free fall satisfies $d^2x^\mu/d\lambda^2 = 0$ where λ parametrises the path. To make this a tensorial equation we use Leibniz and then change the partial by a covariant derivative

$$0 = \frac{d^2x^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda}\partial_\nu \frac{dx^\mu}{d\lambda} \quad \longrightarrow \quad 0 = \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}$$

which gives the geodesic equation.

4.5 Riemann Normal Coordinates

Using geodesics we can now construct explicitly a set of local inertial coordinates. We will construct these using the **Exponential Map**, which is a map from the tangent space T_pM of a point of a manifold to a region of the manifold that contains the point p.

In order to do this consider a vector $k \in T_pM$. We want to solve the geodesic equation with the initial conditions

$$x^\mu(\lambda)\Big|_{\lambda=0} = p \qquad \text{and} \qquad \frac{dx^\mu(\lambda)}{d\lambda}\Big|_{\lambda=0} = k^\mu \tag{4.23}$$

The geodesic is a second order differential equation and so these initial conditions determine a unique geodesic through p. We now define a specific point on that geodesic, the point where $\lambda = 1$ and define the **Exponential Map** as

$$\exp_p(k) = x^\mu(\lambda)\Big|_{\lambda=1} \tag{4.24}$$

It is a map from the Tangent space T_pM at the point p to a point of the manifold, $\exp_p : T_pM \to M$. In words, it takes a tangent vector, uses that as initial condition for a geodesic and then takes a point on the geodesic with $\lambda = 1$.

This exponential map is not always invertible. Indeed different tangent vectors will give different geodesics that may eventually cross, making the map non-invertible. Obviously the range of the map is not necessarily the who manifold as from a given point of the manifold there is no reason that you should be able to go to all the other points of the manifold via a geodesic. Also the domain of the map is not necessarily the whole tangent space because a geodesic may reach a singularity of the manifold – some kind of edge – and we cannot take the geodesic at $\lambda = 1$. Manifolds with such singularities are called **Geodesically Incomplete**.

Using the exponential map we can now build locally inertial coordinates. We take a basis of the tangent space T_pM and make it orthonormal under the metric $g(\cdot,\cdot)$. I.e. the basis $\boldsymbol{e}_{(\mu)}$ satisfies

$$g_{\hat{\mu}\hat{\nu}} = g(\boldsymbol{e}_{(\hat{\mu})}, \boldsymbol{e}_{(\hat{\nu})}) = \eta_{\hat{\mu}\hat{\nu}} \tag{4.25}$$

Let us now find a coordinate system $x^{\hat{\mu}}$ at the point p so that the basis of the tangent space at that point is the coordinate basis $\boldsymbol{e}_{(\hat{\mu})} = \partial_{\hat{\mu}}$ and the partial derivatives of $g_{\hat{\mu}\hat{\nu}}$ vanish.

This is achieved by the exponential map. Any point q close enough to p is linked by a geodesic from p and a unique parametrisation such that at p we have $\lambda = 0$ and at q we have $\lambda = 1$. The tangent vector k^μ at p for this geodesics can be expanded in terms of the basis $k = k^{\hat{\mu}} \boldsymbol{e}_{(\hat{\mu})}$. We now define the coordinates $x^{\hat{\mu}}$ to be the components $x^{\hat{\mu}}(q) = k^\mu$.

In words, we define the coordinates of a nearby point q to be the components (in the normalised basis $\{\boldsymbol{e}_{(\hat{\mu})}\}$) of the the tangent vector $k^{\hat{\mu}}$ that gets mapped by the exponential map to q.

These coordinates are called **Riemann Normal Coordinates** and are locally inertial coordinates, Indeed, we already know from (4.25) that the metric is the flat metric, all we need to show is that the partial derivative of the metric vanishes.

In order to do so, first note that if $\exp_p(k) = x^\mu(\lambda)\big|_{\lambda=1}$ then x^μ is equally well a geodesic map for the tangent vector λk^μ. So a curve of the form $x^{\hat{\mu}} = \lambda k^\mu$ is also a solution of the geodesic equation. In fact, every geodesic through p is of that form for some $k^{\hat{\mu}}$. But as $k^{\hat{\mu}}$ is independent of λ we have

$$\frac{d^2 x^{\hat{\mu}}}{d\lambda^2} = 0 \tag{4.26}$$

As $x^{\hat{\mu}}$ satisfies the geodesic equation in p this implies that

$$\frac{d^2 x^{\hat{\mu}}}{d\lambda^2}\Big|_p = -\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} \frac{dx^{\hat{\rho}}}{d\lambda}\frac{dx^{\hat{\sigma}}}{d\lambda}\Big|_p = -\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}} k^{\hat{\rho}} k^{\hat{\sigma}}\Big|_p \tag{4.27}$$

Since this holds for arbitrary k we find that $\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}}\big|_p = 0$. All that remains to be done is use the metric compatibility

$$0 = \nabla_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}\Big|_p = \partial g_{\hat{\mu}\hat{\nu}}\Big|_p \tag{4.28}$$

as the connections are zero at p. Hence the partial derivative of the metric evaluated at the point p vanishes and the Riemann normal coordinates are indeed locally inertial coordinates.

Given a set of coordinates x^μ with connections $\Gamma^\sigma_{\mu\nu}$ one go to Riemann normal coordinates at the point P with coordinates x^μ_* by the coordinate transformation

$$y^\mu(x) = \frac{\partial y^\mu}{\partial x^\nu}\Big|_{x_*} \left[(x^\mu - x^\mu_*) + \frac{1}{2}\Gamma^\mu_{\nu\sigma}(x^\mu - x^\nu_*)(x^\mu - x^\sigma_*) + o\big((x - x_*)^2\big)\right] \tag{4.29}$$

4.6 About the Conservation of Energy in General Relativity

Somewhat surprisingly, maybe, energy is not conserved in general relativity. A toy model will show this. Consider four dimensional Lorentzian spacetime with flat but expanding space, i.e with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{4.30}$$

The energy-momentum tensor (2.14) in a curved manifold is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \tag{4.31}$$

The energy is given by the space integral of the 00 component

$$E = \int \rho a^3 d^3x \tag{4.32}$$

where the boundaries of the integration are at fixed co-moving coordinates, so the region expands with the coordinates and the $a^3 = \sqrt{-\det g_{\mu\nu}}$. This number is not conserved for general a. We can check this for the three equations of state in (2.15).

1. **Dust**: $\rho \propto a^{-3} \Rightarrow E_{\text{dust}} \propto \int d^3x =$ constant. Dust is essentially matter; the number of particles remains constant and the energy is essentially the rest mass of the particles. It should remain constant.

2. **Radiation**: $\rho \propto a^{-4} \Rightarrow E_{\text{radiation}} \propto \int a^{-1} d^3x \propto a^{-1}$. The energy is not constant, but is inversely proportional to the the expansion coefficient. This is also to be expected as in an expanding universe the wavelength of radiation changes which gives an extra factor of a^{-1} compared to matter.

3. **Vacuum**: $\rho =$ constant $\Rightarrow E_{\text{vacuum}} \propto \int a^3 d^3x \propto a^3$. The vacuum is a constant energy density and so it grows as space expands.

Two comments are in order. First the non-conservation of energy does mot invalidate Noether's theorem. Symmetry under spacetime translations still implies a conserved current $\nabla_\mu T^{\mu\nu} = 0$. Second, the non-conservation of energy clearly follows because the space background changes with time. There is thus a priori no reason to assume energy would be conserved. Mathematically this is a consequence of the fact that there is no timelike Killing vector, as we will see later.

4.7 Curvature

Curvature can be measured in different ways. One way expresses how a vector, when parallel transported around a closed loop does not revert to itself. We will express this with the commutator of covariant derivatives. On a vector we have

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho \tag{4.33}$$

where the **Riemann Tensor** is defined by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho} \tag{4.34}$$

and the **Torsion Tensor**

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \tag{4.35}$$

which, clearly vanishes in torsion free theories. Some points to note :

1. Both the Riemann tensor and the torsion tensors are actually tensors.
2. We defined these for a generic connection, but they are, of course, equally valued for Christoffel connections. This is the only case we will consider henceforth.
3. This definition can be easily extended to general tensors

$$\begin{aligned}[\nabla_\rho, \nabla_\sigma] X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = & - T^{\lambda}{}_{\rho\sigma} \nabla_\lambda X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \\ & + R^{\mu_1}_{\lambda\rho\sigma} X^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} + \cdots + R^{\mu_k}_{\lambda\rho\sigma} X^{\mu_1\cdots\mu_{k-1}\lambda}{}_{\nu_1\cdots\nu_\ell} \\ & - R^{\lambda}{}_{\nu_1\rho\sigma} X^{\mu_1\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_\ell} - \cdots - R^{\lambda}{}_{\nu_k\rho\sigma} X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_{\ell-1}\lambda}\end{aligned} \tag{4.36}$$

4. The torsion and curvature can also be expressed as vector fields. Denote by $\mathfrak{X}(M)$ the vector fields over the manifolds M. The torsion is then a map of $\mathfrak{X}(M)\otimes\mathfrak{X}(M) \to X$:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{4.37}$$

where $\nabla_X = X^\mu \nabla_\mu$ is the covariant derivative along the vector field X. Note that in components this becomes

$$[T(X,Y)]^\mu \partial_\mu = X^\mu \nabla_\mu Y^\nu \partial_\nu - Y^\mu \nabla_\mu X^\nu \partial_\nu - (X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\mu \partial_\mu X^\nu) \tag{4.38}$$

with the covariant and partial derivatives acting on everything to their right. We then set $[T(X,Y)]^\mu = T^{\mu}_{\ \lambda\rho} X^\lambda Y^\rho$ and one recovers the earlier definition of the torsion. Similarly the curvature is a map $\mathfrak{X}(M)\otimes\mathfrak{X}(M)\otimes\mathfrak{X}(M) \to X$:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{4.39}$$

In components we would get $[R(X,Y,Z)]^\mu = R^{\mu}{}_{\rho\sigma\lambda} X^\rho Y^\sigma Z^\lambda$.

5. If there exists a coordinate system in a simply connected region of a manifold where the metric is constant, then the Riemann tensor will vanish in that region and vice versa.

6. Properties of the curvature tensor

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu} \tag{4.40}$$

(a) Symmetry properties

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\ R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\ R_{\rho\sigma\mu\nu} &= +R_{\mu\nu\rho\sigma} \\ R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} &+ R_{\rho\nu\sigma\mu} = 0 \\ R_{\rho[\sigma\mu\nu]} &= 0 \end{aligned} \tag{4.41}$$

As a result the curvature tensor has

$$\frac{1}{12} d^2(d^2 - 1) \quad \text{independent components} \tag{4.42}$$

Note that in one dimension there is no curvature tensor. This means that a line cannot have a curvature. This also mean that a torus, which is topologically equivalent to $S^1 \times S^1$ has no curvature. In $2, 3$ and 4 dimensions there are $1, 6,$ and 20 independent components respectively.

(b) **Bianchi Identity**

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \tag{4.43}$$

This is related the the Jacobi identity of the covariant derivatives.

7. The **Ricci Tensor** is the symmetric tensor defined by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{4.44}$$

8. The **Ricci Scalar** or **Curvature** is defined as

$$R = g^{\mu\nu} R_{\mu\nu} \tag{4.45}$$

9. The **Weyl Tensor** or **Conformal Tensor** is defined by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(d-1)(d-2)}g_{\rho[\mu}g_{\nu]\sigma}R \quad (4.46)$$

and is only defined in three or more dimensions. In three dimensions it vanishes identically. It has the following properties

(a) Symmetry properties

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= C_{[\rho\sigma][\mu\nu]} \\ C_{\rho\sigma\mu\nu} &= C_{\mu\nu\rho\sigma} \\ C_{\rho[\sigma\mu\nu]} &= 0 \end{aligned} \quad (4.47)$$

(b) It is invariant under **Conformal Transformations** $g_{\mu\nu}(x) \to w^2(x) g_{\mu\nu}(x)$.

10. The **Einstein Tensor** is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (4.48)$$

and satisfies

$$\nabla^{\mu} G_{\mu\nu} = 0 \quad (4.49)$$

11. Finally, we mention the **Kretschmann Invariant** which is quadratic in the curvature and is defined as

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \quad (4.50)$$

4.8 Geodesic Deviation

Consider a set of geodesics $\gamma_s(t)$ on a manifold M. Here t is the affine parameter of a given geodesic and $s \in \mathbb{R}$ is a parameter describing the different geodesics. Together these parameters describe a surface of points $x^{\mu}(s,t)$ on M. They provide a coordinate system for that surface provided the geodesics do not cross.

We can now define two vector fields. The vector field tangent to the geodesics T^{μ} and the deviation vectors S^{μ}

$$T^{\mu} = \frac{\partial x^{\mu}}{\partial t} \qquad \text{and} \qquad S^{\mu} = \frac{\partial x^{\mu}}{\partial s} \quad (4.51)$$

Thinking that S^μ is pointing from one geodesic to another, we can introduce a relative velocity and relative acceleration of geodesics

$$V^\mu = (\nabla_T S)^\mu = T^\sigma \nabla_\sigma S^\mu \qquad \text{and} \qquad A^\mu = (\nabla_T V)^\mu = T^\sigma \nabla_\sigma V^\mu \tag{4.52}$$

S and T are just basis vectors in a coordinate system, so $[S,T]=0$. Using $[S,T]^\mu = S^\nu \partial_\nu T^\mu - T^\nu \partial_\nu S^\mu$ this implies that $S^\nu \nabla_\nu T^\mu = T^\nu \nabla_\nu S^\mu$. Using this it follows after some algebra that we can write the acceleration as

$$A^\mu = \frac{D^2 S^\mu}{dt^2} = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \tag{4.53}$$

This is the **Geodesic Deviation Equation**. It says that the relative acceleration between two geodesics, i.e. how they move apart, is proportional with the curvature tensor. The physical interpretation is to consider several adjacent geodesics, e.g. from a beam of photon of particles, and see how the gravitational forces affect that bundle.

Chapter 5

Symmetries

5.1 The Killing Equations and Killing Vectors

In order to find the solution for a metric on a generic manifold, it is, as always, useful to know its symmetries. The symmetries of a metric are known as **Isometries** and they follow from the **Killing equations**

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{5.1}$$

where the vector fields K are known as the **Killing Vectors**. The Killing equation implies that $K_\mu p^\mu$ is conserved along geodesics

$$p^\nu \nabla_\nu (K_\mu p^\mu) = 0 \tag{5.2}$$

We can rewrite the Killing equations (5.1) as

$$g_{\mu\sigma}\partial_\nu K^\sigma + g_{\nu\sigma}\partial_\mu K^\sigma + (\partial_\sigma g_{\mu\nu})K^\sigma = 0 \tag{5.3}$$

which is often the most convenient way to calculate the Killing vectors.

Let us consider two examples

1. Assume the metric is independent of one specific component, say component x^κ, where κ is a fixed, specific index. So $\partial_\kappa g_{\mu\nu} = 0$ for all μ and ν. Let is see what this means for a timelike geodesic. Take the form (4.21), i.e. $p^\mu \nabla_\mu p^\nu = 0$. Use metric

compatibility to write this as $p^\mu \nabla_\mu p_\nu = 0$ and write it out

$$\begin{aligned}
0 &= p^\mu \partial_\mu p_\nu - p^\mu \Gamma^\lambda_{\mu\nu} p_\lambda \\
&= m\frac{dx^\mu}{d\tau}\partial_\mu p_\nu - \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) p^\mu p_\lambda \\
&= m\frac{dp_\nu}{d\tau} - \frac{1}{2}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) p^\mu p^\sigma \\
&= m\frac{dp_\nu}{d\tau} - \frac{1}{2}(\partial_\nu g_{\sigma\mu}) p^\mu p^\sigma
\end{aligned} \tag{5.4}$$

Now set $\nu = \kappa$ and use that $\partial_\kappa g_{\mu\nu} = 0$ to find that p_κ is conserved:

$$\frac{dp_\kappa}{d\tau} = 0 \tag{5.5}$$

The conserved vector $K_\mu p^\mu = K^\mu p_\mu$ thus has $K^\mu = \delta^\mu_\kappa$ and the Killing vector is $K^\mu \partial_\mu = \partial_\kappa$. We can write the conservation law (5.2) as

$$0 = p^\nu \nabla_\nu (K^\mu p_\mu) = p^\nu \nabla_\nu (\delta^\mu_\kappa p_\mu) = p^\nu \nabla_\nu p_\kappa \tag{5.6}$$

which we just showed does indeed corresponds to p_κ being conserved. Let us now show that the Killing equation is satisfied. We rewrite the conservation equation as

$$0 = p^\nu \nabla_\nu (K^\mu p_\mu) = p^\nu K^\mu \nabla_\nu p_\mu + p^\nu p_\mu \nabla_\nu K^\mu = p^\nu p^\mu \nabla_\nu K_\mu \tag{5.7}$$

We have used the geodesic equation $p^\mu \nabla_\mu p^\nu = 0$ and the metric compatibility condition. Symmetrising we find that indeed $\nabla_{(\mu} K_{\nu)} = 0$.

From this we find that if a metric is independent of the time coordinate then $K_t = \partial_t$ is a Killing vector, and if the metric has cylindrical symmetry, i.e. is independent of ϕ then $K_\phi = \partial_\phi$ is a Killing vectors. These are, of course, related to energy and angular momentum as we will soon see.

2. Consider $\mathbb{R}^3$ with metric $ds^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2$. Clearly the metric is invariant under translations $x_i \to x_i + \delta x_i$. The corresponding vector fields are the generators of translations $\partial/\partial x_i$, and so we already have three Killing vectors

$$X^\mu = (1,0,0); \qquad Y^\mu = (0,1,0) \qquad \text{and} \qquad Z^\mu = (0,0,1) \tag{5.8}$$

In order to find the other Killing vectors we rewrite the metric in spherical coordinates $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$. This metric is independent of ϕ and so $\phi \to \phi + \delta\phi$ is an isometry. This is a rotation around the z-axis and the corresponding vector field is $R = -y\partial_x + x\partial_y$. By symmetry we have similar Killing vectors for rotations around the y an x axis and so we have three Killing vectors

$$R^\mu = (-y, x, 0); \qquad S^\mu = (z, 0, -x) \qquad \text{and} \qquad T^\mu = (0, z, -x) \tag{5.9}$$

Let us check that these satisfy the Killing equations. The manifold is flat so the connections are zero and the Killing equations are simply $\partial_i K_j + \partial_j K_i = 0$. Clearly X^μ, Y^μ and Z^μ satisfy this as they are independent of the coordinates. Let check this for the remaining Killing vectors it suffices to do one of them by symmetry

$$\begin{aligned}
&\partial_x R_x + \partial_x R_x = 2\partial_x(-y) = 0 \\
&\partial_x R_y + \partial_y R_x = \partial_x x + \partial_y(-y) = 0 \\
&\partial_y R_z + \partial_z R_y = \partial_y 0 + \partial_z x = 0
\end{aligned} \tag{5.10}$$

We have thus six Killing vectors and recovered the symmetry group of $\mathbb{R}^3$, i.e. $SO(3) \times T_3$.

5.1 Properties of Killing Vectors

1. Killing vectors are in one-to-one correspondence with symmetries of the metric, i.e. with isometries, and imply conserved quantities along geodesics.

2. The Killing equation can be generalised to tensors and this again leads to conserved quantities

$$\nabla_{(\mu} K_{\nu_1 \cdots \nu\ell)} = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu \left(K_{\nu_1 \cdots \nu\ell} p^{\nu_1} \cdots p^{\nu_\ell} \right) = 0 \tag{5.11}$$

3. Killing vectors can be related to the curvature.[1]

[1]For illustration purposes, we prove the first equation here. From the definition of the curvature $[\nabla_\rho, \nabla_\mu] K^\sigma = R^\sigma{}_{\lambda\rho\mu} K^\lambda$ we get by contracting with g_{σ_ν} and using metric compatibility, the symmetry properties of the curvature tensor

$$[\nabla_\rho, \nabla_\mu] K_\sigma = R_{\sigma\lambda\rho\mu} K^\lambda = -R_{\lambda\sigma\rho\mu} K^\lambda = -R^\lambda_{\sigma\rho\mu} K_\lambda$$

Using the Killing equation we rewrite this as

$$\nabla_\rho \nabla_\mu K_\sigma + \nabla_\mu \nabla_\sigma K_\rho = -R^\nu{}_{\sigma\rho\mu} K_\nu$$

We now rewrite this two more times with different indices

$$\begin{aligned}
\nabla_\mu \nabla_\sigma K_\rho + \nabla_\sigma \nabla_\rho K_\mu &= -\, R^\nu{}_{\rho\mu\sigma} K_\nu \\
\nabla_\sigma \nabla_\rho K_\mu + \nabla_\rho \nabla_\mu K_\sigma &= -\, R^\nu{}_{\mu\sigma\rho} K_\nu
\end{aligned}$$

Now take $(1) + (2) - (3)$ and use the one but last equation in (4.41)

$$2\nabla_\mu \nabla_\sigma K_\rho = -(R^\nu{}_{\sigma\rho\mu} + R^\nu{}_{\rho\mu\sigma} - R^\nu{}_{\mu\sigma\rho}) K_\nu = 2R^\nu{}_{\mu\sigma\rho} K_\nu = 2R_{\rho\sigma\mu\nu} K^\nu$$

Raising the index ρ gives the desired result.

$$
\begin{aligned}
\nabla_\mu \nabla_\sigma K^\rho &= R^\rho{}_{\sigma\mu\nu} K^\mu \\
\nabla_\mu \nabla_\sigma K^\mu &= R_{\sigma\nu} K^\nu \\
K^\mu \nabla_\mu R &= 0
\end{aligned}
\tag{5.12}
$$

The last equation tells us that the directional derivative of the curvature along a Killing vector vanishes, and thus that the geometry of the manifold is unchanged along such a vector field.

4. The existence of a timelike Killing vector allows us to define a conserved energy for the entire spacetime. If K_ν is a Killing vector and $T^{\mu\nu}$ is the energy-momentum tensor, then $J_T^\mu = K_\nu T^{\mu\nu}$ is automatically conserved. Now, if K_ν is timelike then we can integrate over a spacelike hypersurface Σ to define a total energy

$$
E_T = \int_\Sigma J_T^\mu \, n_\mu \sqrt{\gamma} \, d^{d-1}x \tag{5.13}
$$

where γ_{ij} is the induced metric on Σ and n^ν is the normal to Σ. It is possible to show that E_T is independent of the hypersurface and hence conserved.

5.2 Maximally Symmetric Spaces

The "flattest" space in d dimensions is Euclidean or Minkowski, depending on the signature. This space has d translation symmetries and $\frac{1}{2}d(d-1)$ rotation and/or boost symmetries for a total of $\frac{1}{2}d(d+1)$ symmetries, i.e. Killing vectors. This is the maximal number of Killing vectors and hence symmetries any manifold can have. A space with this maximum number of Killing vectors is called a **Maximally Symmetric Space**.

We can think of the $\frac{1}{2}d(d+1)$ symmetries of such a manifold as $\frac{1}{2}d(d-1)$ rotations around a fixed point, and d "translations" that move the point around the manifold.

A maximally symmetric space has constant curvature and is thus fully characterised by

1. the dimension of the manifold
2. the signature of the metric
3. the sign of the curvature: positive, zero or negative
4. possible discrete topological characteristics, such as the number of holes of an n-torus

As the curvature is the same at all points of the manifold, so should the curvature tensor "look" the same at all points. In locally inertial coordinates, this is up to Lorentz transformations and the curvature tensor should be build from invariants. In order to satisfy all symmetry properties of the curvature tensor we need to have $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \propto g_{\hat{\rho}\hat{\mu}}g_{\hat{\sigma}\hat{\nu}} - g_{\hat{\rho}\hat{\nu}}g_{\hat{\sigma}\hat{\mu}}$. This is a tensorial relation so it should be valid in any reference frame. Working out the proportionality constant gives in a maximally symmetric space that

$$R_{\rho\sigma\mu\nu} = \frac{R}{d(d-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{5.14}$$

Assuming Euclidean or Lorentzian signature we then have different spaces depending on the sign of the curvature

Lorentzian $(-+\cdots+)$	anti-de Sitter (AdS)	de Sitter (dS)
Eucildean $(++\cdots+)$	Lobachevski	sphere
	$R < 0$	$R > 0$

Figure 5.1: Maximally symmetric spaces

Definitions of de Sitter and Anti-de Sitter spaces will follow later.

5.3 Some Simple Examples

At this point it may be useful to work out some examples in detail.

The Two-Sphere

The Metric, Connection and Curvature

Let us work out a simple example as an illustration. The two-sphere S^2 can be defined by its embedding in $\mathbb{R}^3$ as satisfying $x^2 + y^2 + z^2 = R^2$. Without loss of generality we set

$R = 1$. Using spherical coordinates

$$
\begin{aligned}
x &= \sin\theta\cos\varphi \\
y &= \sin\theta\sin\varphi \\
z &= \cos\theta
\end{aligned}
\tag{5.15}
$$

we find for de invariant line element

$$ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta d\varphi^2 \tag{5.16}$$

Thus the non-zero elements of the metric are

$$g_{\theta\theta} = 1 \qquad \text{and} \qquad g_{\varphi\varphi} = \sin^2\theta \tag{5.17}$$

The non-zero elements of the inverse metric are thus

$$g^{\theta\theta} = 1 \qquad \text{and} \qquad g^{\varphi\varphi} = \sin^{-2}\theta \tag{5.18}$$

This gives for the non-zero components of the connection

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \text{and} \qquad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta \tag{5.19}$$

There are only two non-zero components of the curvature tensor (up to symmetries)

$$R^{\theta}_{\varphi\theta\varphi} = -R^{\theta}_{\varphi\varphi\theta} = \sin^2\theta \qquad \text{and} \qquad R^{\varphi}\theta\varphi\theta = -R^{\varphi}_{\theta\theta\varphi} = 1 \tag{5.20}$$

and of the Ricci tensor

$$R_{\theta\theta} = 1 \qquad \text{and} \qquad R_{\varphi\varphi} = \sin^2\theta \tag{5.21}$$

This gives a constant for the curvature

$$R = 2 \tag{5.22}$$

which agrees with our understanding that the two-sphere has constant positive curvature.

The Geodesic Equation and the Great Circles

Next we turn to the geodesic equations

$$
\begin{aligned}
0 &= \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 \\
0 &= \ddot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi}
\end{aligned}
\tag{5.23}
$$

where $\dot{\theta} = d\theta/d\lambda$ etc. Let us show that these geodesic equations lead to the great circle lines. We set up our coordinate system that the initial condition corresponds to a vector at the north pole in the x-direction. This allows us to write the initial conditions at $\lambda = 0$ as $\theta = \varphi = \dot{\varphi} = 0$ and $\dot{\theta} = 1$. Using the geodesic equations it is straightforward to show that

$$\frac{d}{d\lambda}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) = 0 \tag{5.24}$$

Using the initial conditions this implies that

$$\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 = 1 \tag{5.25}$$

As $\theta = 0$ and $\dot{\theta} = 1$ we can't have $\theta \equiv 0$ everywhere. So we can solve this for $\dot{\varphi}$ as

$$\dot{\varphi}^2 = \frac{1 - \dot{\theta}^2}{\sin^2\theta} \tag{5.26}$$

Using this in the first geodesic equation gives

$$\ddot{\theta} - \cot\theta(1 - \dot{\theta}^2) = 0 \tag{5.27}$$

This is an ordinary differential equation that we can solve as follow. Write

$$\ddot{\theta} = \frac{d\dot{\theta}}{d\lambda} = \frac{d\theta}{d\lambda}\frac{d\dot{\theta}}{d\theta} = \frac{1}{2}\frac{d\dot{\theta}^2}{d\theta} \tag{5.28}$$

which allows us to write (5.27) as

$$-\frac{d(1 - \dot{\theta}^2)}{1 - \dot{\theta}^2} = 2\frac{\cos\theta}{\sin\theta}d\theta = 2\frac{d\sin\theta}{\sin\theta} \tag{5.29}$$

or

$$d\,\ln\left[(1 - \dot{\theta}^2)\sin^2\theta\right] = 0 \tag{5.30}$$

and using the initial conditions

$$(1 - \dot{\theta}^2)\sin^2\theta = 0 \tag{5.31}$$

As we have already seen we can't have $\theta \equiv 0$ everywhere and the same holds for $\sin\theta$. Thus we conclude that $\dot{\theta}^2 = 1$ or $\dot{\theta} = \pm 1$. Using the initial conditions we find that $\theta = \lambda$.

Using this we find that the second geodesic equation reduces to

$$\ddot{\varphi} + 2\cot\lambda\,\dot{\varphi} = 0 \tag{5.32}$$

Clearly $\dot{\varphi} = 0$ is a solution of this equation. Using the initial conditions this gives a solution $\phi = \dot{\phi} = 0$. Let us now assume that there is another solution $\phi(\lambda)$ that has $\dot{\phi} \neq 0$. The previous equation can then be written as

$$\frac{d\dot{\varphi}}{\dot{\varphi}} = -2\frac{\cos\lambda}{\sin\lambda}d\lambda = -2\frac{d\sin\lambda}{\sin\lambda} \tag{5.33}$$

or

$$d[\ln\big(\dot{\varphi}\sin^2\lambda\big)] = 0 \tag{5.34}$$

Again using the initial conditions this implies that

$$\dot{\varphi}\sin^2\lambda = 0 \tag{5.35}$$

But λ is just parameter, so $\sin\lambda$ is certainly not everywhere zero. So we conclude that $\dot{\varphi} = 0$ which is a contradiction with our assumption. Thus there cannot be a solution with $\dot{\varphi} \neq 0$.

We conclude that in that reference frame, the only solution to the geodesic equations is

$$\theta = \lambda \qquad \text{and} \qquad \varphi = 0 \tag{5.36}$$

which is indeed a great circle.

The Killing Vectors

Let us work out the symmetries of S^2 by establishing its Killing vectors. The metric is $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$. As S^2 is just all the points in $\mathbb{R}^3$ at unit distance from we can just take the rotational Killing vectors (5.9). In spherical coordinates

$$R = \partial_\phi; \quad S = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi; \quad T = -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi \tag{5.37}$$

and the isometry group of S^2 is $SO(3)$.

It is useful to deduce these from the Killing equations. We use the form (5.3)

$$g_{\mu\sigma}\partial_\nu K^\sigma + g_{\nu\sigma}\partial_\mu K^\sigma + (\partial_\sigma g_{\mu\nu})K^\sigma \tag{5.38}$$

It is straightforward to work this out

$$\begin{aligned} 0 &= \partial_\theta k^\theta \\ 0 &= \cos\theta k^\theta + \sin\theta\partial_\varphi k^\varphi \\ 0 &= \partial_\varphi k^\theta + \sin^2\theta\partial_\theta k^\varphi \end{aligned} \tag{5.39}$$

and one easily checks that the three Killing vectors (5.37) satisfy these equations.

The Three-Sphere

The metric is

$$ds^2 = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\varphi^2 \tag{5.40}$$

The non-zero Christoffel connections are, up to symmetries

$$\begin{aligned}
\Gamma_{\psi}^{\theta\theta} &= -\cos\psi\,\sin\psi \\
\Gamma_{\psi}^{\varphi\varphi} &= -\cos\psi\,\sin\psi\,\sin^2\theta \\
\Gamma_{\theta}^{\theta\varphi} &= \cot\psi \\
\Gamma_{\theta}^{\varphi\varphi} &= -\cos\theta\,\sin\theta \\
\Gamma_{\varphi}^{\psi\varphi} &= \cot\psi \\
\Gamma_{\varphi}^{\theta\varphi} &= \cot\theta
\end{aligned} \tag{5.41}$$

The non-zero components of the curvature tensor are, up to symmetries

$$\begin{aligned}
R^{\psi}_{\theta\psi\theta} &= \sin^2\psi \\
R^{\psi}_{\varphi\psi\varphi} &= \sin^2\psi\,\sin^2\theta \\
R^{\theta}_{\varphi\varphi\psi} &= -\sin^2\psi\sin^2\theta \\
R^{\varphi}_{\theta\theta\varphi} &= -\sin^2\psi
\end{aligned} \tag{5.42}$$

The non-zero components of the Ricci tensor are, up to symmetries

$$\begin{aligned}
R_{\theta\theta} &= 2\sin^2\psi \\
R_{\varphi\varphi} &= 2\sin^2\psi\sin^2\theta
\end{aligned} \tag{5.43}$$

and the curvature is constant

$$R = 6 \tag{5.44}$$

Chapter 6

Anti-de Sitter Spacetime

6.1 introduction

Anti-de Sitter, or AdS for short, spacetime seem to play an ever increasing important role in theoretical physics. Indeed it is one the legs of the **AdS/CFT Correspondence** , which conjectures an equivalence between gravity in AdS space and $N = 4$ super Yang-Mills theory. This has then lead to the so-called **Holographic Principle** states that certain $d+1$ dimensional field theories can be described by theories on de boundary of a d dimensional theory. This then has lead to the speculation by L. Susskind that "the three-dimensional world of ordinary experience – the universe filled with galaxies, stars, planets, houses, boulders, and people – is a hologram, an image of reality cited on a distant two-dimensional (2D) surface". More recently, the holographic principle seems to have surfaced (pun intended) in a large and varied number of branches of physics and has become one of the hot research topics.

Whilst the ADS/CFT correspondence is still very far from real life applications, it provides a good motivation to spend some time studying AdS spacetime.

6.2 Definition

$(d+1)$- dimensional **Anti-de Sitter spacetime** or **AdS$_{d+1}$** is the hypersurface embedded in $(d+2)$-Minkowski spacetime with a $(2, d)$ signature, i.e. metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d}(dX^i)^2 - (dX^{d+1})^2 = \eta_{MN} dX^M dX^N \tag{6.1}$$

that satisfies

$$\eta_{MN}X^M X^N = -(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = -L^2 \tag{6.2}$$

where L is the radius of curvature of AdS.

AdS$_{d+1}$ has an $O(2,d)$ isometry and so $(d+1)(d+2)/2$ Killing vectors. It is therefore maximally symmetric.

6.3 The Boundary of AdS

For very large X^M we can set $L \approx 0$ and then we approach a boundary that is a light-cone

$$\eta_{MN}X^M X^N = -(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = 0 \tag{6.3}$$

We can write this boundary as

$$\partial\text{AdS}_{d+1} = \left\{[X] | X \in \mathbb{R}^{2,d}, X \neq 0, \eta_{MN}X^M X^N = 0\right\} \tag{6.4}$$

Where $[X]$ means that we identify $X \equiv \lambda X$ for $\lambda \in \mathbb{R}$.

Let us take a point on the boundary of AdS that satisfies $\sum_{i=1}^{d}(X^i)^2 = 1$. Being on the boundary then implies that $(X^0)^2 + (X^{d+1})^2 = 1$. As X and $-X$ give the same point on the boundary, we thus have topologically that

$$\partial\text{AdS}_{d+1} = \frac{S^1 \times S^{d-1}}{\mathbb{Z}_2} \tag{6.5}$$

We can also view AdS$_{d+1}$ as a compactification of d-dimensional Minkowski spacetime as follows. Take a point $X \neq 0$ and define $u^{\pm} = X^{d+1} \pm X^d$. The boundary condition then becomes $u_+ u_- = \eta_{\mu\nu}X^\mu X^\nu$ where $\mu, \nu = 0, \cdots, d-1$ is the d-dimensional Minkowski metric. As long as $u_- \neq 0$ we can rescale it to one. For a given point X^μ in d-dimensional Minkowski spacetime, we can then simply solve this for u_+. When $u_- = 0$, we need $u_+ = \infty$ and this thus adds points at infinity and so we have a compactification of d-dimensional Minkowski spacetime.

6.4 Coordinate Systems

There are several useful coordinate systems to parametrise AdS.

6.4 Poincaré Patch Coordinates

The Metric and the Curvature

The **Poincaré Coordinates** are given by

$$
\begin{aligned}
X^0 &= \frac{L^2}{2r}\left[1+\frac{r^2}{L^4}\left(\vec{x}^2-t^2+L^2\right)\right] \\
X^i &= \frac{rx^i}{L} \qquad (i=1,\cdots,d-1) \\
X^d &= \frac{L^2}{2r}\left[1+\frac{r^2}{L^4}\left(\vec{x}^2-t^2-L^2\right)\right] \\
X^{d+1} &= \frac{rt}{L}
\end{aligned}
\tag{6.6}
$$

Here $t, x^i \in \mathbb{R}$ and $r \in \mathbb{R}_+$. The metric becomes

$$
ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(-dt^2 + d\vec{x}^2) = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{6.7}
$$

The non-zero connections are, up to symmetry.

$$
-\Gamma^r_{rr} = \Gamma^t_{rt} = \Gamma^i_{ri} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{tt} = -\Gamma^r_{ii} = \frac{r^3}{L^4} \tag{6.8}
$$

The non-zero components of the curvature tensor are, up to symmetries

$$
R^r{}_{trt} = R^t{}_{iit} = R^t{}_{iir} = R^i{}_{tit} = R^i_{jji} = \frac{r^2}{L^4} \quad \text{and} \quad R^t{}_{rrt} = R^i_{rr-} = \frac{1}{r^2} \tag{6.9}
$$

The non-zero components of the Ricci tensor are

$$
R_{rr} = -\frac{d+1}{r^2} \qquad \text{and} \qquad R_{tt} = -R_{rr} = \frac{(d+1)r^2}{L^4} \tag{6.10}
$$

and the curvature is given by

$$
R = -\frac{d(d+1)}{L^2} \tag{6.11}
$$

The curvature is negative so it is a Lorentzian hyperbolic space.

From (6.7) we see that we can view the Poincar'e patch of AdS$_{d+1}$ as a flat Minkowski space with d coordinates $(t, \vec{x})$ and an "warped" coordinate r. With a little bit of imagination this looks like

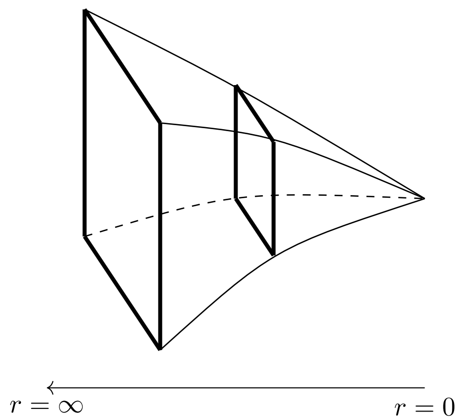


Figure 6.1: Poincar'e patch of AdS$_{d+1}$

The Conformal Boundary

Note that the metric has poles at $r = 0$ and $r = \infty$. Let us first consider the pole at $r = 0$. This is a singularity due solely to the coordinate choice, as the curvature remains constant and we could just extend the Poincaré patch from $r > 0$ to $r < 0$. The hypersurface at the pole at $r = \infty$ is called the **Conformal Boundary**.[1]

If we change coordinates $r = 1/x^0$, with also $x_0 \in \mathbb{R}_+$, then we can rewrite the metric as

$$ds^2 = \frac{L^2}{x_0^2}\left(-dt^2 + (dx^0)^2 + (d\vec{x})^2\right) \tag{6.12}$$

From this we can see that the Poincaré metric does not cover the whole of AdS$_{d+1}$, and so it is only a patch of that space. In order to see this, redefine $x_0 = Le^{-y}$ to write the metric as

$$ds^2 = e^{2y}\left(-dt^2 + (d\vec{x})^2\right) + L^2 dy^2 \tag{6.13}$$

Now $y \in \mathbb{R}$. Consider now a light ray going to $+\infty$ in the y-direction, hence to $x_0 = 0$ and $r = +\infty$, whilst $\vec{x}$ is held constant. This is lightlike so $ds^2 = 0$ and from (6.13) we find

[1]It can be shown that any spacetime that is asymptotically AdS has a quadratic divergence at a point r^* in the radial direction. Here $r^* = \infty$.

$0 = -e^{2y}dt^2 + L^2dy^2$. We can integrate this to find the time t_∞ to go to infinity from, say $y = 0$ at $t = 0$

$$t = \int_0^{t_\infty} dt = L \int_0^{+\infty} e^{-y} dy = L \tag{6.14}$$

This is a finite number, so it takes only a finite time reach that boundary. But t is not finite so the light ray can go further. It can "reflect" from the boundary and travel to another part of the AdS spacetime. Thus the Poincaré coordinates only cover a patch of AdS$_{d+1}$ and we should be able to find coordinates that cover the whole of it.

One way to traverse the boundary is by multiplying the metric by a positive smooth function $g(r, t, \vec{x})$ that has a second order zero at $r = +\infty$. As an example take $g = (L^2/r^2)\omega(t, \vec{x})$. In the limit $r \to +\infty$ the metric then becomes $ds^2 = \omega(t, \vec{x})(-dt^2 + d\vec{x}^2)$ and the singularity has gone. The metric of the bulk thus determines the metric on the boundary. Different choices of $\omega(t, \vec{x})$ for the metric in the bulk, give different choices of the metric on the boundary, but these are related by a conformal factor, i.e. an overall multiplication of $g_{\mu\nu}$. Metrics related by such a conformal factor are a class forming a **Conformal Structure**. Because of this one sometimes refers to the boundary of AdS as being conformal.

Whilst the isometry group of AdS$_{d+1}$ is $SO(2, d)$, the isometry group of the metric (6.7) is only a subgroup of this, viz. $ISO(d-1, 1) \times SO(1, 1)$. Where $ISO(d-1, 1)$ is the Poincaré group acting on $(t, \vec{x})$, i.e. on the conformal boundary of AdS, and $SO(1, 1) \equiv U(1)$ acts on the coordinates as $(r, t, \vec{x}) \to (\lambda^{-1}r, \lambda t, \lambda\vec{x})$.

It can be shown that the other generators of $SO(2, d)$ act on the conformal boundary as the conformal group of Minkowski space and that, in particular the subgroup $SO(1, 1)$ corresponds to the dilations of the conformal group on $\mathbb{R}^{1,d-1}$.

Coordinate Systems Related to Poincaré Coordinates

The replacement $z = L^2/r$ gives the metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 - dt^2 + d\vec{x}^2) = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{6.15}$$

and the conformal boundary now sits at $z = 0$.

Transforming $z = e^{-r/L}$ gives

$$ds^2 = dr^2 + L^2 e^{2r/L}\eta_{\mu\nu}dx^\mu dx^\nu) \tag{6.16}$$

The conformal boundary now sits at $r \to \infty$.

Transforming $\rho = z^2$ gives

$$ds^2 = L^2 \left(\frac{1}{4\rho^2} d\rho^2 + \frac{1}{\rho} \eta_{\mu\nu} dx^\mu dx^\nu \right) \tag{6.17}$$

This is known as the **Fefferman-Graham Metric**. The conformal boundary sits at $\rho \to 0$.

6.4 Global Coordinates

The Metric

Global coordinates for AdS_{d+1} are

$$\begin{aligned} X^0 &= L \cosh \rho \cos \tau \\ X^{d+1} &= L \cosh \rho \sin \tau \\ X^i &= L \Omega_i \sinh \rho \qquad (i = 1, \cdots, d) \end{aligned} \tag{6.18}$$

Here $\rho \in \mathbb{R}_+$, $\tau \in [0, 2\pi[$ and Ω_i are angular coordinates with $\sum_i \Omega_i^2 = 1$, i.e. they parametrise S^{d-1}. These coordinates are called, not surprisingly, **Global Coordinates**.

We can visualise AdS_2 as follows. We have coordinates ρ, τ and Ω_1. We extend ρ to $\rho \in \mathbb{R}$. AdS_2 is then a hyperboloid that is rotated around its axis.

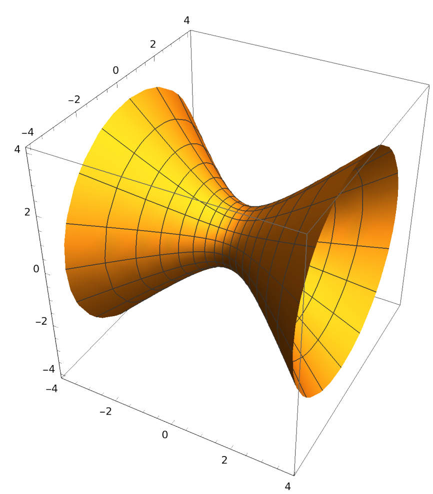


Figure 6.2: Visualisation of AdS_2

The corresponding metric is

$$ds^2 = L^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-1}) \tag{6.19}$$

This metric is reminiscent[2] of the metric on S^{d-1}.

The Boundary

In order to understand the boundary in terms of global coordinates, we change variable

$$\sinh\rho = \tan\theta \qquad \Rightarrow \qquad \cosh\rho = \frac{1}{\cos\theta} \tag{6.20}$$

The metric becomes

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}) \tag{6.21}$$

This is known as the **Einstein Static Universe** $\mathbb{R} \times \boldsymbol{S^d}$. But since $\theta \in [0, \pi/2[$ we only cover half of $\mathbb{R} \times \boldsymbol{S^d}$. We can always rescale the metric to get rid of the pre-factor and we can add the point $\theta = \pi/2$, which corresponds to $\sinh\rho = +\infty$ and hence to $X^i = \infty$, i.e. spatial infinity. We can thus consider the metric

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}, \qquad \theta \in [0, \pi/2], \qquad \tau \in [0, 2\pi[\tag{6.22}$$

The hypersurface $\theta = \pi/2$ is a Cauchy surface, i.e. setting boundary conditions on it gives a well-defined initial value problem on AdS$_{d+1}$.

6.4 Geodesics

The timelike coordinate τ is periodic and so AdS$_{d+1}$ has closed timelike curves. One can avoid these by taking $\tau \in \mathbb{R}$ without identifying poits, This amounts to unwrapping the timelike circle, i.e. taking the universal covering group of AdS$_{d+1}$.

Consider now a radially directed light ray starting at $\rho = \rho_0$ wit time $\tau(\rho_0) = 0$. The metric becomes $0 = -\cosh^2\rho d\tau^2 + d\rho^2$ or $d\tau = d\rho/\cosh\rho$. This has as solution

$$\tau = 2\left(\arctan\tanh\frac{\rho}{2} - \arctan\tanh\frac{\rho_0}{2}\right) \tag{6.23}$$

[2] The metric on S^{d-1} is

$$ds^2 = R^2(\cos^2\rho\, dw^2 + d\rho^2 + \sin^2\rho\, d\Omega^2_{d-1})$$

so AdS is obtained by the analytical continuation in θ.

The time it takes light to go to infinity is a finite time.

$$\tau(\rho = \infty) = \frac{\pi}{2} - 2\arctan\tanh\frac{\rho_0}{2} \tag{6.24}$$

For a massive particle the path is given by

$$\rho(\tau) = \Big|\arcsin[\sin\rho_0 \cos(\tau - \tau_0)]\Big| \tag{6.25}$$

The path is given below for $\rho_0 = 1$ and $\tau_0 = 0$

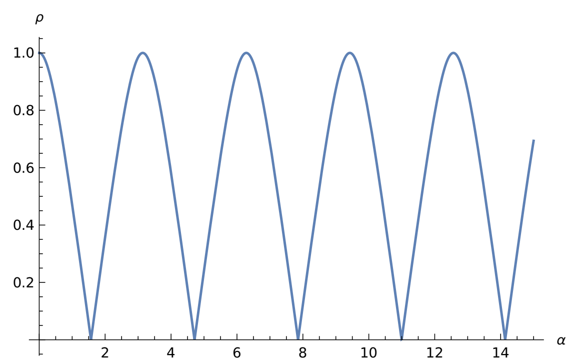


Figure 6.3: Radial path of massive particle in AdS

The particle oscillates and never reaches the boundary.

Chapter 7

Einstein's Equations

7.1 Introduction

In classical mechanics, the equations for a body in a gravitational field are

1. Newton's second law $a = -\nabla\Phi$
2. with the gravitational potential derived from the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, with ρ the mass density and G Newton's constant.

Our goal is to find the corresponding equations in general relativity.

7.2 The Einstein Field Equations

The **Einstein Field Equations** or **EFE** for short are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{7.1}$$

where G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor. Some remarks

1. The EFE relates the matter content of spacetime via $T_{\mu\nu}$ with the behaviour of the metric via the curvature.
2. Recall that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor so the EFE can be written as $G_{\mu\nu} = 8\pi GT_{\mu}$.
3. The curvature tensor has terms that are second order derivatives of the metric, so the EFE can be viewed as a generalisation of the Poisson equation for Newton's gravitational potential, $\nabla^2\Phi = 4\pi G\rho$.
4. Contract the EFE as $R - \frac{d}{2}R = 8\pi GT$, with $T = g^{\mu\nu}T_{\mu\nu}$. Hence
$$R = -\frac{16\pi G}{d-2}T \tag{7.2}$$

Plugging this back in the EFE gives

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{d-2} \eta_{\mu\nu} T \right) \tag{7.3}$$

Thus, in vacuum, $T_{\mu\nu} = 0$, the Ricci tensor vanishes

$$R_{\mu\nu} = 0 \qquad \text{in vacuum} \tag{7.4}$$

5. A more naive form of the EFE could be $R_{\mu\nu} \propto T_{\mu\nu}$ but this gives a problem as conservation of energy-momentum $\nabla^\mu T_{\mu\nu} = 0$ would lead to $\nabla^\mu R_{\mu\nu} = 0$. From (4.49), i.e. $\nabla^\mu G_{\mu\nu} = 0$ we then get that $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$ and form the naive field equation we would get $\nabla T^\mu_\mu = \partial T^\mu_\mu = 0$, as T^μ_μ is just a scalar. This would imply that the trace of the energy-momentum tensor is constant through spacetime which we have no reason to believe should be true.

6. In four dimensions, the EFE are 10 equations for the unknown components of the Einstein tensor $G_{\mu\nu}$, which itself contains up to second partial derivatives of the metric. We thus have 10 non-linear coupled second order partial differential equations that determine the metric in terms of the matter sector. But the Einstein tensor satisfies the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, so we only have six independent PDEs. The metric has 10 components but due to the required diffeomorphism under four coordinate transformations, only six components are independent. This matches the number of equations of the EFE. A similar reasoning holds, of course, in d dimensions.

7. The EFE relates the Ricci tensor, i.e. the traceless part of the curvature tensor, to the matter sector. The remaining components of that curvature tensor are encoded in the Weyl tensor. From the Bianchi identity and the EFE one can show that this tensor satisfies, in four dimensions,

$$\nabla^\rho C_{\rho\sigma\mu\nu} = 8\pi G \left(\nabla_{[\mu} T_{\nu]\sigma} + \frac{1}{3} g_{\sigma[\mu} \nabla_{\nu]} T \right) \tag{7.5}$$

This is sufficient to determine the six independent metric components

7.3 Newtonian Limit

Newtonian gravity is recovered in the limit of slowly moving particles in a static weak field. Consider the perfect fluid energy-momentum tensor (2.14) with, as the particles are slow-moving the pressure set to zero, so that most of the energy sits in the mass of the particles, i.e. $T_{\mu\nu} = \rho U_\mu U_\nu$. Let the fluid be some massive body, say the earth. In the rest frame of the massive body we have $U^\mu = (U^0, 0, 0, 0)$. We can find U^0 from the requirement that $-1 = g_{\mu\nu}U^\mu U^\nu$. In the weak field limit we an expand to first order

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \Rightarrow \qquad g^{\mu\nu} = \eta^{\mu\nu} - h_{\mu\nu} \tag{7.6}$$

Thus

$$-1 = (-1 + h_{00})(U^0)^2 \qquad \Rightarrow \qquad U^0 = (1 - h_{00})^{-1/2} = 1 + \frac{1}{2}h_{00} \tag{7.7}$$

In the rest frame,

$$U_0 = g_{00}U^0 = (-1 + h_{00}) \times \left(1 + \frac{1}{2}h_{00}\right) = -1 + \frac{1}{2}h_{00} \tag{7.8}$$

This gives

$$T_{00} = \rho U_0 U_0 = \rho \times \left(-1 + \frac{1}{2}h_{00}\right)^2 = \rho(1 - h_{00}) \tag{7.9}$$

and thus

$$T = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} = (-1 - h_{00})\rho(1 - h_{00}) = -\rho + o(h^2) \tag{7.10}$$

as all other components of $T_{\mu\nu}$ are zero in the rest frame. Use this with (7.2) to find

$$R = \frac{16}{d-2}\pi\rho G + o(h^2) \tag{7.11}$$

Now use all of this in the 00 component of the EFE

$$R_{00} - \frac{1}{2}(-1 + h_{00})\frac{16}{d-2}\pi\rho G = 8\pi\rho G(1 - h_{00}) \tag{7.12}$$

from which

$$R_{00} = 8\pi\rho G \times \frac{d-3}{d-2} \times (1 - h_{00}) = 8\pi\rho G \times \frac{d-3}{d-2} \tag{7.13}$$

where we have only kept the lowest order contribution.

Now $R_{00} = R^j{}_{0j0}$ as $R^0{}_{000}$ by antisymmetry. We need

$$R^j{}_{0j0} = \partial_j \Gamma^j_{00} - \partial_0 \Gamma^j_{j0} + \Gamma^j_{j\lambda}\Gamma^\lambda_{00} - \Gamma^j_{0\lambda}\Gamma^\lambda_{j0} \tag{7.14}$$

The second term is zero because we are assuming a static universe. The last two terms contain give a contribution of the order h_{00}^2 and can be ignored. Thus

$$\begin{aligned} R^j{}_{0j0} &= \partial_j \Gamma^j_{00} = \partial_j \left[\frac{1}{2}g^{j\lambda}(2\partial_0 g_{\lambda_0} - \partial_\lambda g_{00})\right] = -\frac{1}{2}\partial_j(g^{j\lambda}\partial_\lambda g_{00}) = -\frac{1}{2}\partial_j \partial^j g_{00} \\ &= -\frac{1}{2}\nabla^2 h_{00} \end{aligned} \tag{7.15}$$

This means that (7.13) becomes

$$\nabla^2 h_{00} = -16\pi\rho G \times \frac{d-3}{d-2} \stackrel{d=4}{=} -8\pi\rho G \tag{7.16}$$

If we associate h_{00} with the Newtonian gravitational potential by $h_{00} = -2\Phi$ then we recover Poisson's equation $\nabla^2\Phi = 4\pi G\rho$.

In order to justify this we use the generalisation of Newton's second law, which is the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{7.17}$$

where we are considering a massive particle and the parameter is the proper time. Slowly moving particles implies that $dx^i/dt \ll 1$ or hence $dx^i/d\tau \ll dt/d\tau$ with $x^0 = t$ the classical time. So all $dx^i/d\tau$ can be neglected in the geodesic equation and we are left with

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0 \tag{7.18}$$

Now, using the static assumption,

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}) = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \tag{7.19}$$

In the last equation we used the weak field assumption (7.6). The geodesic equation thus becomes

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} \tag{7.20}$$

The $\mu = 0$ component reduces to $d^2t/d\tau^2 = 0$, so that $dt/d\tau$ is constant and t is an affine parameter. The space components are

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00} \qquad \Rightarrow \qquad \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \tag{7.21}$$

This is Newton's second law provided we indeed identify $\frac{1}{2}\partial_i h_{00}$ by $-\partial_i\Phi$ or $h_{00} = -2\Phi$ plus an irrelevant constant. Note that this implies that

$$g_{00} = -(1+2\Phi) = -\left(1 - \frac{2GM}{r}\right) \tag{7.22}$$

with M the mass of the object causing the gravitational field.

The limit of general relativity of slowly moving bodies in a static weak field thus reduces to Newtonian gravity.

7.4 The High-Energy Limit of General Relativity

Using Planck's constant $\hbar$, Newton's gravitational constant G and the speed of light one can construct four unique constants with dimensions of mass, length, time and energy respectively

name	symbol	formula	value
Planck mass	m_P	$(\hbar c/G)^{1/2}$	2.18×10^{-5} g
Planck length	ℓ_P	$(\hbar G/c^3)^{1/2}$	1.62×10^{-35} m
Planck time	t_P	$(\hbar G/c^5)^{1/2}$	5.39×10^{-44} s
Planck energy	E_P	$(\hbar c^5/G)^{1/2}$	1.22×10^{19} GeV

Even before we reach the value of these parameters we are well into the realm of quantum theory, and it is still largely unclear how gravity changes in that realm, though string theory gives us hopeful glimpses of that. Still, these values are a strong indication that at these levels, something fundamental changes, possibly, if not likely, including a breakdown of how we model spacetime.

7.5 The Einstein Equations from the Action Principle

The Einstein equations follow from the extremisation of the following action.

$$S = \frac{1}{16\pi G} S_H + S_M \tag{7.23}$$

Where S_H is the **Einstein-Hilbert Action**

$$S_H = \int d^d x \sqrt{-g}\, R \tag{7.24}$$

and S_M is the matter action, which depends on the fields under consideration.

Nowadays it is more fashionable to derive the EFEs from the action than to posit them and check that they correspond to Newtonian gravity in the weak field limit. The requirements are then that the action is a scalar and contains at most two derivatives of the metric. It turns out that there is only one such scalar, viz. the curvature R. Combining with the invariant volume element $d^d x \sqrt{-g}$ gives the Einstein-Hilbert action.

The EFE follow most easily from the variation under the inverse metric $g^{\mu\nu}$. This is a rather lengthy calculation, but we can easily find its general form. We need $\delta S_H = \int d^d x\, K_{\mu\nu} \delta g^{\mu\nu}$ for some tensor $K_{\mu\nu}$ that contains not more than two derivatives of the metric. There are only two such tensors $g_{\mu\nu}$ which we can multiply by R and $R_{\mu\nu}$. Thus, without doing any calculation we know that the EFE must be of the form $a_1 R_{\mu\nu} + a_2 R g_{\mu\nu} = 0$. Contact this with $g^{\mu\nu}$ to get $(a_1 + d a_2) R = 0$. Provided $a_1 + d a_2 \neq 0$, which turns out to be the case, we see that $R = 0$ and hence $R_{\mu\nu} = 0$. So without any calculation we have found that in vacuum the EFE are $R_{\mu\nu} = 0$!

Let us now do the detailed calculation as we will need the coefficients a_1 and a_2 if there is matter involved. The detailed calculation has four different parts

1. $\int d^d x\, (\delta\sqrt{-g}) R_{\mu\nu} g^{\mu\nu}$. Use $\ln\det M = \operatorname{tr}\ln M$ to find $\delta(\det M) = \det M \operatorname{tr} M^{-1}\delta M$ and therefore

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{7.25}$$

2. $\int d^d x\, \sqrt{-g}(\delta R_{\mu\nu}) g^{\mu\nu}$. Consider a general variation of a connection. This is the difference between two connections and hence transforms as a tensor, giving the standard form for the covariant derivative $\nabla_\lambda(\delta\Gamma^\sigma_{\mu\nu}) = \cdots$. One then readily finds that

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\sigma_{\lambda\mu}) \tag{7.26}$$

which, amongst aficionados, is known as the Palatini identity. From this it follows that

$$\begin{aligned}\int d^d x\, (\delta\sqrt{-g}) R_{\mu\nu} g^{\mu\nu} &= \int d^d x\, \sqrt{-g}\nabla_\sigma \left(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu}\right) \\ &= \int d^d x\, \sqrt{-g}\nabla_\sigma \Big[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\mu(\delta g^{\sigma\mu})\Big]\end{aligned} \tag{7.27}$$

where in the last equation we used the variation of the connection. From Stokes theorem this reduces to a surface integral, and vanishes as long as the variation of the metric and its first derivative banish at infinity.

3. $\int d^d x\, \sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu}$. Nothing needs to be done as this is in the desired form already.

4. δS_M. We simply define the energy-momentum tensor as

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{7.28}$$

Bringing the four contributions together recovers the EFE. Some comments are in order

- We assumed that the variation of the metric and its first derivative vanish on the boundary. This may not always be the case and so we must recognise the possibility of boundary terms.

- That $T_{\mu\nu}$ is the energy-momentum tensor can be checked from e.g. scalar field theory. The action in covariant form is

$$S_\phi = \int d^d x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] \tag{7.29}$$

This leads to the energy-momentum tensor

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi) \tag{7.30}$$

which is, up to an allowed total derivative, equal to the canonical energy-momentum tensor obtained by Noether's theorem. This definition is however more convenient in general as it automatically ensures symmetry and gauge invariance of the energy-momentum tensor.

7.6 The Cosmological Constant

It is not unconceivable that spacetime has an isotropic and homogenous energy density. This is a special form of the energy-momentum tensor

$$T^{(\text{vac})}_{\mu\nu} = -\rho_{\text{vac}} g_{\mu\nu} \tag{7.31}$$

Comparing to a perfect fluid $T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p \eta^{\mu\nu}$ this corresponds to an isotropic pressure $p_{\text{vac}} = \rho_{\text{vac}}$ and we can write the EFE as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} \tag{7.32}$$

where

$$\Lambda = 8\pi G \rho_{\text{vac}} \tag{7.33}$$

is the **Cosmological Constant** and now $T_{\mu\nu}$, excludes vacuum energy.

This form of the EFE can be obtained form the action

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right] \tag{7.34}$$

with $\mathcal{L}_M$ the matter Lagrangian.

We can make a heuristic calculation of the value of the cosmological constant from quantum field theory arguments. When quantised, the simple harmonic oscillator has zero-point energy $\hbar\omega/2$, with $\omega = \sqrt{\vec{k}^2 + m^2}$ the frequency of each oscillator. If we go to field theory and sum over all oscillators, which becomes an integration over continuous momenta, we get, in four dimensions, a vacuum energy proportional to $\hbar\int d^3k\,\omega_k = \hbar\int k^2dk\sqrt{\vec{k}^2+m^2}$. Assuming our theory to be valid up to the Planck scale we can put a cut-off on the integration and we get a cosmological constant of the order of $\rho_{\text{vac}} = \hbar k_P^4$ with value

$$\rho_{\text{vac}} \approx \left(10^{18}\text{GeV}\right)^4 \approx 10^{72}\,\text{GeV}^4 \tag{7.35}$$

One can also put a bound on this vacuum energy from observations

$$|\rho_{\text{obs}}| \leq \left(10^{-12}\text{GeV}\right)^4 \approx 10^{-48}\,\text{GeV}^4 \tag{7.36}$$

That is discrepancy of the order of 10^{120} with our simple calculation. It is clear that the simple calculation is to simple and that there can be plenty more contributions, a.o. from other fields, but it is still a puzzle how these large values can combine to the observed value.

7.7 Energy Conditions

When there are many different matter sectors present, it is useful to be able to make statements based on general properties of the combined energy-momentum tensor. Energy conditions are coordinate invariant restrictions on the energy-momentum tensor. These are obtained by turning the energy-momentum tensor into a scalar by contracting it with timelike (t^μ) or lightlike (ℓ^μ) vectors. Physical intuition can be obtained by applying these condition to a perfect fluid with

$$T^{\mu\nu} = (\rho+p)U^\mu U^\nu + p\eta^{\mu\nu} \tag{7.37}$$

One usually considers five such conditions

1. The **Weak Energy Condition** or **WEC** requires

$$T_{\mu\nu}\,t^\mu t^\nu \geq 0 \quad \text{for all timelike vectors } t^\mu \tag{7.38}$$

   This will be the case if $T_{\mu\nu}U^\mu U^\nu \geq 0$ and $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ for some lightlike vector ℓ^μ.[1] Now go to in the rest frame of the perfect fluid with locally inertial coordinates we

[1] We first argue that any timelike vector t^μ can be written as $t^\mu = \lambda U^\mu + \ell^\mu$, with U^μ the velocity vector and ℓ^μ a lightlike vector. Let us go to the rest frame of the perfect fluid so that $U^\mu = (1,\mathbf{0})$ and $t = t_\mu t^\mu$. We need to show that it $t<0$ we can always find a λ such that ℓ^μ is lightlike. So we solve

$$0 = \ell_\mu\ell^\mu = (t_\mu - \lambda U_\mu)(t^\mu - \lambda U^\mu) = t + 2\lambda t^0 - \lambda^2$$

have

$$0 \leq T_{\mu\nu}U^\mu U^\nu = T_{00} = \rho \tag{7.39}$$

and

$$0 \leq T_{\mu\nu}\ell^\mu \ell^\nu = \ell_0^2(T_{00} + T_{11}) = \ell_0^2(\rho + p) \tag{7.40}$$

These are inequalities on scalars so reference frame independent and we conclude that the weak energy condition is equivalent to

$$\rho \geq 0 \qquad \text{and} \qquad \rho + p \geq 0 \tag{7.41}$$

The energy density should be non-negative and the pressure should not be too large compared to the energy density. It is generally believed that these conditions are satisfied for matter systems.

2. The **Null Energy Condition** or **NEC** requires

$$T_{\mu\nu}\,\ell^\mu \ell^\nu \geq 0 \quad \text{for all lightlike vectors } \ell^\mu \tag{7.42}$$

or equivalently

$$\rho + p \geq 0 \tag{7.43}$$

It is a less restrictive version of the WEC: the energy density may be negative as long as there is a compensating positive pressure.

3. The **Dominant Energy Condition** or **DEC** requires

$$T_{\mu\nu}\,t^\mu t^\nu \geq 0 \text{ for all timelike vectors } \ell^\mu \text{ and } T_{\mu\nu}t^\mu \text{ is non-spacelike} \tag{7.44}$$

For a perfect fluid this is equivalent to

This is a quadratic equation that has real solutions provided the discriminant is positive, i.e. $4t_0^2 + t \geq 0$. This is the case as $t_0^2 + t = t_0^2 - t_0^2 + \boldsymbol{t}^2 \geq 0$. Now

$$T_{\mu\nu}t^\mu t^\nu = \lambda^2 T_{\mu\nu}U^\mu U^\nu + T_{\mu\nu}\ell^\mu \ell^\nu + 2\lambda T_{\mu\nu}U^\mu \ell^\nu$$

which will be greater than zero if both inequalities in the main text are satisfied.

$$\rho \geq |p| \tag{7.45}$$

the energy density is positive and not smaller than the absolute value of the pressure.

4. The **Null Dominant Energy Condition** or **NDEC** requires

$$T_{\mu\nu}\,\ell^\mu \ell^\nu \geq 0 \text{ for all ligthlike vectors } \ell^\mu \text{ and } T_{\mu\nu}\ell^\mu \text{ is non-spacelike} \tag{7.46}$$

For a perfect fluid this is equivalent to

$$\rho \geq |p| \tag{7.47}$$

just as for the DEC, but negative energy densities are allowed as long as $p = -\rho$.

5. The **Strong Energy Condition** or **SEC** requires

$$T_{\mu\nu}\,t^\mu t^\nu \geq \frac{1}{2} T^\mu_\mu t^\nu t_\nu \text{ for all timelike vectors} \tag{7.48}$$

For a perfect fluid this is equivalent to

$$\rho + p \geq 0 \quad \text{and} \quad \rho + 3p \geq 0 \tag{7.49}$$

Note that the SEC does not imply the WEC. It implies the WEC with very large negative pressures. It turns out that the SEC is responsible for gravity to be attractive.

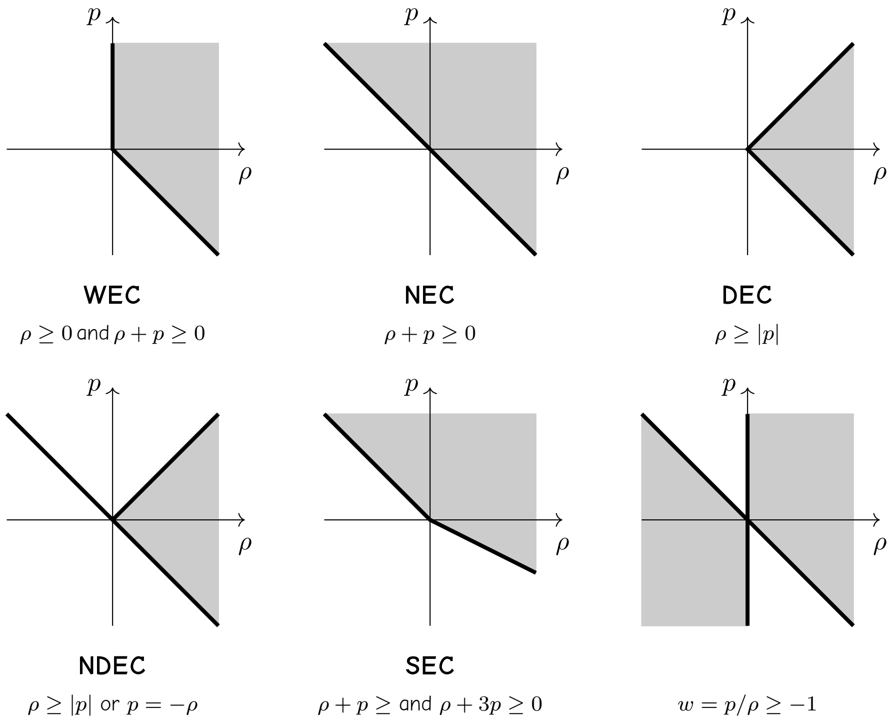


Figure 7.1: Energy Conditions: Weak, Null, Dominant, Null Dominant, Strong and the equation of state condition $w \geq -1$ and the conditions for a perfect fluid

7.8 Alternative Theories of Gravity

We will consider four alternative theories only and refer to Carroll for details. It turns out non of these four ideas are paradigm shifting,

1. **Gravitational scalar fields**: One adds one (or more) scalar fields λ. In the Einstein Hilbert Lagrangian we replace R by $f(\lambda)R$ for some function $f(\lambda)$ and adds a Lagrangian for a scalar field. One assumes that the matter sector does not couple to λ. Essentially one can make a transformation of the fields to rewrite the Lagrangian as a traditional Einstein-Hilbert Lagrangian with an additional scalar field and the matter sector now couples to the scalar field. The metric and curvature is now, of course, completely different.

2. **Kaluza-Klein Compactification**: We consider the extra spatial dimensions to be compactified and the metric to be of the form

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + b^2(x)\gamma_{ij}(y)dy^i dy^j \tag{7.50}$$

We can integrate out the extra terms in the action and arrive at an effective four-dimensional action. Besides massless states this action has an infinite tower of massive states that we ignore as they are very heavy if the compactified dimensions are very small. The action can then basically be rewritten as a four-dimensional metric coupled to a scalar field, the so-called **Dilaton Field**.

3. **Higher Order Curvature Terms**: These would include higher order derivatives of the metric and would thus require many more initial conditions to give a well-defined problem. One would also assume that these corrections need to be small, as the current theory works so well. However, at high energy they may become important. Also viewing a quantum gravity theory as an effective theory, one should, in principle, consider all possible interaction terms. An issue is that these theories lead to negative energy solutions.
4. **Non-Christoffel Connections and Torsion**: In the **Palatini Formalism** we consider a theory where the connection and the metric are independent but is torsion free. The equations of motion dictate that the connections are the Christoffel connections. If we drop the requirement that the theory is torsion free, then, as torsion is characterised by a tensor, we are just adding a new type of matter to the theory. Similarly if we forget about metric compatibility, we can always write the metric as a sum of a metric compatible one and a remaining tensor, which can fall in the matter sector.

There are more exotic theories like loop quantum gravity and emergent gravity, but these are out of scope here.

7.9 Linearised Gravity

7.9 First Order Approach

We consider a more systematic approach than for the Newtonian limit. We set

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.51}$$

with $h_{\mu\nu}$ "small" compared to $\eta_{\mu\nu}$. It isn't entirely clear what smallness of a tensor means but we take it to mean that its values are much smaller than one in some global inertial coordinate system. The inverse metric is then

$$g^{\mu\nu} = \eta^{mu\nu} - h^{\mu\nu} \tag{7.52}$$

We expand everything to first order in $h_{\mu\nu}$ and its derivatives. The Christoffel symbols are

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \tag{7.53}$$

For the Riemann tensor we can ignore the terms quadratic in the connections and thus

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + o(h^2) \tag{7.54}$$

The Ricci tensor becomes

$$R^{(1)}_{\mu\nu} = \partial^\sigma \partial_{\{\nu} h_{\mu\}\sigma} - \frac{1}{2}\partial^\sigma \partial_\sigma h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h^\sigma_\sigma \tag{7.55}$$

and the Einstein tensor becomes

$$G^{(1)}_{\mu\nu} = \partial^\sigma \partial_{\{\nu} h_{\mu\}\sigma} - \frac{1}{2}\partial^\sigma \partial_\sigma h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h^\sigma_\sigma - \frac{1}{2}\eta_{\mu\nu}(\partial^\sigma \partial^\rho h_{\sigma\rho} - \partial^\sigma \partial_\sigma h^\rho_\rho) \tag{7.56}$$

It is convenient to introduce

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\sigma_\sigma \tag{7.57}$$

so that the Einstein equations become

$$-\frac{1}{2}\partial^\sigma \partial_\sigma \gamma_{\mu\nu} + \partial^\sigma \partial_{\{\nu} \gamma_{\mu\}\sigma} - \frac{1}{2}\eta_{\mu\nu}\partial^\sigma \partial^\rho \gamma_{\sigma\rho} = 8\pi T_{\mu\nu} \tag{7.58}$$

We can now use the diffeomorphism invariance that transforms the metric as $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi \mu$ to chose a gauge

$$\partial^\nu \gamma_{\mu\nu} = 0 \tag{7.59}$$

and the linear Einstein equations are

$$\partial^\sigma \partial_\sigma \gamma_{\mu\nu} = -16\pi T_{\mu\nu} \tag{7.60}$$

In vacuum (7.59) and (7.60) are the equations for a massless spin two particle, which we would identify with a graviton. Note that the 00 component of (7.60) gives rise to (7.16).

7.9 Second Order and the Concept of Energy

Because the metric describes both the background spacetime structure and the dynamics of the gravitational field, there is no concept of local energy density in general relativity. Indeed, one cannot decompose the gravitational energy in a background and in a dynamical part as both are interconnected. A local energy density would only depend on the dynamical part, but this cannot be disentangled from the background.

We can illustrate this with second order corrections from the weak field limit. The linearised Einstein equations in vacuum are of the form

$$G^{(1)}_{\mu\nu}[h] = 0 \tag{7.61}$$

Consider now the second order corrections to the Ricci tensor

$$R^{(2)}_{\mu\nu} = \frac{1}{2}h^{\sigma\rho}\partial_\mu\partial_\nu h_{\sigma\rho} - h^{\sigma\rho}\partial_\sigma\partial_{\{\mu}h_{\nu\}\rho} + \frac{1}{4}\partial_\mu h_{\sigma\rho}\partial_\nu h^{\sigma\rho} + \partial^\rho j^\sigma_\nu\partial_{[\rho}h_{\sigma]\mu} + \frac{1}{2}\partial_d(h^{\rho\sigma}\partial_\sigma h_{\mu\nu}) - \frac{1}{4}\partial^\sigma h^\rho_\rho\partial_\sigma h_{\mu\nu} - \left(\partial_\rho h^{\sigma\rho} - \frac{1}{2}\partial^\sigma h^\rho_\rho\right)\partial_{\{\mu}h_{\nu\}\sigma} \tag{7.62}$$

This would result in a second order correction of the Einstein equations. These come from the second order metric $g = h + h^{(2)}$ in $G^{(1)}[h^{(2)}]$ and the first order metric in $G^{(2)}_{\mu\nu}[h]$ and are of the form

$$G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h] = 0 \tag{7.63}$$

We can rewrite this as

$$G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi t_{\mu\nu} \qquad \text{with} \qquad t_{\mu\nu} = -\frac{1}{8\pi}G^{(2)}_{\mu\nu}[h] \tag{7.64}$$

Clearly $t_{\mu\nu}$ is symmetric and one can also check that it is conserved $\partial_\mu t_{\mu\nu} = 0$. This suggest that $t_{\mu\nu}$ is an effective energy-momentum that causes a correction to the weak field spacetime metric, valid to second order. But this not entirely correct. To start with it turns out that $t_{\mu\nu}$ is not invariant under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi\mu$. In addition to that, one can always add a tensor of the form $\partial^\sigma\partial^\rho U_{\mu\nu\sigma\rho}$ to it that is local, quadratic in $h_{\mu\nu}$ and satisfies $U_{\mu\nu\sigma\rho} = U_{[\mu\sigma]\nu\rho} = U_{\mu\sigma[\nu\rho]} = U_{\nu\rho\mu\sigma}$ and the symmetry of $t_{\mu\nu}$ and its conservation law is unaffected. This is a reflection of the fact that local energy density is not well-defined in general relativity.

Chapter 8

Conformal Diagrams

Conformal Diagrams, also known as **Penrose Diagrams**, are representations of spacetimes in a finite diagram that are particularly suited for understanding the causal structure of the spacetime.

They are obtained by performing specific conformal transformations. Because such transformations preserve the angle, the light cones in conformal diagrams remain at $45°$.

8.1 Minkowski Space

The metric is, in spherical coordinates and four dimensions

$$ds^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{8.1}$$

with

$$t \in \mathbb{R}, \quad r \in \mathbb{R}_+, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi[\tag{8.2}$$

We first replace t and r by coordinates u and v

$$u = t - r \quad \text{and} \quad v = t + r \qquad \text{with} \quad u \leq v \in \mathbb{R} \tag{8.3}$$

so that the metric becomes

$$ds^2 = -\frac{1}{2}(du\,dv + dv\,du) + \frac{1}{4}(v - u)^2 d\Omega^2 \tag{8.4}$$

with $d\Omega$ the metric on the unit two-sphere. Now we bring the u and v to a finite range

$$U = \arctan u \quad \text{and} \quad V = \arctan v \qquad \text{with} \quad U \leq V \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \tag{8.5}$$

Straightforward algebra leads to

$$ds^2 = \frac{1}{4\cos^2 U\, \cos^2 V}\left[-4dU\,dV + \sin^2(V - U)\,d\Omega^2\right] \tag{8.6}$$

Finally we go back to

$$T = V + U \quad \text{and} \quad R = V - U \qquad \text{with} \quad R \in [0, \pi] \quad \text{and} \quad |T| + R \leq \pi \tag{8.7}$$

The metric finally becomes

$$ds^2 = \omega^{-2}\left(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2\right) \tag{8.8}$$

with $\omega = \cos T + \cos R$ and is of the form $\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$ and is thus a conformal transformation and thus preserves angles. The coordinates T and R now only take finite values

If we momentarily forget the finite range of T and consider the metric

$$\widetilde{ds}^2 = \omega^2 ds^2 = -dT^2 + dR^2 + \sin^2 R\, d\Omega^2 \tag{8.9}$$

then this describes not a flat space but $\mathbb{R} \times S^3$. This is a manifold with a curvature. This should not worry us as it is not the Minkowski space anymore, due to the extension of the coordinates. This space is known as **Einstein Static Universe** can be represented as a cylinder in which each circle of constant T is actually a three sphere. The part of that space where also $|T| + R \leq \pi$ represents Minkowski space.

The complete transformations are thus

$$\begin{aligned} T &= \arctan(t+r) + \arctan(t-r) \\ R &= \arctan(t+r) - \arctan(t-r) \end{aligned} \tag{8.10}$$

The "$\arctan$" function transforms the infinite coordinates t and r to the finite coordinates T and R.

The conformal diagram of Minkowski space is shown below. The boundaries of this diagram are referred to as **Conformal Infinity**. For convenience let us recapitulate the causal structures:

symbol	name	(T, R)	(t, r)
i^+	future timelike infinity	$(-\pi, 0)$	$(+\infty, r)$
i^-	past timelike infinity	$(-\pi, 0)$	$(-\infty, r)$
i^0	spatial infinity	$(0, \pi)$	$(t, +\infty)$
$\mathcal{J}^+$	future null infinity	$(\pi - R, R)$	$(\frac{1}{2}\cot R, -\frac{1}{2}\cot R)$
$\mathcal{J}^-$	past null infinity	$(-\pi - R, R)$	$(\frac{1}{2}\cot R, -\frac{1}{2}\cot R)$

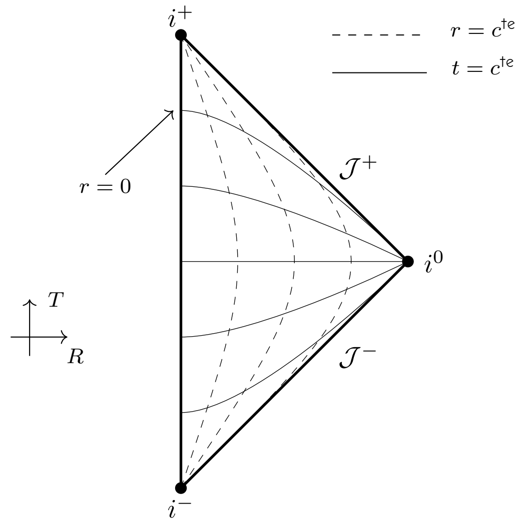


Figure 8.1: Conformal diagram for Minkowski spacetime in the R–T plane. Lines of constant t are the lines starting at the vertical axis at $R = 0$ and ending at the point i^0. Lines of constant r start and end at $i^\pm$.

Because the diagram is conformal, all null geodesics are at $\pm 45°$. All timelike geodesics start at the point i^- and end at the point i^+. All lightlike geodesics starts on the line $\mathcal{J}^-$ and end on the line $\mathcal{J}^+$. All spacelike geodesics end at the point i^0.

Spacetimes (or regions of it) that are asymptotically flat share the structure of $\mathcal{J}^\pm$ and i^0 with the conformal diagram of Minkowski space.

8.1 Two-Dimensional Minkowski Space

The case of two-dimensional Minkowski space is slightly different because the metric is $ds^2 = -dt^2 + dx^2$ and here x can take on all real values, not only positive one. The consequence is that the conformal map is not a triangle, but a square

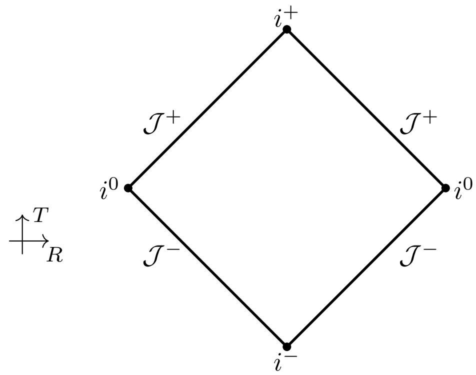


Figure 8.2: Conformal diagram for $2d$ Minkowski spacetime in the R–T plane

This c

8.2 Robertson-Walker Spacetime

Consider the metric in four dimensions

$$ds^2 = -dt^2 + t^{2q}(dr^2 + r^2 d\Omega^2) \tag{8.11}$$

for some $q \in]0,1[$. We will encounter a straightforward generalisation of this metric, describing the **Robertson-Walker Spacetime**, in section 16.1 when we discuss a basic cosmological model for the universe. It is easily checked that the curvature of this space is

$$R = \frac{6q(2q-1)}{t^2} \tag{8.12}$$

There is a real singularity at $t = 0$ so the restrictions are $t > 0$ and $r \geq 0$. We can bring this into a familiar form by transforming

$$\eta = \frac{1}{1-q} t^{1-q} \qquad \text{with} \quad \eta > 0 \tag{8.13}$$

to get

$$ds^2 = [(1-q)\eta]^{2q/(1-q)}(-d\eta^2 + dr^2 r^2 d\Omega^2) \tag{8.14}$$

We can now perform the same type of transformations from (η, r) to (R, T) as in the Minkowski case and obtain

$$ds^2 = \omega^{-2}\left(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2\right) \tag{8.15}$$

with now

$$\omega(T,R) = \left(\frac{\cos T + \cos R}{2\sin T}\right)^{2q} (\cos T + \cos R) \tag{8.16}$$

But this time the new coordinates are restricted to

$$T > 0; \qquad R \geq 0; \qquad T + R < \pi \tag{8.17}$$

The conformal diagram is therefore half that of Minkowski spacetime:

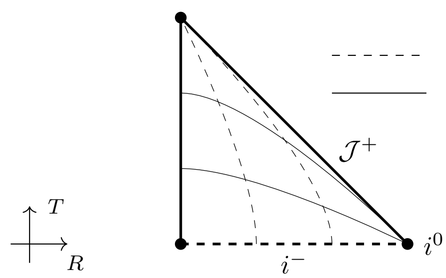


Figure 8.3: Conformal diagram for the Robertson-Walker spacetime in the R–T plane

8.3 Anti-de Sitter Space

We start from the metric in the form (6.21)

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}) \qquad \text{with} \quad \theta \in [0, \pi/2], \qquad \tau \in [0, 2\pi[\tag{8.18}$$

but then go to the universal cover to avoid closed timelike curves, see section 6.4.3 so that $\tau \in \mathbb{R}$.

The conformal diagram of the universal cover of AdS_2 is then an infinite strip as shown in fig. 8.4. Without the universal cover it is the rectangle bounded by the dashed lines at $\tau = 0$ and $\tau = 2\pi$. The Poincaré patch, that only covers part of AdS_2 is the shaded triangle. The conformal diagram of AdS_{d+1} can be obtained from this by adding a sphere S^{d-1} to each point.

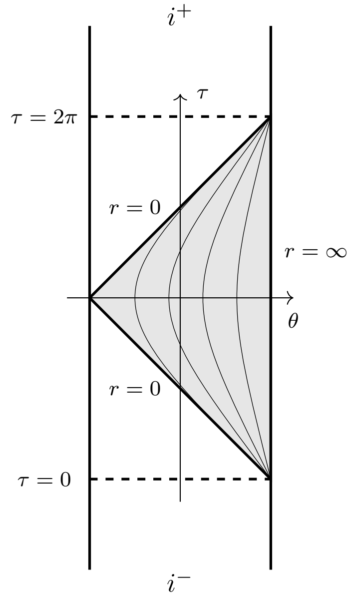


Figure 8.4: Conformal diagram for AdS_2 in the R–T plane

Chapter 9

The Schwarzschild Metric

9.1 The Schwarzschild Metric

The **Schwarzschild Metric** is a simple static spherically symmetric solution to Einstein's equations. In d dimensions the metric is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-2}^2 \qquad \text{with} \qquad f(r) = 1 - \frac{2\mu}{r^{d-3}} \tag{9.1}$$

Birkhoff's Theorem states that this is the unique spherically symmetric solution to Einstein's equations, and in particular that there are no time-dependent solutions of this form.

For simplicity we restrict ourselves to four dimensions and will often ignore the angular part. We set $\mu = GM$ and the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2 \tag{9.2}$$

with the metric on S^2 being $d\Omega_2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The interpretation of M follows from (7.22), i.e. form the weak field limit of a gravitational field created by an object: M is just the mass of that object.[1]It is not necessarily the sum of the masses of different objects as there would be some binding energy in that case. But it is valid in the weak field limit. As $r \to \infty$ the metric resembles the Minkowski metric, i.e. we recover flat space. This is known as **Asymptotic Flatness**..

There are two special points, $r = 0$ and $r = 2GM$. These special points are coordinate

[1]in chapter 12 we will derive a formula for conserved integrals for general black holes and show that M is indeed the conserved integral corresponding to the mass of the energy, i.e. the mass of the black hole.

dependent and do not necessarily lead to singularities. To find whether they are singularities, we need to see if scalars, which are coordinate free, diverge in these points. We find, using e.g. Mathematica, that the **Kretschmann Invariant** is given by

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{9.3}$$

so $r = 0$ is a genuine singularity.

The point $r = 2GM$ turns out not to be a singularity. This radius is of little day-to-day consequences. For example if we consider this radius for the sun $r = 2GM_\odot$ and compare this to the sun's radius which is $R_\odot \approx 10^6 GM_\odot$ then we find that $r \ll R_\odot$ so this radius sits well within the sun. But the Schwarzschild solution is only valid in vacuum, so outside of the sun this special radius is irrelevant. It can only be relevant for an object with a very small radius and a large mass; a black hole as we will see.

9.2 Geodesics of the Schwarzschild Metric

The non-vanishing Christoffel connections for the Schwarzschild metric are

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{GM}{r(r-2GM)} \\
\Gamma^r_{tt} &= \frac{GM}{r^3}(r-2GM) \\
\Gamma^r_{rr} &= -\frac{GM}{r(r-2GM)} \\
\Gamma^r_{\theta\theta} &= -(r-2GM) \\
\Gamma^r_{\phi\phi} &= -(r-2GM)\sin^2\theta \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{r\phi} &= \frac{1}{r} \\
\Gamma^\phi_{\theta\phi} &= -\cot\theta
\end{aligned} \tag{9.4}$$

This gives four geodesic equations

$$
\begin{aligned}
0 &= \frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} \\
0 &= \frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 \\
&\quad - (r-2GM)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] \\
0 &= \frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 \\
0 &= \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda}
\end{aligned} \tag{9.5}
$$

A sane person would not think of trying to solve them. Rather we will use the symmetries and the Killing vectors to gain insight. There are four Killing vectors, three for the spherical symmetry and one for time translations, that will each lead to a conserved quantity: if K^μ is a Killing vector then(5.2) is satisfied, $p^\nu\nabla_\nu(K_\mu p^\mu) = 0$. From Leibniz $p^\mu\nabla_\mu = (dx^\mu/d\lambda)\nabla_\mu = d/d\lambda$ and so the Killing vectors satisfy $d(K_\mu p_\mu)/d\lambda = 0$ and thus

$$K_\mu \frac{dx^\mu}{d\lambda} = c^{\text{te}} \tag{9.6}$$

Moreover we also have that

$$-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon = c^{\text{te}} \tag{9.7}$$

Indeed for timelike geodesics this is just $U^\mu U_\mu = -1$ so the $\epsilon = +1$. For lightlike geodesics we have $\epsilon = 0$ by definition.

Spherical symmetry implies conservation of angular momentum. Conservation of the direction of angular momentum means that it moves in a plane. We can always chose this plane to be the equatorial plane, which means that we can always choose $\theta = \pi/2$, which fixes two components of the angular momentum. The two remaining Killing vectors are then the magnitude of angular momentum, and from the timelike Killing vector, energy. The latter has the timelike Killing vector, with some abuse of notation,

$$K^\mu = \partial_t = (1, 0, 0, 0) \tag{9.8}$$

and the Killing vector for the conservation of the magnitude of angular momentum is

$$R^\mu = \partial_\phi = (0, 0, 0, 1) \tag{9.9}$$

The corresponding conserved quantities are

$$
\begin{aligned}
E &= -K^\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} \\
L &= +R^\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda}
\end{aligned} \tag{9.10}
$$

where we have lowered the indices of the Killing vectors and have used $\theta = \pi/2$. We can view E and L as the energy and angular momentum (density) of the test particle.[2]

Insert the metric in (9.7), using $\theta = \pi/2$

$$
-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\Phi}{d\lambda}\right)^2 = -\epsilon \tag{9.11}
$$

Multiplying by $1 - 2GM/r$ and using the definitions of E and L immediately gives

$$
-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2} = -\epsilon\left(1 - \frac{2GM}{r}\right) \tag{9.12}
$$

We can rewrite this as

$$
\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}\mathcal{E} \tag{9.13}
$$

with

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2}E^2 \\
V(r) &= \frac{1}{2}\epsilon - \frac{\epsilon GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}
\end{aligned} \tag{9.14}
$$

Eq. (9.13) is similar to the equation for a massive particle for the conservation of energy, being kinetic plus potential energy, in a one-dimensional potential $V(r)$ with total energy $\mathcal{E}$. Of course, we also need $t(\lambda)$ and $\phi(\lambda)$ to get a full picture, but we can already learn a lot now.

Figures 9.1 and 9.2 Show the Newtonian (thick line) and general relativity potential (dashed line) for massive and massless particles respectively. The analysis can be done in the same way as for Newtonian gravity.

[2] In (4.22) we saw that $-p_\mu U^\mu$ was the energy of a particle as measure by an observer with four-velocity U^μ. The difference with $E = -p_\mu K^\mu$ is that $-p_\mu U^\mu$ is the inertial/kinetic energy of the test particle, whilst $-p_\mu K^\mu$ includes the potential energy. Recall that a curved spacetime only has a well-defined conserved energy if there is a timelike Killing vector, i.e. symmetry under time translations.

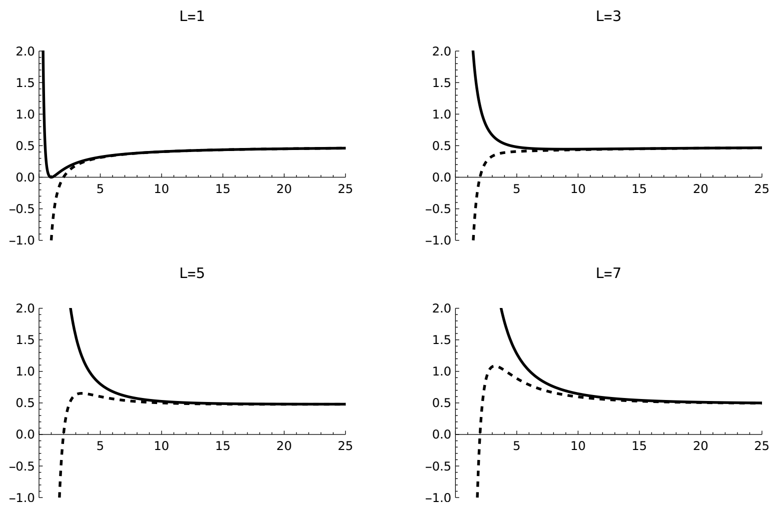


Figure 9.1: Newtonian (thick line) vs GR (dashed line) potential for massless particles

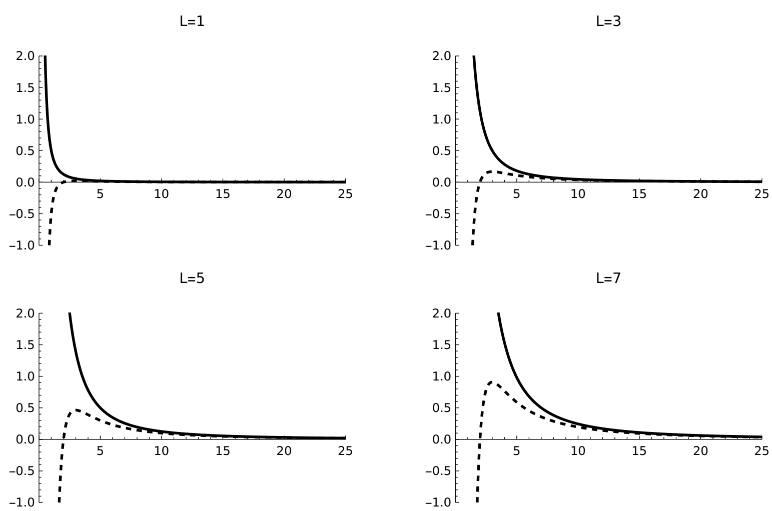


Figure 9.2: Newtonian (thick line) vs GR (dashed line) potential for massless particles

Massive Particles

Consider first massive particles, i.e $\epsilon = 1$. Recall that in Newtonian mechanics the central force problem for a massive particle can be rewritten as a one dimensional radial problem with an effective potential

$$V_{\text{Newton}} = -\frac{GM}{r} + \frac{\ell^2}{2r^2} \tag{9.15}$$

with ℓ the angular momentum. The potential V for a massive particle is now

$$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{9.16}$$

The difference is the constant term, which we can always add in classical mechanics, and the last term GML^2/r^3, which hence provides a general relativity correction to Newtonian gravity.

The potential has extrema when $dV(r)/dr = 0$, i.e. when

$$GMr_c^2 - L^2 r_c + 3GML^2\gamma = 0 \tag{9.17}$$

where $\gamma = 0$ for Newtonian gravity and $\gamma = 1$ for general relativity. For Newtonian gravity this becomes $r_c = 0$ or $r_c = L^2/\epsilon GM$. However we now have

$$r_\pm = \frac{L^2}{2GM}\left[1 + \mp\sqrt{1 - \frac{3(2GM)^2}{L^2}}\right] \tag{9.18}$$

There is a critical value for the extrema when the square root vanishes

$$L = \sqrt{3}(2GM) \tag{9.19}$$

and we distinguish three cases

$\boldsymbol{L < \sqrt{3}(2GM)}$: the potential has no real minimum or maximum and fall straight to $r = 0$. An observer falling radially towards $r = 0$ will reach the Schwarzschild radius and the $r = 0$ point in a finite proper time, as we will see later.

$\boldsymbol{L > \sqrt{3}(2GM)}$: the potential has a real maximum and minimum. There are now three potential cases depending on the initial energy $\frac{1}{2}(E^2 - 1)$ as shown in fig, 9.3:

(a) $\frac{1}{2}(E^2 - 1) > V(r_+)$: the particle sails over the potential barrier and spirals towards $r = 0$.

(b) $V(r_+) > \frac{1}{2}(E^2 - 1) > 0$: the particle bumps into the potential barrier and moves back to infinity.

(c) $0 > \frac{1}{2}(E^2 - 1) > V(r_-)$: the particle is trapped in the potential and follows an elliptical orbit around $r = 0$ with a precessing perihelion.

$\boldsymbol{L = \sqrt{3}(2GM)}$: The minimum and the maximum coincide. There is an unstable point at $r = L^2/2GM$ but any disturbance will result in the particle moving to $r = 0$.

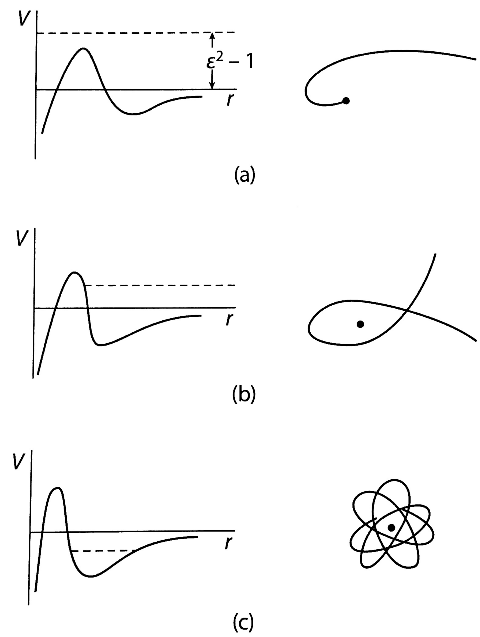


Figure 9.3: Schwarzschild orbits of a massive particle with$L > \sqrt{3}(2GM)$. (a) $\frac{1}{2}(E^2 - 1) > V(r_+)$: the particle falls into the singularity; (b) $V(r_+) > \frac{1}{2}(E^2 - 1) > 0$ the particle bumps into the barrier and is ejected; (c) $0 > \frac{1}{2}(E^2 - 1) > V(r_-)$: the particle follows an elliptical orbit with a precessing perihelion.

Elliptical orbits occur at distances (9.18). Expanding this for large L gives

$$r_- = \frac{L^2}{GM} \quad \text{and} \quad r_+ = 3GM \tag{9.20}$$

The former is stable and is farther away for larger L. The latter is unstable. As L is smaller the orbits are closer together and coincide when $L = \sqrt{12}GM$, i.e. at radius $r_c = 6GM$. For smaller L there are no real solutions and hence no stable orbits. Thus $r_c = 6GM$ is the smallest possible stable circular orbit for an object in the Schwarzschild metric. This radius is known as the **Innermost Stable Circular Orbit** or **ISCO**.

Recall that in Newtonian gravity the orbits are conic sections (hyperbola, parabola, ellipse, circle). This is not the case in general relativity. But interestingly it turns out that Kepler's third law that the squares of the orbital periods of the planets are directly proportional to the cubes of the semi-major axes of their orbits remains valid.

Massless Particles

For a massless particle, the potential is

$$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3} = \frac{L^2}{2r^2}\left(1 - \frac{2GM}{r}\right) \tag{9.21}$$

Setting $dV/dr = 0$ we find only one extremum at $r_c = 3GM$, unless $L = 0$ in which case $V = 0$. We see that for $L \neq 0$ there is always a barrier to surmount. There are three cases

(a) If the photon does not have sufficient energy it will continue on, after being deflected.

(b) If the photon has sufficient energy, it will go over the barrier and be dragged towards the center $r = 0$.

(c) Circular orbits for massless particle occur at $r_c = 3GM$.

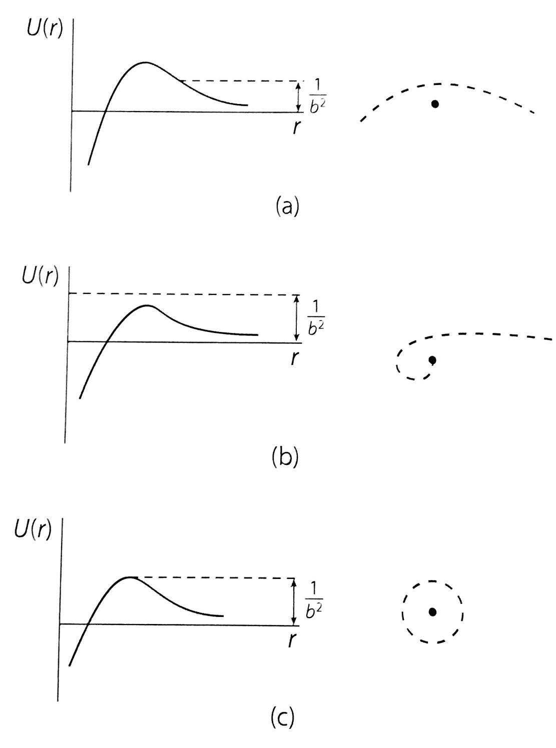


Figure 9.4: Schwarzschild orbits of a massless particle

9.3 The Precession of Perihelia

The **Perihelion** is the point of a planet closest to the sun. In general relativity, the perihelion is not fixed, but precesses.

We use the geodesic (9.13) for a massive particle and divide it by the square of the second equation of (9.10) to get an equation for $r(\phi)$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GMr^3}{L^2} + r^2 - 2GMr = \frac{2\mathcal{E}r^4}{L^2} \tag{9.22}$$

Rename $x = L^2/GMr$ and differentiate w.r.t. ϕ to get

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2}x^2 \tag{9.23}$$

We expand $x = x_0 + x_1$, where x_0 is the Newtonian circular orbit, which we know satisfies $d^2x_0/d\phi^2 = 1 - x_0$ and has solution $x_0 = 1 + e\cos\phi$ witth e the eccentricity of the ellipse. Using this we find that x_1 satisfies, using the fact that x_1^2 is a higher order correction and that $6G^2M^2x_0^2/L^2 \ll 1$

$$\begin{aligned}\frac{d^2x_1}{d\phi^2} + x_1 &= \frac{3G^2M^2x_0^2}{L^2} = \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2 \\ &= \frac{3G^2M^2}{L^2}\left(1 + \frac{1}{2}e^2 + \frac{1}{2}e^2\cos 2\phi + 2e\cos\phi\right)\end{aligned} \tag{9.24}$$

One easily checks that a solution is given by

$$x_1 = \frac{3G^2M^2}{L^2}\left[\left(1 + \frac{1}{2}e^2\right) + e\phi\,\sin\phi - \frac{1}{6}e^2\cos 2\phi\right] \tag{9.25}$$

The figure below shows the precession of the orbits for different values of L and e.

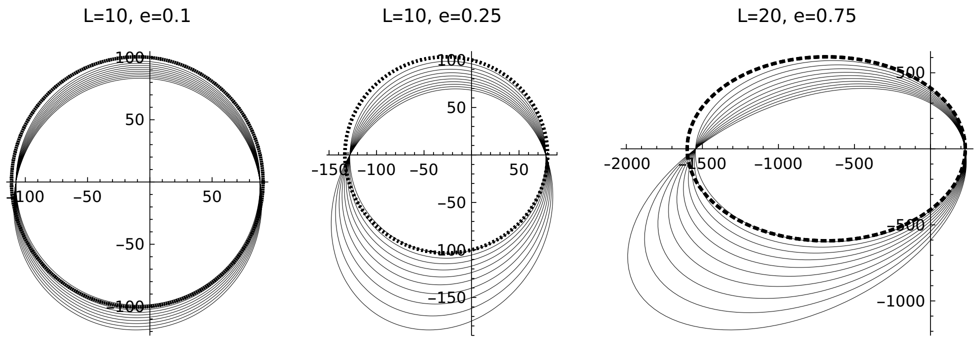


Figure 9.5: Perihelion precession for different values of L and e, with $GM = 1$. The dotted line is the Newtonian orbit.

The first term (9.25) is a constant displacement and the third term oscillates around zero. But the second terms makes the correction increase with the successive orbits. We will consider the simplified solution

$$x_S = 1 + e\cos\phi + \frac{3G^2M^2e}{L^2}\phi\sin\phi = 1 + e\cos[(1-\alpha)\phi] \quad \text{with} \quad \alpha = \frac{3G^2M^2}{L^2} \tag{9.26}$$

where we have used a Taylor expansion for the second equality. This contains all the contributions we are interested in.

This implies that at each successful orbit the perihelion advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6G^2M^2}{L^2} \tag{9.27}$$

The angular momentum L can be related to parameters of Newtonian gravity as corrections would be of higher order. This was successfully confirmed by the precession of Mercury and provided one of the very first confirmations of general relativity.[3]

9.4 Gravitational Redshift

Consider an observer who is stationary in Schwarzschild metric and thus has four velocity $(U^0, 0, 0, 0)$. From $g_{\mu\nu}U^\mu U^s\nu = -1$ we find that $U^0 = (1 - 2GM/r))^{-1/2}$. From (4.22) we

[3]As always, real life is a bit more complicated. The theoretical prediction turns out to be 43"/century and the experimental value is 5601"/century. But the latter comprises Newtonian effects from the precession of equinoxes in our geocentric coordinate system and gravitational perturbations form other planets. These still gave a gap of 43"/century which is precisely the precession due to general relativity.

know that this observer will measure the energy, and hence frequency of a photon with momentum p^μ as $-g_{\mu\nu}p^\mu U^\nu$ and thus, using the definition of E in (9.10)

$$\omega = \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{dt}{d\lambda} = \left(1 - \frac{2GM}{r}\right)^{-1/2} E \tag{9.28}$$

As E is a conserved quantity, the frequency will take different values at different radial distances. A photon emitted at r_1 and observed at r_o will have frequencies related as

$$\frac{\omega_o}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_o}\right)^{1/2} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_o} = 1 + \Phi_1 - \Phi_o \tag{9.29}$$

where the approximation is valid for $r \ll 2GM$ and Φ_i is the Newtonian potential.

As we climb out of a potential field, Φ_o increases and the frequency decreases: this is gravitational redshift. Photons that fall into a gravitational fields are blueshifted.

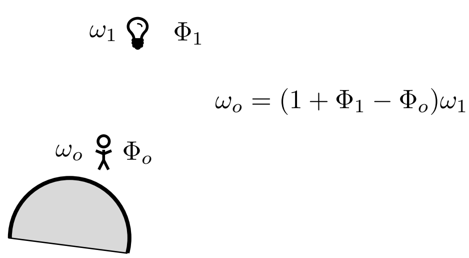


Figure 9.6: Gravitational redshift

9.5 Birkhoff's Theorem

We prove Birkhoff's theorem in four dimension by explicit calculation. I.e. we want to show that the most general spherical symmetric metric in vacuum is the Schwarzschild solution. We can write the most general spherical symmetric solution as

$$ds^2 = -U(t,r)dt^2 - 2V(t,r)dt\,dr + W(t,r)dr^2 + X^2(t,r)d\Omega^2 \tag{9.30}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Here U, V, W and X are functions of t and r that we will determine by solving the EFE in vacuum.

First we show we can get rid of the function X. Define $\tilde{r} = X(t,r)$ so that $d\tilde{r} = \partial_t X dt + \partial_r X dr$. The angular part of the metric is now simply $\tilde{r}^2 d\Omega^2$ and the other terms receive an additional contribution depending on X. Their form is not important, but the fact is that we can now write the metric as

$$ds^2 = -U(t,r)dt^2 - 2V(t,r)dt\,dr + W(t,r)dr^2 + r^2 d\Omega^2 \tag{9.31}$$

where we have deleted the ˜ on r and U, V and W are functions different form the original ones.

In order to get rid of the cross term, define $d\tilde{t} = \zeta(t,r)U(dt + V dr)$ and define a function $\Phi(t,r)$ that satisfies $\partial_t\Phi = \zeta U$ and $\partial_r\Phi = \zeta V$, so that $d\tilde{t} = \partial_t\Phi dt + \partial_r\Phi dr$ is a total differential. From $\partial_t\partial_r\Phi = \partial_r\partial_t\Phi$ we get $\partial_t(\zeta V) = \partial_r(\zeta U)$ which can be solved for ζ. This shows that such a redefinition to $\tilde{t}$ is always possible. Again the detailed form need not worry us. Now insert $dt = \zeta^{-1}U^{-1}d\tilde{t} - U^{-1}V dr$ in $U(t,r)dt^2 + 2V(t,r)dt\,dr$ and get

$$U\left(\zeta^{-1}U^{-1}d\tilde{t} - U^{-1}V dr\right)^2 + 2V\left(\zeta^{-1}U^{-1}d\tilde{t} - U^{-1}V dr\right)dr \tag{9.32}$$

The cross term is

$$\left(-2U\zeta^{-1}U^{-1}U^{-1}V + 2V\zeta^{-1}U^{-1}\right)d\tilde{t}\,dr = 0 \tag{9.33}$$

and thus disappears.

We have thus shown that the most general spherical symmetric metric in four dimensions is of the form

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2 \tag{9.34}$$

for some functions A and B. Note that so far we have only used symmetry arguments to obtain this form. We will now use the EFE to determine these functions. The Ricci tensor is found to be

$$\begin{aligned}
R_{tt} &= \frac{\partial_r A}{rB} + \frac{\partial_r^2 A}{2B} - \frac{\partial_r A}{4B}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}\right) - \frac{\partial_t^2 B}{2B} + \frac{\partial_t B}{4B}\left(\frac{\partial_t A}{A} + \frac{\partial_t B}{B}\right) \\
R_{tr} &= \frac{\partial_t B}{rB} \\
R_{rr} &= \frac{\partial_r B}{rB} - \frac{\partial_r^2 A}{2A} + \frac{\partial_r A}{4A}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}\right) + \frac{\partial_t^2 B}{2A} - \frac{\partial_t B}{4A}\left(\frac{\partial_t A}{A} + \frac{\partial_t B}{B}\right) \\
R_{\theta\theta} &= 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{\partial_r A}{A} - \frac{\partial_r B}{B}\right) \\
R_{\varphi\varphi} &= \sin^2\theta\, R_{\theta\theta}
\end{aligned} \tag{9.35}$$

For completeness we also show the curvature scalar.

$$R = \frac{\partial_r A \partial_r B}{2AB^2} - \frac{\partial_t A \partial_t B}{2A^2 B} + \frac{(\partial_r A)^2}{2A^2 B} - \frac{2\partial_r A}{rAB}$$
$$- \frac{\partial_r^2 A}{AB} - \frac{(\partial_t B)^2}{2AB^2} + \frac{\partial_t^2 B}{AB} + \frac{2\partial_r B}{rB^2} - \frac{2}{r^2 B} + \frac{2}{r^2} \quad (9.36)$$

The EFE $R_{tr} = 0$ implies that

$$\partial_t B = 0 \quad (9.37)$$

Plugging this into $R_{tt} = 0, R_{rr} = 0$ and $R_{\theta\theta} = 0$ we get

$$0 = \frac{\partial_r A}{rB} + \frac{\partial_r^2 A}{2B} - \frac{\partial_r A}{4B}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}\right)$$
$$0 = \frac{\partial_r B}{rB} - \frac{\partial_r^2 A}{2A} + \frac{\partial_r A}{4A}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}\right)$$
$$0 = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{\partial_r A}{A} - \frac{\partial_r B}{B}\right)$$
$$(9.38)$$

From the last equation we get

$$\partial_r \ln A = \frac{2(B-1)}{r} + \partial_r \ln B \quad (9.39)$$

As B is independent of t so is the RHS and hence $\partial r \ln A$ is a function of r only, call it $h(r)$. We can integrate this as $\ln A = \int h(r)dr + g(t)$ where g is an arbitrary function of t. Thus A is the product of a function of r and of a function of t, say $A(r,t) = a(r)f^2(t)$. But we can then always redefine $d\tilde{t} = f(t)dt$ and this will get rid of the $f(t)$ in the metric. We thus conclude that the most general spherical symmetric solution to the EFE in vacuum is time independent and of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \quad (9.40)$$

In order to find A and B take the combination $R_{tt}/A + R_{rr}/B = 0$ which eliminates the second derivative and gives

$$\frac{1}{rB}\left(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}\right) = 0 \quad (9.41)$$

which implies that $\partial_r \ln AB = 0$ or $AB = k$ for some constant k. Plugging this into $R_{\theta\theta} = 0$ gives

$$0 = 1 - \frac{A}{k} - \frac{rA}{2k} 2\frac{\partial_r A}{A} \quad (9.42)$$

or $k = r\partial_r A + A$. This is solved by $A = k + \alpha/r$ and thus the metric becomes

$$ds^2 = -\left(k + \frac{\alpha}{r}\right) dt^2 + k\left(k + \frac{\alpha}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{9.43}$$

Requiring that this becomes the Minkowski metric at $r \to \infty$ fixes $k = 1$ and requiring that this reduces to Newtonian gravity in the weak limit sets $\alpha = -2GM$ so that finally

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{9.44}$$

This proves the Birkhoff theorem in four dimensions: Schwarzschild metric is the only solution to the EFE equations that is spherically symmetric. Note that we did not have to require the metric to be static, it is a consequences of the EFE and the symmetry.

Chapter 10

The Schwarzschild Black Hole

10.1 To the Boundary or not to the Boundary? That is the Question.

Consider radial lightlike curves in a Schwarzschild metric, ignoring the angular part,

$$0 = ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \tag{10.1}$$

Thus

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \tag{10.2}$$

As $r \to \infty$ we have $dt/dr \to \pm 1$ and the lightcones are at a $45°$ angle. As r becomes smaller dt/dr becomes smaller and the lightcones "close up". At the **Schwarzschild Radius** $r = 2GM$, we have $dt/dr = 0$ and the lightcone is completely flattened, see the figure below.

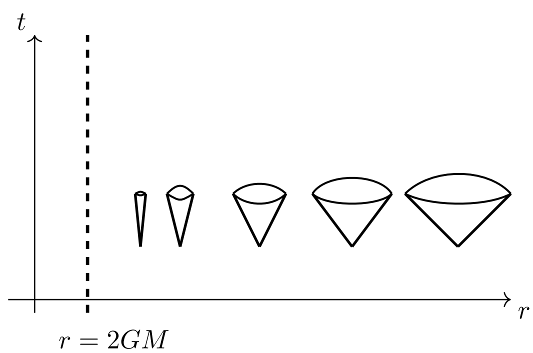


Figure 10.1: Lightcones in the Schwarzschild metric close up as $r \to 2GM$. It seems like particles, massive or massless, are never able to reach that radius.

One would think that particles, massive or massless, are never able to reach the radius $r = 2GM$ as they have to stay in their lightcone. This is however a direct result of the choice of coordinate system. Equal time intervals for a traveller going towards a the Schwarzschild radius are perceived by an outside observer to increase as the traveller comes nearer to that radius. It turns out that the traveller will not notice the boundary at all and can travel straight through it. It is just the outside observer that will never see this happen.

10.2 The Event Horizon

But there is still something special going on. This is best seen by change coordinate system from (t, r) to (v, r) where

$$v = t + r^* \qquad \text{with} \qquad r^* = r + 2GM \, \ln\left(\frac{r}{2GM} - 1\right) \tag{10.3}$$

r^* is known as the **Tortoise Coordinate** and the (v, r) are the **Eddington-Finkelstein Coordinates**. The radius $r = 2GM$ has now moved to $r^* = -\infty$. The metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dv\, dr + r^2 d\Omega^2 \tag{10.4}$$

Note that the metric has no singularities at $r = 2GM$ and the determinant of the metric is well behaved as well, $g = -r^4 \sin^2\theta$. Whilst the tortoise coordinate is only defined for $r > 2GM$ we can analytically extend the Eddington-Finkelstein coordinates to $r \leq 2GM$.

We can now find how the lightcones change when we approach $r = 2GM$. First note that the solution to the lightlike equation (10.2) is simply $t = \pm r^* + c^{\text{te}}$.[1] We now determine the lightcones in the Eddington-Finkelstein coordinates. For the plus sign we have $dt = +dr^*$ or hence $dv - dr^* = dr^*$ and thus $dv = 2dr^* = 2(1 - 2GM/r)^{-1} dr$. For the minus sign we have $dt = -dr^*$ or hence $dv - dr^* = -dr^*$ and thus $dv = 0$. Thus the radial null curves are

[1] Indeed, we have

$$dr^* = dr + \frac{2GM}{r/2GM - 1} \frac{dr}{2GM} = \left(1 - \frac{2GM}{r}\right)^{-1} dr$$

and thus

$$dt = \pm dr^* = \pm\left(1 - \frac{2GM}{r}\right)^{-1} dr$$

which is the equation for lightlike paths.

given by

$$\frac{dv}{dr} = 0 \qquad \text{and} \qquad \frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \tag{10.5}$$

The former corresponds to incoming paths, the latter to outgoing paths. In the (r, v) plane one boundary of the lightcone is at $dv = 0$, i.e. $v = c^{te}$ and the other one starts at a slope $dv/dr = 2$ at infinity, tilts and becomes infinity at $r = 2GM$ but then continues to tilt inwards so that for $r < 2GM$ all particles, massive or massless will remain in the $r < 2GM$ region as per the fig. 10.2.

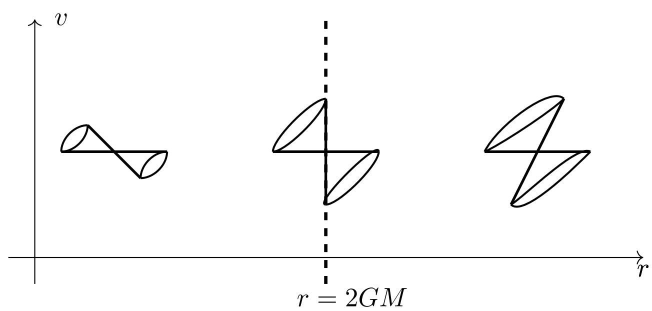


Figure 10.2: Lightcones in the Schwarzschild metric using Eddington-Finkelstein coordinates (r, v), I. The lightcones tilt inwards in $r < 2GM$ and no particles can escape to $r > 2GM$ from there.

The surface $r = 2GM$ is now locally perfectly regular, but it gives a global boundary such that once a test particle, massive or massless, crosses it, it cannot go back. Such a surface is called an **Event Horizon**, a surface past which particles can never escape to infinity. The region within the event horizon is called a **Black Hole** as nothing can escape from it.[2]

Contrary to popular belief and science fiction a black hole does not suck in all matter. The exterior of a black hole is just the same Schwarzschild metric as outside of a star or a planet. It is just that if you move beyond a certain point, there is no return.

Note that had we taken a coordinate $u = t - r^*$ in stead of v the metric would have become

$$ds^2 = \left(1 - \frac{2GM}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2 \tag{10.6}$$

[2]We will see that this needs to be qualified in the quantum theory.

and the lightcones would tilt as in fig. 10.3

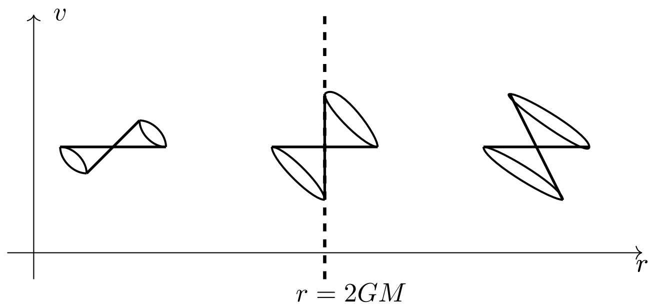


Figure 10.3: Lightcones in the Schwarzschild metric using Eddington-Finkelstein coordinates (r, u), II. The lightcones tilt outwards in $R < 2GM$ and we can follow past directed paths to this region.

What happened here? The tortoise coordinate r^* is only defined for the region $r > 2GM$. By using the coordinates (r, v) we have been able to follow a particle into the future in the region $r < 2GM$ and by using the coordinates (r, u) we have been able to follow a particle from the past in the region $r < 2GM$. So we have been able to extend spacetime into past and future in the region $r < 2GM$. Are there more region to discover?

10.3 Moving through the Schwarzschild Radius

We consider a massive test particle that moves radially from a point at distance R from the singularity towards the singularity. We set the proper time τ and the time measured by an outside observer t to be zero when the particle starts from rest.

We only consider radial movement, so that (9.10) tells us that $L = 0$. We can then write (9.12)

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} - 1 + E^2 \tag{10.7}$$

As the particle is massive, we have used the proper time as the parameter and set $\epsilon = +1$. The first equation of (9.10) becomses

$$\frac{dt}{d\tau} = \frac{E}{1 - 2GM/r} \tag{10.8}$$

The particle starts from rest so we have $dr/d\tau = 0$ at $r = R$ and thus

$$R = \frac{2GM}{1 - E^2} \qquad \Rightarrow \qquad E = 1 - \frac{2GM}{R} \tag{10.9}$$

In order to solve the equations, we introduce a parameter η defined by

$$r = \frac{GM}{1 - E^2}(1 + \cos\eta) = R\cos^2\eta/2 \tag{10.10}$$

Note that at the horizon $r_H = 2GM$ we have $2GM = GM/(1 - E^2)\cos^2\eta_H$ from which we get

$$\eta_H = 2\arcsin E \tag{10.11}$$

At the singularity we have $r = 0$ and hence $\eta_S = \pi$, whereas at the initial point $r = R$ we have $\eta_i = 0$.

Straightforward algebra allows us to write (10.7) in terms of η as

$$\left(\frac{dr}{d\tau}\right)^2 = (1 - E^2)\tan^2\eta/2 \tag{10.12}$$

and (10.8) as

$$\frac{dt}{d\tau} = \frac{E\cos^2\eta/2}{\cos^2\eta/2 - \cos^2\eta_H/2} \tag{10.13}$$

From the definition of η, we also have

$$\frac{dr}{d\eta} = -R\cos\eta/2\,\sin\eta/2 \tag{10.14}$$

Because we are considering an in-falling test particle we have $dr/d\tau < 0$ and (10.12) becomes

$$\frac{dr}{d\tau} = -\sqrt{1 - E^2}\tan\eta/2 = -\sqrt{\frac{2GM}{R}}\tan\eta/2 \tag{10.15}$$

and thus

$$\frac{d\tau}{d\eta} = \frac{d\tau}{dr}\frac{dr}{d\eta} = R\sqrt{\frac{R}{2GM}}\cos^2\eta/2 = \frac{R}{2}\sqrt{\frac{R}{2GM}}(1 + \cos\eta) \tag{10.16}$$

which we can integrate as

$$\tau = \frac{R}{2}\sqrt{\frac{R}{2GM}}(\eta + \sin\eta) \tag{10.17}$$

which satisfies the boundary condition that $\tau = 0$ at $\eta = 0$.

Similarly we have

$$\frac{dt}{d\eta} = \frac{dt}{d\tau}\frac{d\tau}{d\eta} = E\sqrt{\frac{R}{2GM}}\frac{\cos^4\eta/2}{\cos^2\eta/2 - \cos^2\eta_H/2} \tag{10.18}$$

which solves as

$$t = E\sqrt{\frac{R}{2GM}}\left[\frac{1}{2}(\eta + \sin\eta) + (1 - E^2)\eta\right] + 2GM\log\frac{\tan\eta_H/2 + \tan\eta/2}{\tan\eta_H/2 - \tan\eta/2} \tag{10.19}$$

Finally

$$\frac{dr}{d\eta} = \frac{dr}{d\tau}\frac{d\tau}{d\eta} = -R\sin\eta/2\,\cos\eta/2 = -\frac{R}{2}\sin\eta \tag{10.20}$$

which solves as

$$r = \frac{R}{2}(1 + \cos\eta) \tag{10.21}$$

Eqs (10.17), (10.19) and (10.21) are what we need to analyse the trajectory. Fig.10.4 shows the trajectory for $GM = 1$ and $R = 10$.

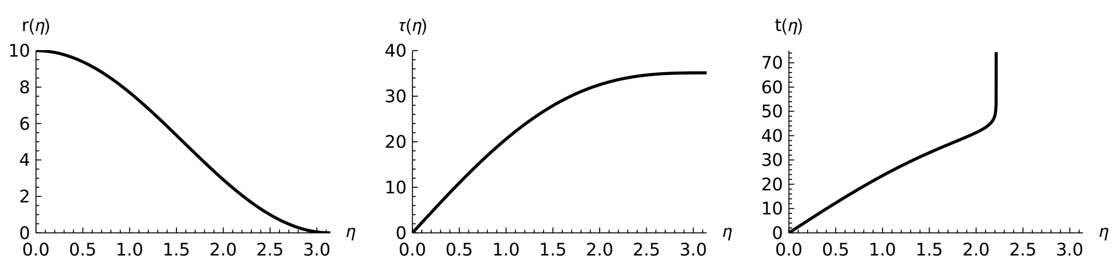


Figure 10.4: Trajectory for a massive test particle radially moving towards the singularity of a Schwarzschild black hole, I

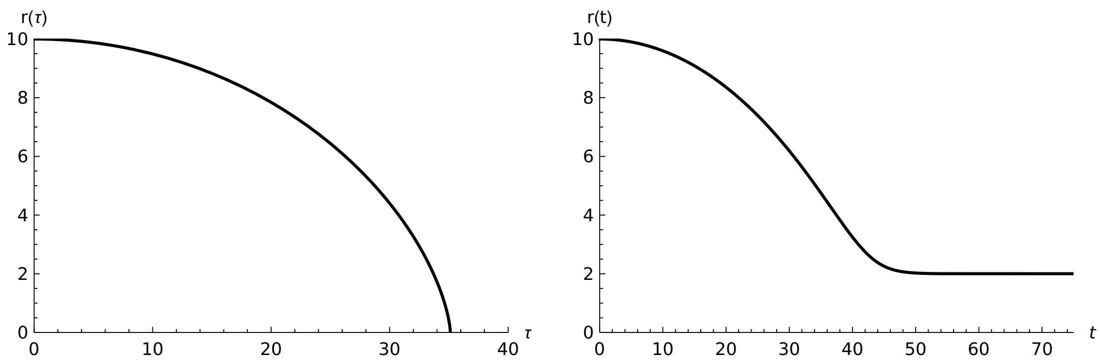


Figure 10.5: Trajectory for a massive test particle radially moving towards the singularity of a Schwarzschild black hole, II

There are three special points:

- $\eta = 0$ corresponds to the initial point with $r = R = 10$ and $t = \tau = 0$.
- $\eta = \pi$ corresponds to the singularity. It is reached in a finite proper time

$$\tau_S = \frac{\pi R}{2}\sqrt{\frac{R}{2GM}} \tag{10.22}$$

in our example this is $\tau_S = 5\sqrt{5}\pi \approx 35.12$. But the singularity is never reached for the external observer: as $\eta \to \eta_H$ we see that the observer time goes to infinity. Fig. 10.5 shows that in that case r approaches the horizon, $r = 2GM$

- $\eta = \eta_H$ with

$$\eta_H = 2\arcsin\sqrt{1 - \frac{2GM}{r}}\Big|_{r=2GM} = \infty \tag{10.23}$$

It takes an infinite amount of time for the the external observer to see the test particle reach that point. For the test particle itself it takes only a finite amount of proper time

$$\tau_H = \frac{R}{2\sqrt{2}}\sqrt{\frac{R}{2GM}}\left[2\arcsin\sqrt{1-\frac{2GM}{R}} + \sin\left(2\arcsin\sqrt{1-\frac{2GM}{R}}\right)\right] \tag{10.24}$$

in our example this is $\tau_H \approx 33.70$.

Let us assume that the test particle is a space traveller and she is moving towards the singularity for research purposes and is asked to send us as much information as possible during his travel. Let us first consider what happens when she crosses the event horizon. In fact, she will notice nothing in particular when this happens. This is just a smooth transition that will happen at her proper time τ_H. But we already know that once she has crossed the event horizon, she, or any information she would like to send cannot go beyond the horizon. So she will notice that any question she may ask an observer outside the event horizon will remain unanswered, but that she can still receive information for that observer.

But what happens a bit earlier as she reaches the event horizon? Let us assume that she sends a signal to the external observer with a fixed frequency, i.e. every $\Delta\tau$ seconds. For the external observer, these will arrive not with constant frequency, but with longer and longer time spans between them as illustrated in fig. 10.6

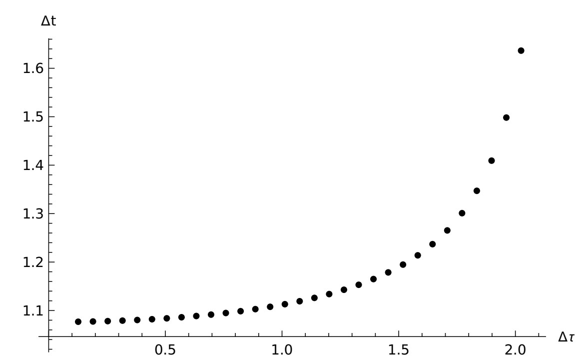


Figure 10.6: Trajectory for a massive test particle radially moving towards the singularity of a Schwarzschild black hole, III

In fact, it will take an infinite amount of time for the external observer to see her reach the event horizon, so he will never receive the signal sent at the exact point of crossing, and the signals received before crossing will be extremely wide spread. The space traveller can provide more information by increasing the frequency of the signal as she approaches the event horizon. But sending signal requires al least one bit of energy, and so the higher the frequency of her signals the higher energy she needs. This energy must be with her somewhere from the start and if it is high enough it will have an impact on spacetime itself, altering its curvature and invalidating the assumption that we are working with a Schwarzschild metric.

This is a theoretical approach and will not happen. In reality, tidal forces are likely to rip her apart before she reaches that point.

10.4 The Kruskal Coordinates

First we go from the coordinates (r, t) to (u, v). The metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv + r^2 d\Omega^2 \tag{10.25}$$

where r is defined implicitly via

$$v - u = 2r^* = 2r + 4GM \ln\left(\frac{r}{2GM} - 1\right) \tag{10.26}$$

It turns out that the horizon $r = 2GM$ is still at an infinite distance, i.e at $v = -\infty$ or $u = +\infty$, so these coordinates are not useful yet. We thus bring them to finite values using

$$\tilde{u} = +e^{+v/4GM} \qquad \text{and} \qquad \tilde{u} = -e^{-u/4GM} \tag{10.27}$$

Next we move to coordinates

$$\begin{aligned} T &= \frac{1}{2}(\tilde{v} + \tilde{u}) = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh \frac{t}{4GM} \\ R &= \frac{1}{2}(\tilde{v} - \tilde{u}) = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh \frac{t}{4GM} \end{aligned} \tag{10.28}$$

which we have expressed in terms of the (r, t) coordinates as well. The metric is now

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2 \tag{10.29}$$

with now r implicitly defined from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \tag{10.30}$$

The coordinates (T, R, θ, ϕ) are called **Kruskal Coordinates** or **Kruskal-Szekeres Coordinates**. The allowed values of the (R, T) coordinates are

$$R \in \mathbb{R} \qquad \text{and} \qquad T^2 < R^2 + 1 \tag{10.31}$$

The latter condition comes from (10.30) and the realisation that the function $(1-x)e^x$ has a maximum at one.

From the metric we see that the radial lightlike curves have $dT^2 = dR^2$ and thus $T = \pm R + c^{\text{te}}$: lightcones have $45°$ angles. From (10.30) we also see that $r = c^{\text{te}}$ curves are hyperbolae $T^2 - R^2 = c^{\text{te}}$ and from the fact that $T/R = \tanh(t/4GM)$ curves of constant t are straight lines with slope $\tanh(t/4GM)$.

The fact that R and T seem to become imaginary for $r < 2GM$ should not worry us. We only relate them to (r,t) for $r > 2GM$ but continue them for all values of (R,T) satisfying (10.31). All of this is summarised in the **Kruskal Diagram** in fig. 10.7.

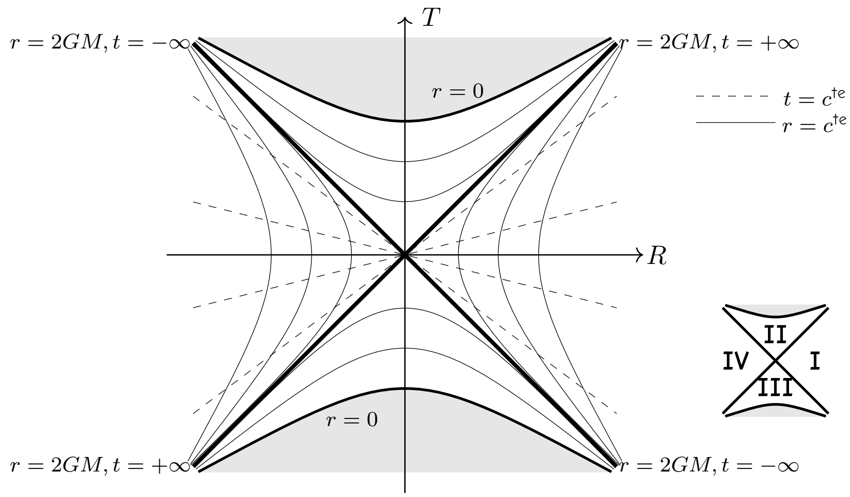


Figure 10.7: The Kruskal diagram for the Schwarzschild metric

Each point on the Kruskal diagram has a two-sphere. The Schwarzschild spacetime ends at the $r = 0$ boundary, so the grey areas do not form part of it. The thick black diagonals are the event horizon. Lines of constant t are straight dashed lines through the origin. Lines of constant r are the hyperbolae. Recall that lightcones are at $45°$ angles in the Kruskal diagram. We also identify four different regions

I. This is the spacetime region outside of the event horizon $r > 2GM$, the patch where we defined the original (r,t) coordinates.

II. This is the region within the event horizon $r < 2GM$ that we would reach if we would follow future directed paths from region I. Note that because lightcones are at

$45°$ angles, once we have crossed into region II all paths automatically lead to $r = 0$ i.e. to the singularity. Remark that this is not the case when you are in region I.

III. This is the region within the event horizon $r < 2GM$ that we would reach if we could follow past directed paths from region I. Whilst things can escape from this region to us in region I, nothing from our region can get into there. It is therefore known as a **White Hole**.

IV. This region could be reached from I if we could follow spacelike geodesics. It is theoretically possible to connect it with region I via a **Wormhole** or a so-called **Einstein-Rosen Bridge**.

Note that the Kruskal coordinates (10.32) require $r > 2GM$ for the square root to be real. Hence these coordinates are only valid outside the event horizon, and so let us denote them by T_o and R_o. Inside the event horizon we can define them as

$$\begin{aligned} T_i &= \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \cosh \frac{t}{4GM} \\ R_i &= \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \sinh \frac{t}{4GM} \end{aligned} \tag{10.32}$$

At $R = 2GM$ we have $T_o = R_o = T_i = R_i = 0$

10.5 The Conformal Diagram of the Schwarzschild Spacetime

The conformal diagram is obtained by transforming the $(\tilde{u}, \tilde{v})$ coordinates in (10.27) to

$$U = \arctan \frac{\tilde{U}}{\sqrt{2GM}} \qquad \text{and} \qquad V = \arctan \frac{\tilde{v}}{\sqrt{2GM}} \tag{10.33}$$

with ranges

$$U, V, U + V \in \left] -\frac{\pi}{2}, +\frac{\pi}{2} \right[\tag{10.34}$$

and is

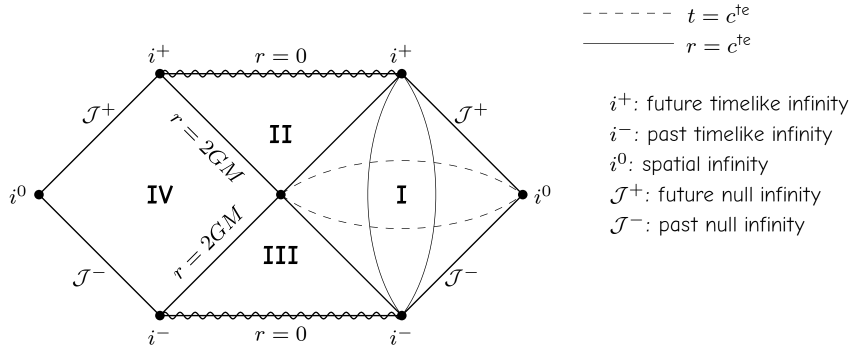


Figure 10.8: Conformal diagram for the Schwarzschild spacetime

The $r = 0$ singularities are straight lines stretching from timelike infinities from region I to region IV. Note that the future and past timelike infinities i^+ and i^- are different from $r = 0$. Indeed there are many timelike geodesics that do not end in $r = 0$. Lightcones here are again at $45°$.

Chapter 11

Black Holes from Stars

11.1 The Schwarzschild Black Hole from Star Collapse

The Schwarzschild radius is typically much smaller than the radius of a star, so it is irrelevant as within a star there is no vacuum and no Schwarzschild metric. But if a star shrinks by gravitational collapse and lies within the Schwarzschild radius then this becomes relevant and the star has become a black hole. The conformal diagram of a collapsing star is given below.

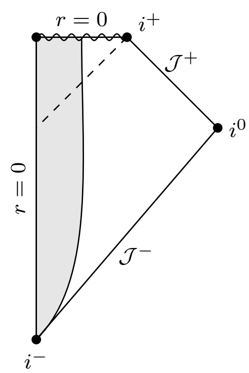


Figure 11.1: Conformal diagram for a black hole from a collapsing star, I. The dashed line is the event horizon.

The shaded area is not vacuum; it is the interior of the star and bounded by the radius of the star $r = r_{\text{star}}$. A massive particle will start from the far past in $i^-$and will follow a timelike geodesic. It may either end at future timelike infinity i^+, or if it crosses the event horizon, represented by the dashed line, it will end in the singularity $r = 0$.

Sewing Spacetimes together

We can understand this conformal diagram by sewing together conformal diagrams of Minkowski and Schwarzschild spacetime. Let us assume that we have a thin spherical shell of photons in Minkowski spacetime moving radially towards the origin. We can represent this by the following diagram, where the spherical shell of photons is shown as a double line. The double line is at $45°$ and so represents a photon propagating. It starts at a past null infinity $\mathcal{J}^-$ and propagates towards the origin, which is the vertical line in the Minkowski conformal diagram.

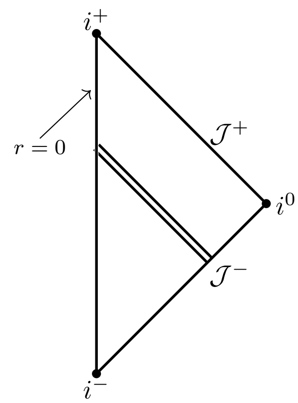


Figure 11.2: Spherical shell of photons in Minkowski spacetime

Suppose now that the spherical shell of photons has sufficient energy to form a black hole. We then need to use the conformal diagram for a spherically symmetric solution around of a black hole as shown below. Here again the double line is the photons propagating at a $45°$ angle, crossing the event horizon ($r = 2GM$) and finally hitting the singularity (the squiggly line $r = 0$).

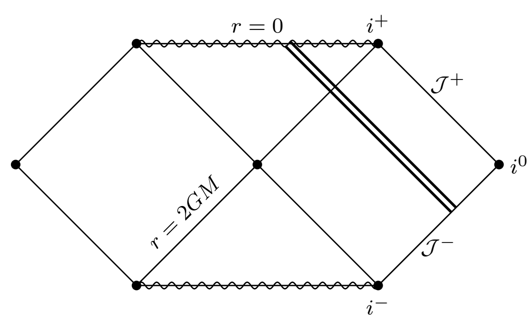


Figure 11.3: Spherical shell of photons in a Schwarzschild spacetime

But this picture is not entirely correct as we are trying to describe the formation of a black hole. Before the shell of photons arrives, spacetime is Minkowski and thus for the region to the left and below of the double line of the Schwarzschild conformal diagram we need to use the Minkowski conformal diagram. And above and to the right of the Minkowski conformal diagram we need to use the Schwarzschild conformal diagram. We thus need to cut and sew the two diagrams with the appropriate parts, which will give something of the form

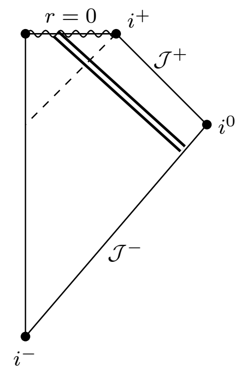


Figure 11.4: Conformal diagram for a black hole from a collapsing star, II

Before the shell arrives we have a Minkowski spacetime and afterwards we have a Schwarzschild spacetime with a black hole. This spacetime diagram of a black hole forming reveals something quite interesting and surprising, as shown in fig. 11.5. Suppose you are living at the origin and the field is so weak that spacetime is Minkowski. Your worldline is then the vertical line at $r = 0$. Assume you are at a point P_1 at a time t_1 on that line. You can take a rocket and blast off and stay within your lightcone always beyond the dashed line. But if you are at a point P_2 at a time t_2 then you will always stay left and above the dashed line.

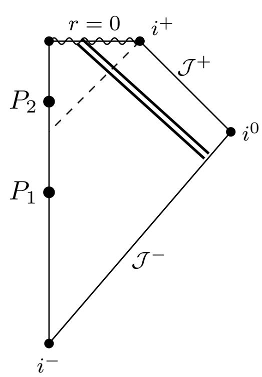


Figure 11.5: Doomed by your blissful ignorance

But the dashed line is the event horizon of the black hole. Thus at time t_1 you may still escape outside the event horizon, but at time t_2 you will stay within the event horizon and as we have seen you will be inevitably drawn to the singularity and get ripped by the tidal forces. This is the case even though at that time you have no idea that the spherical shell of photons is coming at you and you are still enjoying the quiet life of Minkowski spacetime. Indeed, no photons from the spherical shell can have reached you at that time. But you are doomed to be crushed in your blissful ignorance.

11.1 The Tolman-Oppenheimer-Volkoff Equation

In order to understand the gravitational collapse of a star, we need to start with understanding solutions of Einstein's equations in a star. Thus we look for static spherical symmetric solutions with a non-zero energy-momentum tensor. The most general form of such a metric is

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{11.1}$$

One could add a prefactor $e^{2\gamma(r)}$ to the $d\Omega^2$ terms, but by defining $\tilde{r} = e^{\gamma(r)}r$ one can get rid of the exponential in the coefficient of $d\Omega^2$ and so equivalently one can set $\gamma(r) = 0$. Using Mathematica we find for the non-vanishing components of the Einstein tensor

$$\begin{aligned}
G_{tt} &= e^{2(\alpha-\beta)}r^{-2}\left(e^{2\beta} + 2r\beta' - 1\right)\\
G_{rr} &= r^{-2}\left(-e^{2\beta} + 2r\alpha' + 1\right)\\
G_{\theta\theta} &= e^{-2\beta}r\left[r\alpha'' + \alpha'\left(1 - r\beta'\right) + r\alpha'^2 - \beta'\right]\\
G_{\phi\phi} &= e^{-2\beta}r\sin^2\theta\left[r\alpha'' + \alpha'\left(1 - r\beta'\right) + r\alpha'^2 - \beta'\right] = \sin^2\theta\, G_{\theta\theta}
\end{aligned} \tag{11.2}$$

We take the energy-momentum tensor to be that of a perfect fluid $T_{\mu\nu} = (\rho+p)U_\mu U_\nu + pg_{\mu\nu}$ with $p = p(r)$ and $\rho = \rho(r)$. We take the four velocity in the rest frame of the star, $U_\mu = (u, 0, 0, 0)$. From $-1 = g^{\mu\nu}U_\mu U_\nu$ we find $u = e^{\alpha}$. Thus $T_{\mu\nu} = \mathrm{diag}(e^{2\alpha}\rho, e^{2\beta}p, r^2 p, r^2\sin^2\theta\, p)$. There are then three independent Einstein equations

$$8\pi\, G\,\rho = e^{-2\beta}r^{-2}\left(e^{2\beta} + 2r\beta' - 1\right) \tag{11.3}$$

$$8\pi\, G\,p = e^{-2\beta}r^{-2}\left(-e^{2\beta} + 2r\alpha' + 1\right) \tag{11.4}$$

$$8\pi\, G\,p = e^{-2\beta}\left[\alpha'' + +\alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')\right] \tag{11.5}$$

Replace $\beta(r)$ by a new function $m(r)$ defined by

$$e^{2\beta(r)} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1} \tag{11.6}$$

Eq. (11.3) then becomes very simple

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{11.7}$$

which can be integrated as

$$m(r) = 4\pi \int_0^r \rho(s) s^2 \, ds \tag{11.8}$$

If we wish to match the metric at the boundary of the star R_* then we need $m(R_*) = M_*$, with M_* the mass of the star. Thus $M_* = \int_0^{R_*} \rho(r) r^2 dr$ and it looks like $m(r)$ is the integral of the energy density over the interior of the star and $m(r)$ is the mass within radius r.[1] Eq. (11.4) now becomes

$$\frac{d\alpha(r)}{dr} = \frac{Gm(r) + 4\pi G r^3 p}{r[r - 2Gm(r)]} \tag{11.10}$$

Rather than using Eq. (11.5) we use momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. It turns out only the $\nu = r$ equation is nontrivial

$$(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr} \tag{11.11}$$

Eliminating $d\alpha/dr$ from the last two equations gives an equation for hydrostatic equilibrium, the **Tolman-Oppenheimer-Volkoff Equation**

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]} \tag{11.12}$$

Since $m(r)$ is a function of ρ this is an equation that links the energy density and the pressure inside a star. We can now add an equation of state of the firm $p = p(\rho)$ to eliminate one of the variables. A simple choice is the polytropic equation $p = K\rho^\gamma$ for constants K and γ. An even more simple assumption is that the star is incompressible, so has constant energy density ρ_*. This implies that

$$m(r) = \frac{4}{3}\pi r^3 \rho_* \;\text{ for }\; r < R \qquad \text{and} \qquad m(r) = \frac{4}{3}\pi R^3 \rho_* = M \;\text{ for }\; r \leq R \tag{11.13}$$

[1] This is not really correct as the three dimensional volume elements is $e^{2\beta} r^2 \sin\theta dr\, d\theta d\phi$ and the actual integrated energy density is

$$\tilde{M} = 4\pi \int_0^R \rho(r) r^2 e^{\beta(r)} dr = 4\pi \int_0^R \frac{\rho(r) r^2}{\sqrt{1 - 2Gm(r)/r}} dr \tag{11.9}$$

The difference is the binding energy from the mutual gravitational interaction of the fluid components of the star, $E_B = \tilde{M} - M > 0$.

The solution to the hydrostatic equilibrium equation turns out to be

$$p(r) = -\frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{3R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}} \times \rho_* \tag{11.14}$$

Charts for pressure in such a star are shown in fig. 11.6 for fixed mass and varying radius and for fixed radius and varying mass. We see that for $R = 9GM/4$ the pressure at $r = 0$ becomes infinite as the denominator of (11.14) becomes zero. This means that for a given radius there is no static solution when $M \geq 4R/9G$. A star that shrinks to that level must inevitably keep on shrinking, eventually reaching $R < 2G$ and forming a black hole.

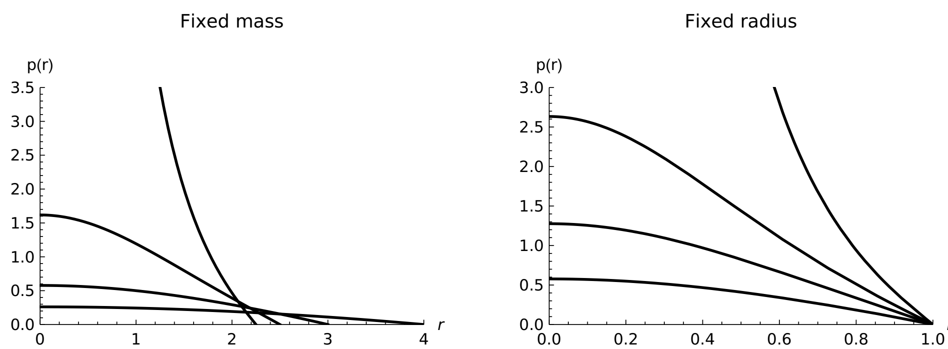


Figure 11.6: Pressure in star of constant energy density. The left chart is for constant Mass, $GM = 1$, $\rho_* = 1$ and $R = 9/4, 10/4, 3$ and 4. The value R for a given curve is where it crosses the x-axis. The right chart is for constant radius $R = 1$ and varying mass, from top to bottom $M = 4/9G, (4-1/4)/9G, (4-1/2)/9G$ and $(4-1)/9G$.

The assumptions for this to be valid are pretty strict, but it turns out they can be considerably weakened and the **Buchdahl's Theorem** states that any reasonable static spherical symmetric solution of the interior of a star must have $M < 4R/9G$.

11.2 General Considerations

Not everything turns into a black hole. Planets are typically too small for gravitational collapse to occur. Stars can be large enough, but the heat of the fusion provides the pressure to balance this. When the fusion stops, gravitational collapse starts. There are then two possible outcomes

- Fermions are pushed so closely together that Fermi degeneracy, i.e. the Pauli exclusion principle, stops any further collapse. The star becomes a **White Dwarf** and this is how the star ends. Heavier fermions need higher densities for the Fermi principle to play a role, so it is mainly electrons that are in play. Most stars end like that and their size is typically like that of the earth.

- If the star is heavier than the **Chandrasekhar Limit** of approx $1.4 M_\odot$ then gravitational pressure can overcome the electron degeneracy and the star shrinks further. Electrons combine with protons to form neutrons and the result is a **Neutron Star**, of radius typically 10 km. These are low luminosity, rapidly spinning objects with strong magnetic fields with jets of particles coming from their magnetic poles. These are **Pulsars**. Heavy neutron stars may collapse further and it is believed that there can be no neutron stars heavier than the **Oppenheimer-Volkoff Limit** of 3-$4 M_\odot$. We know of no denser structure than neutron stars, so it is believed that the end result of this last collapse is a black hole.

These ideas are elaborated in more details in the next section.

Note however that black holes can come in many guises. The most common seem to be from the collapse of heavy stars as described above. There are also supermassive black holes with masses of 10^6 to 10^9 solar masses that are found at the center of galaxies. There are also black holes with masses between those ranges. There is also some speculation about the existence of micro black holes that originated from the very high density just after the big bang.

11.3 The Collapse of a Star: The Chandrasekhar Limit

Consider a star to be a self-gravitating ball of hydrogen atoms supported by thermal equilibrium.[2] The total energy is then $E = E_{\text{grav}} + E_{\text{kin}}$, with $E_{\text{grav}} \sim -GM^2/R$ and $E_{\text{kin}} \sim nR^3 \langle E \rangle$. Here R is the radius of the star, M its mass, n the total number of hydrogen atoms and $\langle E \rangle$ the average kinetic energy of the atoms. The pressure is given by $P \sim nkT$.

When fusion stops the star cools and contracts. However the total pressure does not go to zero due to the Fermi degeneracy; the Pauli exclusion principle dictates that fermions cannot all lump together. The degeneracy occurs first with the lightest fermions, i.e. the electrons; each needs to occupy a cube with side of the order of its Compton wavelength. Thus for the electron density we have $n_e^{-1/3} \sim \hbar / \langle p_e \rangle$ with p_e the average electron momentum.

First assume the electrons are non-relativistic, $\langle E \rangle \sim \langle p_e \rangle^2 / m_e$. Since the degeneracy

[2] For simplicity we will ignore numerical pre-factors and only add them at the end when we need to work out specific numbers.

comes from the electrons we have $n = n_e$ and

$$E_{\text{kin}} \sim n_e R^3 \frac{\langle p_e \rangle^2}{m_e} \sim n_e R^3 \frac{\hbar^2 n_e^{2/3}}{m_e} \sim \frac{n_e^{5/3} \hbar^2 R^3}{m_e} \tag{11.15}$$

The bulk of the mass sits in the protons so that $M \sim n_p R^3 m_p$ and as $n_e = n_p$ we have $n_e \sim M/m_p R^3$. This gives

$$E_{\text{kin}} \sim \frac{\hbar^2}{m_e} \left(\frac{M}{m_p}\right)^{5/3} \frac{1}{R^2} \tag{11.16}$$

and thus the total energy

$$E \sim -\frac{\alpha}{R} + \frac{\beta}{R^2} \tag{11.17}$$

with α and β positive constants for fixed M. The general form of the energy is shown in fig. 11.7

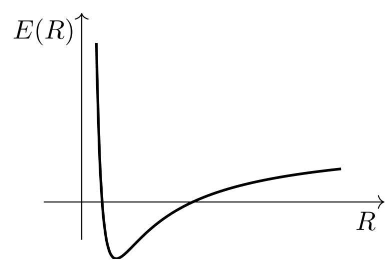


Figure 11.7: Total energy in a star

The energy has a minimum, i.e a stable configuration, when $R_{\text{min}} \sim \beta/\alpha$, i.e.

$$R_{\text{min}} = \frac{\hbar^2}{m_e} \left(\frac{M}{m_p}\right)^{5/3} /GM^2 = \frac{\hbar^2}{M^{1/3} m_e m_p^{5/3}} \tag{11.18}$$

This stable configuration is a **White Dwarf**. At equilibrium, the electron density is

$$n_e \sim \frac{M}{m_p R_{\text{min}}^3} \sim \frac{G^3 M^2 m_e^3 m_p^4}{\hbar^6} \tag{11.19}$$

We assumed that the electrons where non-relativistic. This means that $\langle p_e \rangle \ll m_e c$, i.e. $\langle p_e \rangle / m_e \sim \hbar n_e^{1/3} / m_E \ll c$ or $n_e \ll (m_e c/\hbar)^3$. For a white dwarf at equilibrium (11.19) this means

$$\frac{G^3 M^2 m_e^3 m_p^4}{\hbar^6} \ll \frac{m_e^3 c^3}{\hbar^3} \qquad \Rightarrow \qquad M \ll \frac{1}{m_p^2} \left(\frac{c\hbar}{G}\right)^{3/2} \tag{11.20}$$

If M is larger than the electrons are relativistic and their kinetic energy is then $\langle E\rangle = \langle p_e\rangle c = \hbar c n^{1/3}$. The kinetic energy is then

$$E_{\text{kin}} \sim n_e R^3 \langle E\rangle \sim \hbar c R^3 n_e^{4/3} \sim \hbar c R^3 \left(\frac{M}{m_p R^3}\right)^{4/3} \sim \hbar c \left(\frac{M}{m_p}\right)^{4/3} \frac{1}{R} \tag{11.21}$$

The total energy then becomes

$$E = -\frac{\alpha}{R} + \frac{\tilde{\beta}}{R} \tag{11.22}$$

with α and $\tilde{\beta}$ positive constants for fixed M. Equilibrium is now only possible if $\alpha = \tilde{\beta}$ or

$$M \sim \frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \tag{11.23}$$

If M is smaller than this value that equilibrium can only exist if the radius is larger and the electrons becomes non-relativistic and the star is supported by electron degeneracy pressure. If M is larger than this value, then the radius of the star must continue to decrease as electron degeneracy cannot support the star. This thus defines a critical mass, and corresponding critical radius

$$M_C \sim \frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \quad \text{and} \quad R_C \sim \frac{1}{m_e m_p}\left(\frac{\hbar^3}{Gc}\right)^{1/2} \tag{11.24}$$

above which the radius of the star continues to decrease and a white dwarf cannot exist. This is the **Chandrasekhar Limit**. Re-instating all numerical values gives $M_C \approx 1.4\, M_\odot$.

11.3 Neutron Stars

The energy of the electrons in a white dwarf are of the order of the Fermi energy, i.e. the energy difference between the highest and lowest occupied single particle states in a system of non-interacting fermions. This Fermi energy must be smaller than $m_e c^2$ or the electrons would be relativistic and cannot support the star. This means that there is not sufficient energy in a white dwarf for inverse β decay $e^- + p^+ \to n + \nu_e$ to occur. Indeed this reaction would need an energy of at least $\Delta m_{np} c^2$ with Δm_{np} the mass difference between the neutron and the proton and we know that $\Delta m_{np} \approx 3 m_e > m_e$. So for inverse β decay to occur we need energies of at least $3 m_e c^2$, which is not available in white dwarfs.

But if the mass of the collapsing star is larger than the Chandrasekhar limit then the star will continue to collapse until that energy level is reached. Inverse β decay will then happen and protons and electrons will combine into neutrons and neutrinos, and the latter will just fly out of the star as they react with only extremely weakly with anything. Without neutrinos, there is no reaction of the form $n + \nu_e \to e^- + p$ that can create an equilibrium. Also β decay, i.e. $n \to e^- + p^+ + \bar{\nu}_e$ cannot occur as they would need an energy level $\Delta m_{np} c^2$ for the electron that was created, but all these energy levels are already filled if the Fermi energy, i.e. the level of the highest occupied state, is higher than that. In other words, all protons and electrons convert into neutrons and the star will continue to collapse up to the point where neutron degeneracy pressure kicks in.

We can use the same approximation as for electron degeneracy. The critical mass is independent of the electron mass remains the same, but for the critical radius we need to replace the electron mass by the proton mass

$$R_C \sim \frac{1}{m_p^2} \left(\frac{\hbar^3}{Gc} \right)^{1/2} \sim \frac{GM_C}{c^2} \tag{11.25}$$

which is basically the Schwarzschild radius. This basically means that our approximation of flat spacetime that was implicit in our earlier Newtonian is not valid anymore, and we have to give it a fuller general relativity treatment. In addition treating nuclear matter as an ideal gas is not justified. However one can use a perfect fluid approximation and it then turns out that there is a critical mass below which neutron stars are stable, $M \sim 3 - 4M_\odot$. This is the **Oppenheimer-Volkoff Limit**. More massive stars continue to collapse and as we know of no mechanism to stop it, we can only assume that they become very high density object, eventually black holes,

Chapter 12

Black Holes: General Considerations

12.1 The No-Hair Theorem

The Schwarzschild black hole is just one example of a black hole and illustrates nicely certain properties of black holes. One would naively think that there may be a plethora of different types of black holes, but that is misleading. It turns out that black hole solutions are fully determined by only a small number of parameters. This is encompassed by a so-called **No-Hair Theorem** that says that

> stationary, asymptotically flat black hole solutions of gravity coupled to electromagnetism that are non-singular outside the event horizon are completely characterised by four parameters: mass, angular momentum and electric and magnetic charge.

This deserves some comments.

- The restriction to stationary is not irrelevant as we expect this to be the end-state of gravitational collapse. Non-stationary black holes will emit gravitational waves and have there energy damped until they are stationary.

- We only mention electromagnetism because, as far as we know, this is the only long range force – besides gravity of course – and hence the only force for which this is relevant. For the weak and strong force we would need a quantum theory of gravity and it is possible that the black hole would have different sorts of hair.

- This no-hair theorem is at the crux of the information paradox . We can throw in all the stuff we want in a black hole but we only need a small number of parameters to describe it. Classically we might live with that as an outsider can't know what happens inside the event horizon, but quantum mechanically this causes a paradox as we discuss in more detail in section 17.3 on Hawking radiation.

12.2 Definition of a Black Hole

The most important feature of a black hole is actually not the singularity but the event horizon: the fact that there is a hypersurface beyond which nothing can escape. More formally this means that there is a hypersurface that separates spacetime points that are connected to infinity by a timelike curve from those that are not. By infinity we mean far enough from the black hole in such a way that spacetime can be approximated by Minkowski spacetime, i.e. is asymptotically flat. This means that its conformal diagram future and past null infinity (where light rays start an end) $\mathcal{J}^{\pm}$ and spatial infinity i^0 have the same structure as the conformal diagram of Minkowski spacetime, i.e. it must be of the form

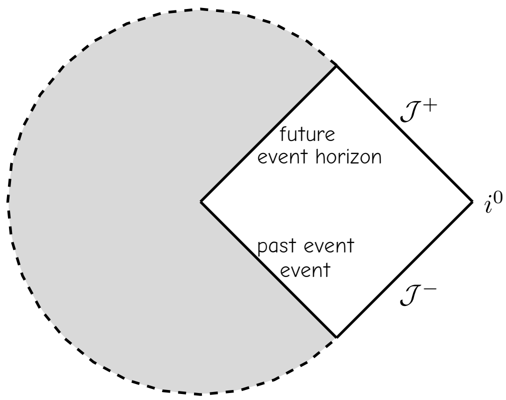


Figure 12.1: Conformal diagram of an asymptotically flat spacetime

Recall from our discussion in section 3.3 that we defined the causal future $J^+(S)$ the subset of a spacetime that can be reached by the causal curves starting in S, where causal curves are worldlines that are either timelike or lightlike. Similarly the causal past $J^-(S)$ is the subset of a spacetime that with causal curves ending in S. The future event horizon can thus be viewed as the boundary of the causal past of future null infinity, i.e. $\partial J^-(\mathcal{J}^+)$. Similarly, the past event horizon is the boundary of the causal future of the past null infinity, $\partial J^+(\mathcal{J}^-)$.

12.3 Event Horizons as Null Hypersurfaces

It follows from this that an event horizon is a **Null Hypersurface**, i.e. a hypersurface Σ defined by $f(x) = 0$ whose normal $\partial_\mu f$ is a null vector, $\partial^\mu f \partial_\mu f = 0$, and hence also a tangent to Σ. This is the case because the boundary is lightlike. Null hypersurfaces are generated by null geodesics $x^\mu(\lambda)$, whose tangent vectors ξ^μ are necessarily a linear combination of the normal vectors, as they are also tangent vectors

$$\xi^\mu = \frac{dx(\lambda)}{d\lambda} = h(x) g^{\mu\nu} \partial_\nu f \tag{12.1}$$

where $g^{\mu\nu}$ appears because it is the only covariant tensor available. If we chose the function $h(x)$ so that the geodesics are affinely parametrised we then have $\xi^2 = 0$ and $\xi \cdot \nabla \xi^\mu = 0$. For a future event horizon, the generators of the null hypersurface may have an end in the past, but they will always continue indefinitely into the future and vice versa for a past event horizon.

12.4 The Singularities of a Black Hole

As we have seen with the Schwarzschild metric a singularity in the metric could very well be a coordinate singularity that disappears by a judicious choice of coordinates. So how do we find the real singularities of a given metric. We could, of course, compute scalars such as the Kretschmann invariant $K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ and many others, until we find a singularity, but this is computationally too hard.

For the metrics we will consider, stationary and asymptotically flat metrics with event horizons that have a spherical topology, there is a neat way to find the singularities. As the metric is stationary there is a Killing vector ∂_t that is asymptotically timelike and we can chose our metric to be independent of t. On a hypersurface $t = c^{\text{te}}$ we can chose coordinates such that the metric at infinity looks like the Minkowski metric in spherical coordinates. Hypersurfaces with $r = c^{\text{te}}$ are as $r \to \infty$ now timelike cylinders with topology $\mathbb{R} \times S^2$, described by coordinates $t \times (\theta, \phi)$.

If our coordinates are such that as we decrease r from infinity the $r = c^{\text{te}}$ hypersurface remains timelike until we come to given value of r_* where the hypersurface is null, then this is necessarily an event horizon since in the region $r < r_*$ timelike paths will not be able to go to infinity, and we have $r_* = r_H$.

In other words we can find event horizons by determining the points at which the $r = c^{\text{te}}$ hypersurface becomes a null hypersurface. But from the definition of a null hypersurface in the previous section, we know that this happens when $(\partial r)^2 = 0$. Writing this out we find $g^{\mu\nu} \partial_\mu r \partial_\nu r = g^{rr}$. Hence we conclude that for our cases of metrics an event horizon is a hypersurface defined by

$$g^{rr}(r_H) = 0 \tag{12.2}$$

which is a much easier check than calculating quantities as the Kretschmann invariant. One readily checks that this indeed works out for the Schwarzschild metric.

12.5 Singularity Theorems, the Cosmic Censorship Conjecture and Naked Singularities

Singularities are nearly unavoidable in general relativity and are generally hidden behind event horizons, although it is possible to find metrics with singularities from which signals can reach null future infinity $\mathcal{J}^+$, and so are not hidden behind event horizons.

Singularity Theorems

There is a whole list of **Singularity Theorems**, initiated by Hawking and Penrose, that basically say that once a star collapses behind a certain point, it will necessarily evolve to a singularity. We can find out that there is a singularity via **Geodesic Incompleteness** which is the concept that there exists some geodesics that cannot be extended within the manifold and end at a finite value of the affine parameter.

Collapse of a star has reach a point of no return if we have a **Trapped Surface**, which is basically a two-dimensional submanifold from which nothing can escape. To illustrate this consider a two-sphere in Minkowski spacetime at a given time. Light rays from this two-sphere can travel outwards or inwards, and will describe a shrinking respectively an expanding sphere of light. But now consider such a two-sphere in of fixed radius $r < GM$ well within the event horizon. Both sets of light rays will evolve to smaller r, i.e. to the singularity, as we have seen, and thus two a smaller two-sphere. That is we have a compact spacelike two-dimensional surface such that all future directed light rays converge in all directions. That is precisely a trapped surface.

With these concepts out of the way we can give an example of a singularity theorem:

Assume that the metric on a manifold satisfies the EFEs,
some generic conditions and the strong energy condition.
If the manifold has a trapped surface then there must
either a closed timelike curve or a singularity.

Before we elaborate a bit on the conditions, let us take a moment to understand what this means. A trapped surface means that no timelike or lightlike geodesics can escape. So any geodesic will either have to return on itself (i.e. it is closed) or it has to end somewhere in the trapped surface, i.e. in a singularity.

Let us now elaborate on the conditions

1. Obviously the metric needs to satisfy the Einstein field equations.

2. The generic conditions that the metric needs to satisfy are merely to ensure that

certain metrics with special properties of the curvature tensor are excluded. They are that (a) on every timelike geodesics with tangent vector U^μ there must be at least one point on which $R_{\mu\nu\rho\sigma}U^\mu U^\sigma \neq 0$ and (b) on every lightlike geodesic with tangent vector k^μ there must be at least one point where $k_{[\mu}R_{\nu]\sigma\rho[\kappa}k_{\xi]}k^\sigma k^\rho \neq 0$

3. The energy-momentum tensor must satisfy the strong energy condition $T_{\mu\nu}\, t^\mu t^\nu \geq \frac{1}{2}T^\mu_\mu t^\nu t_\nu$ for all timelike vectors t^μ. Recall that for a perfect fluid this is equivalent to $\rho + p \geq 0$ and $\rho + 3p \geq 0$.

There are different singularity theorems in general relativity, but they all point out to the fact that singularities are near unavoidable in all but the most time-dependent basic spacetimes. This is a bit of a quandary because the equations of general relativity predict singularities, but the theory is not valid at these singularities. Hopefully a theory of quantum gravity can shed some light on this.

The Cosmic Censorship Conjecture

One fortunate fact is that it may turn out that singularities originating from gravitational collapse in generic asymptotically flat spacetimes obeying the dominant energy condition cannot be naked. Here a **Naked Singularity** is a singularity that is not hidden behind an event horizon, i.e. a singularity from which signals can reach null future infinity $\mathcal{J}^+$. Recall that the dominant energy condition is that $T_{\mu\nu}\, t^\mu t^\nu \geq 0$ for all timelike vectors ℓ^μ and that $T_{\mu\nu} t^\mu$ is non-spacelike, which for a perfect fluid means that $\rho \geq |p|$. This statement is known as the **Cosmic Censorship Conjecture** and remains to be proven. Note that this doesn't say anything about the actual existence of naked singularities, just about their formation. In fact we will see that there is a regime of a charged black hole that can have just such a naked singularity.

If this conjecture is correct, then one can show that under the weak energy condition and cosmic censorship the area of a future event horizon in an asymptotically flat spacetime is non-decreasing. This is known as the **Area Theorem**.

Note that this means that the mass of a Schwarzschild black hole, for which the area is a monotonically increasing function of the radius, $A \propto R_S^2 \propto M^2$, cannot decrease. This is a statement of classical general relativity. We will see later that if gravity is coupled to quantum fields, then the black hole can radiate and loose mass, via the so-called Hawking radiation.

Note also that not all black holes have necessarily increasing mass. Rotating black holes have, as we will see, an area of the event horizon that is a funtion of both the mass and the angular momentum and we can extract energy and hence decrease the mass of the black hole by decreasing its spin.

12.6 Killing Horizons

A Killing vector has a **Killing Horizon** if there exists a null hypersurface along which that Killing vector is null. Take as as example the Schwarzschild metric with the Killing vector[1] $K^\mu = \partial_t = (1, 0, 0, 0)$. We have $K^2 = g_{\mu\nu} K^\mu K^\nu = -(1 - 2GM/r)$. As we enter into the event horizon K^2 turns from negative, i.e. timelike, to positive, i.e. spacelike, and on the event horizon it is null. So this event horizon is a Killing horizon for the Killing vector ∂_t.

Generally, one should not expect that event horizons and Killing horizons are closely linked, but if the metric has time-translation symmetry we can, under certain generic conditions, classify spacetimes.

First, it turns out that in a stationary, asymptotically flat spacetime every event horizon is a Killing horizon for some Killing vector field. Next, if spacetime also happens to be static then that Killing vector is $K^\mu = \partial_t$ and represents time translations in that asymptotically flat part. If however, spacetime is not static, but only stationary, then the metric will be axisymmetric with a rotational Killing vector $R^\mu = \partial_\phi$ and the Killing vector of the Killing horizon will be a linear combination of $K^\mu = \partial_t$ and $R^\mu = \partial_\phi$.

As an example we will see in chapter 13 on rotating or Kerr black holes that the event horizon is a Killing horizon for a combination of $K^\mu = \partial_t$ and $R^\mu = \partial_\phi$ and that in particular the hypersurface on which ∂_t becomes null is timelike and not null, and hence not a Killing horizon.

We will not dwell into the conditions for when this classification is valid, bar it to say that one can show that in the static case one does not need the EFE to show that the event horizon is a Killing horizon for K^μ, i.e. this is a purely geometric fact and that for the stationary case we need to make certain assumptions on the matter fields for this to be valid.

Note that every Killing horizon does not imply an event horizon. Take e.g. Minkowski spacetime $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and consider the Killing vector for a boost in the x-direction, i.e. $K = x\partial_t + t\partial_x$. It has norm $K^2 = -x^2 + t^2$ and thus becomes null on the surface $x = \pm t$, which are therefore Killing horizons. But we know, of course, that Minkowski spacetime has no event horizons. In fact for Minkowski spacetime we can combine the boost Killing vectors with Killing vectors from translations and rotations and we can move

[1] We will be forgiven for the loose, but clear, notation we are using here.

the Killing horizon all over spacetime. Non-flat spacetimes will have less Killing vectors and the associated Killing horizons may have a more significant physical interpretation.

Killing Vectors as Generators of Geodesics on Killing Horizons

Let us approach Killing horizons in a slightly different way. From (12.1) we know that the normal vector ξ to a null hypersurface Σ defined by $f(x) = 0$ can be written as

$$\xi^\mu = \frac{dx(\lambda)}{d\lambda} = h(x) g^{\mu\nu} \partial_\nu f \tag{12.3}$$

This hypersurface $f(x) = 0$ is thus a Killing horizon for a Killing vector k^μ if it's normal is null, $\xi^2 = 0$ and if the Killing vector is proportional to that normal

$$k^\mu = \psi(x) \xi^\mu \tag{12.4}$$

Let us illustrate this with an example. Consider the Killing vector $k = \partial_t$ of the Schwarzschild metric. It becomes null when $0 = k^2 = g_{\mu\nu} k^\mu k^\nu = g_{tt} = -(1 - 2GM/r)$, thus the Killing horizon is the same as the event horizon and is defined by the hypersurface $f(x) = 1 - 2GM/r$. The normal to this horizon can then be written as

$$\begin{aligned} \xi &= \xi^\mu \partial_\mu = h g^{\mu r} \partial_r \left(-1 + \frac{2GM}{r} \right) \Big|_\Sigma \partial_\mu \\ &= h g^{\mu r} \frac{2GM}{r^2} \Big|_\Sigma \partial_\mu = h g^{\mu r} \frac{1}{2GM} \partial_\mu \end{aligned} \tag{12.5}$$

as Σ is defined by $r = 2GM$. We seem to have an issue because we would expect that the normal to Σ as defined by $f(x) = 1 - 2GM/r$ is ∂_r and that is certainly not proportional to our Killing vector ∂_t. But we are saved by the realisation that the Schwarzschild coordinates have a singularity at the event horizon, as $g_{rr} \to \infty$ and so are not a good set of coordinates to describe that horizon. We thus need to go to coordinates that are not singular at the event horizon, e.g. the Eddington-Finkelstein coordinates of (10.4)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dv\, dr + r^2 d\Omega^2 \tag{12.6}$$

The relevant components of the inverse metric are then $g^{vr} = 1, g^{vv} = 0$ and $g^{rr} = (1 - 2GM/r)$, and the Killing vector is now $k = \partial_v$. We now have

$$\begin{aligned} \xi &= h g^{\mu r} \frac{1}{2GM} \partial_\mu = \frac{h}{2GM} (g^{rr} \partial_r + g^{vr} \partial_v) \\ &= \frac{h}{2GM} \left[(1 - 2GM/r)) \partial_r + \partial_v \right] \\ &= \frac{h}{2GM} \partial_v \end{aligned} \tag{12.7}$$

as on the horizon $r = 2GM$ and the normal ξ is indeed proportional to the Killing vector k when using good coordinates.

Recall as well that ξ is not just normal to Σ but it is also tangent to it. Indeed tangents vectors t are by definition orthogonal to normal vectors, $\xi \cdot t = 0$ but as ξ is null by construction we have $\xi^2 = 0$ and so ξ is also tangent to Σ. This means that there must exist a curve $x(\lambda)$ in Σ such that

$$\xi^\mu = dx^\mu/d\lambda \tag{12.8}$$

It turns out that these curves are geodesics. In order to show this we first calculate $\xi^\rho \nabla_\rho \xi^\mu$, but not restricted to Σ:

$$\begin{aligned}
\xi^\rho \nabla_\rho \xi^\mu &= \xi^\rho \nabla_\rho (h g^{\mu\nu} \partial_n f) && \text{(use (12.3))}\\
&= \xi^\rho (\partial_\rho h) g^{\mu\nu} \partial_\nu f + \xi^\rho h g^{\mu\nu} \nabla_\rho \partial_\nu f && \text{(Leibniz)}\\
&= \xi^\rho (\partial_\rho h) h^{-1} \xi^\mu + \xi^\rho h g^{\mu\nu} \nabla_\nu \partial_\rho f && \text{((12.3) and } \partial_{[\rho}\partial_{\mu]} = 0)\\
&= \xi^\rho \xi^\mu \partial_\rho \log h + \xi^\rho h g^{\mu\nu} \nabla_\nu \partial_\rho f\\
&= \xi^\mu \frac{d \log h}{d\lambda} + h \xi^\rho \nabla^\mu (h^{-1} \xi_\rho) && \text{((12.8) and (12.3))}\\
&= \xi^\mu \frac{d \log h}{d\lambda} + \xi^\rho \nabla^\mu \xi_\rho - \xi^2 \partial^\mu \log h && \text{(Leibniz)}\\
&= \xi^\mu \frac{d \log h}{d\lambda} + \frac{1}{2} \partial^\mu \xi^2 - \xi^2 \partial^\mu \log h && (12.9)
\end{aligned}$$

Let us now evaluate this on Σ. By definition we have $\xi^2 = 0$ so the last term vanishes. Now, $\partial^\mu \xi^2$ does not necessarily vanish on Σ, but as ξ^2 is constant it means that $t^\mu \partial_\mu \xi^2 = 0$ for any tangent vector t^μ of Σ and so $\partial_\mu \xi^2$ is normal to Σ and hence we have $\partial_\mu \xi^2 = \alpha \xi_\mu$ for some function α. We can thus write

$$\xi^\rho \nabla_\rho \xi^\mu = \xi^\mu \frac{d \log h}{d\lambda} + \frac{1}{2} \alpha \xi^\mu \tag{12.10}$$

As h is an arbitrary function we can select a specific choice by requiring $0 = 2\frac{d \log h}{d\lambda} + \alpha$ which implies that

$$\xi^\rho \nabla_\rho \xi^\mu = 0 \tag{12.11}$$

The reader will recognise this as the geodesic equation for $x(\lambda)$. Explicitly, by using (12.8) we find

$$0 = \xi^\rho \nabla_\rho \xi^\mu = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} \tag{12.12}$$

Eq. (12.10) is still a geodesic equation but with a non-affine parametrisation.

We have thus shown that the geodesics $x^\mu(\lambda)$ with affine parameter λ of a Killing vector with tangent vectors $\xi^\mu = dx^\mu/d\lambda$ are normal to the Killing horizon and generate it.

12.7 Surface Gravity

To every Killing horizon we can associate a **Surface Gravity** which will be related to how an external observer sees the acceleration of a static observer at the horizon.

Consider a Killing vector k associated with some Killing horizon. Along the horizon we have by construction $k^\mu k_\mu = 0$, so it is a constant on the horizon and therefore $\nabla^\nu(k^\mu k_\mu)$ is also normal to the horizon. There must therefore be a function $\kappa(x)$, called the surface gravity, such that

$$\frac{1}{2}\nabla^\nu(k^\mu k_\mu) = k^\mu \nabla_\mu k^\nu = -\kappa k^\nu \tag{12.13}$$

We can also find this in another way. We know that the Killing horizon is a null hypersurface and therefore a normal ξ^μ satisfies (12.11), i.e. $\xi^\nu \nabla_\nu \xi^\mu = 0$. But we also know that the corresponding Killing vector k^μ is related to the normal ξ^μ by (12.4), i.e. $k^\nu = \psi(x)\xi^\nu$. Let us now work out

$$\begin{aligned} k^\nu \nabla_\nu k^\mu &= k^\nu \nabla_\nu(\psi \xi^\mu) = k^\nu(\partial_\nu \psi)\xi^\mu + k^\nu \psi \nabla_\nu \xi^\mu \\ &= k^\nu(\partial_\nu \psi)\psi^{-1} k^\mu + \psi^2 \xi^\nu \nabla_\nu \xi^\mu \\ &= (k^\nu \partial_\nu \log\psi) k^\mu = -\kappa k^\mu \end{aligned} \tag{12.14}$$

with

$$\kappa = -k^\nu \partial_\nu \log\psi \tag{12.15}$$

We can write the surface gravity in a number of nice forms by using the fact that k satisfies the Killing equation $\nabla_{(\mu} k_{\nu)} = 0$ and Frobenius theorem that states that $k_{[\mu}\nabla_\nu k_{\sigma]} = 0$ is a necessary and sufficient condition for k to be orthogonal to the horizon.

1. Combining these two equations we find that $k_\mu \nabla_\nu k_\sigma - k_\nu \nabla_\mu k_\sigma + k_\sigma \nabla_\mu k_\nu = 0$. We now contract thus with $\nabla^\mu k^\nu$ to find

$$(\nabla^\mu k^\nu)k_\mu(\nabla_\nu k_\sigma) - (\nabla^\mu k^\nu)k_\nu(\nabla_\mu k_\sigma) + (\nabla^\mu k^\nu)k_\sigma(\nabla_\mu k_\nu) = 0 \tag{12.16}$$

We use the Killing equation on the second term to rewrite it as $(\nabla^\nu k^\mu)k_\nu(\nabla_\mu k_\sigma) = (\nabla^\mu k^\nu)k_\mu(\nabla_\nu k_\sigma)$ where we have also interchanged the dummy indices μ and ν. This gives

$$\begin{aligned} (\nabla^\mu k^\nu)k_\sigma(\nabla_\mu k_\nu) &= -2(\nabla^\mu k^\nu)k_\mu(\nabla_\nu k_\sigma) = -2(k^\mu \nabla_\mu k^\nu)\nabla_\nu k_\sigma \\ &= +2\kappa k^\nu \nabla_\nu k_\sigma = -2\kappa^2 k_\sigma \end{aligned} \tag{12.17}$$

where we have also used the definition of κ. From this we deduce a simple formula for the surface gravity valid at the Killing horizon

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu k^\nu)(\nabla_\mu k_\nu) \tag{12.18}$$

2. We can also use Frobenius theorem and the Killing equation to write

$$3k_{[\mu}\nabla_\nu k_{\rho]} = k_\mu \nabla_\nu k_\rho + k_\nu \nabla_\rho k_\mu + k_\rho \nabla_\mu k_\nu \tag{12.19}$$

Multiply this with $k^\mu \nabla^\nu k^\rho$. for the second term on the RHS use $\nabla_\rho k_\mu = -\nabla_\mu k_\rho$ and for the third term first interchange the dummy indices ρ and ν and then use $\nabla^\rho k_\nu = -\nabla_\nu k^\rho$. We then get

$$3k^{[\mu}\nabla^\nu k^{\rho]} k_{[\mu}\nabla_\nu k_{\rho]} = k_\mu k_\mu (\nabla^\nu k^\rho)(\nabla_\nu k_\rho) - 2(k^\mu \nabla^\nu k^\rho)(k_\nu \nabla^\mu k_\rho) \tag{12.20}$$

We now notice that the gradient of the LHS vanishes on the Killing horizon Σ. Indeed the LHS is of the form $3W^{\mu\nu\rho}W_{\mu\nu\rho}$ with $W^{\mu\nu\rho}$ totally antisymmetric. Therefore $\nabla_\sigma(3W^{\mu\nu\rho}W_{\mu\nu\rho}) = 6W^{\mu\nu\rho}\nabla_\sigma W_{\mu\nu\rho}$ and this vanishes on the Killing horizon because $W_{\mu\nu\rho} = k_{[\mu}\nabla_\nu k_{\rho]} = 0$ on Σ by Frobenius theorem. Let us now divide the LHS of the above equation by $|k|^2$ and take the limit of being on Σ

$$\lim_{\Sigma} \frac{3k^{[\mu}\nabla^\nu k^{\rho]} k_{[\mu}\nabla_\nu k_{\rho]}}{|k|^2} \to \frac{0}{0} \tag{12.21}$$

We can thus use l'Hospital's rule. But the gradient of the numerator is zero as we have just seen and the gradient of $|k|^2$ does not vanish for non-zero κ by (12.15). We therefore find that on Σ

$$0 = \frac{1}{|k|^2}\left[k_\mu k_\mu (\nabla^\nu k^\rho)(\nabla_\nu k_\rho) - 2(k^\mu \nabla^\nu k^\rho)(k_\nu \nabla^\mu k_\rho)\right] \tag{12.22}$$

or

$$(\nabla^\nu k^\rho)(\nabla_\nu k_\rho) = \frac{2(k^\mu \nabla^\nu k^\rho)(k_\nu \nabla_\mu k_\rho)}{|k|^2} \tag{12.23}$$

By (12.18) the LHS is just $-2\kappa^2$. In order to rewrite the RHS we work out

$$\begin{aligned}\frac{\partial^\rho(k^\nu k_\nu)\partial_\rho(k^\mu k_\mu)}{2|k|^2} &= \frac{\nabla^\rho(k^\nu k_\nu)\nabla_\rho(k^\mu k_\mu)}{2|k|^2} = \frac{(2k_\nu \nabla^\rho k_\nu)(2k^\mu \nabla_\rho k_\mu)}{2|k|^2} \\ &= \frac{2(k^\mu \nabla^\rho k^\nu)(k_\nu \nabla_\rho k_\mu)}{|k|^2} = \frac{2(k^\mu \nabla^\nu k^\rho)(k_\nu \nabla_\mu k_\rho)}{|k|^2}\end{aligned} \tag{12.24}$$

Therefore

$$\kappa^2 = -\frac{g^{\mu\nu}\partial_\mu k^2 \partial_\nu k^2}{4k^2} \tag{12.25}$$

which is often the most useful form to compute the surface gravity. Note finally that introducing $\sigma^2 = -k^2$ we can also write this in the compact form

$$\kappa^2 = (\partial\sigma)^2\Big|_\Sigma \tag{12.26}$$

where we have explicitly reinstated the fact that this is to be taken on the Killing horizon.

As Killing vectors can always be rescaled, the surface horizon is actually arbitrary, in principle. However in a static, asymptotically free spacetime with time translation Killing vector $K = \partial_t$ we can set the normalisation $\lim_{r\to\infty} K^2 = -1$ and this fixes the surface gravity of any associated Killing horizon. In a stationary asymptotically flat spacetime the Killing horizon is associated with a linear combination of $K = \partial_t$ and $R = \partial_\phi$ and fixing the normalisation of K fixes the linear combination and so also the surface gravity.

Surface Gravity as the Acceleration of a Static Observer on the Horizon

We now claim that

in a static asymptotically flat spacetime the surface gravity is the acceleration of a static observer near the horizon as seen by a static observer at infinity.

In other words, if you as an external observer far away from the horizon will see an observer that is static at and w.r.t. to a horizon as having an acceleration equal to the surface gravity of the horizon.

In order to show this, consider a particle near the horizon Σ, moving on an orbit of the Killing vector k^μ, i.e. following a path $x(\tau)$, where we take τ to be the proper time. Its four-velocity is $u^\mu = dx^\mu/d\tau$ and is proportional to k^μ as the Killing vector is normal to the horizon, i.e. $u^\mu = \alpha k^\mu$. To determine α we recall that the four-velocity satisfies $u^2 = -1$ so that $\alpha^2 = -k^2 = \sigma^2$. Note that is the same σ that we used in (12.26). Thus

$$u^\mu = \sigma^{-1} k^\mu \tag{12.27}$$

The four-acceleration a^μ is given by

$$\begin{aligned} a^\mu &= Du^\mu/D\tau = (dx^\mu/d\tau)\nabla_\nu u^\mu = u^\nu \nabla_\nu u^\mu \\ &= \sigma^{-1}k^\nu\nabla_\nu(\sigma^{-1}k^\mu) = \sigma^{-1}k^\nu\left[-\sigma^{-2}(\nabla_\nu\sigma)k^\mu + \sigma^{-1}k^\nu\nabla_\nu k^\mu\right] \\ &= \sigma^{-2}k^\nu\nabla_\nu k^\mu - \sigma^{-3}k^\mu k^\nu\nabla_\nu\sigma \end{aligned} \tag{12.28}$$

Consider now the second term, it is

$$
\begin{aligned}
-\sigma^{-3}k^\mu k^\nu \nabla_\nu \sigma = & -\sigma^{-3}k^\mu k^\nu \nabla_\nu \sqrt{-k^2} = \frac{1}{2}\sigma^{-4}k^\mu k^\nu \nabla_\nu k^2 \\
= & \sigma^{-4}k^\mu k^\nu k_\rho \nabla_\nu k^\rho = 0
\end{aligned} \tag{12.29}
$$

by symmetry arguments. Using this and rewriting the first term we find

$$
a^\mu = -\frac{1}{2}\sigma^{-2}\partial^\mu \sigma^2 = \sigma^{-1}\partial^\mu \sigma \tag{12.30}
$$

The magnitude of the acceleration is thus given by

$$
a = \sqrt{g^{\mu\nu}a_\mu a_\nu} = \sigma^{-1}\sqrt{\partial_\mu \sigma \partial_\nu \sigma} \tag{12.31}
$$

As the particle approaches the horizon, the square root becomes the surface gravity κ but $\sigma^{-1} = (-k^2)^{-1/2}$ diverges as $k^2 = 0$ on the horizon. Thus the proper acceleration of a particle following a path $x(\tau)$ diverges on the horizon. However that acceleration measured by a static observer at infinity will have a scale factor relating the proper time of the particle to the coordinate time at infinity. Take as an example a Schwarzschild black hole, where the Killing vector is simply $k = \partial_t$, we would then have $d\tau^2 = -g_{00}dt^2$, but we can write this as $d\tau^2 = -k^\mu k^\nu g_{\mu\nu}dt^2$. As this is a covariant expression it provides a natural way of expressing the scaling factor in all cases and so $d\tau = \sigma dt$. Note that σ is nothing else but the redshift factor.

Thus the acceleration of a test particle near the horizon as seen by a static observer at infinity is equal to the surface gravity, which explains why it is called that way.

This consideration breaks down if spacetime is stationary and not static. In the stationary case $K = \partial_t$ is still a Killing vector, but it won't become zero at the Killing horizon. The surface where $K^2 = 0$ is the **Stationary Surface** and corresponds to $g_{tt} = 0$. We will work this out explicitly when we consider rotating black holes and see that on the stationary surface one of the light rays is indeed stationary, and results in an infinite redshift. We will see that in the spacetime between the event horizon and the stationary surface, called the **Ergosphere**, timelike paths are necessarily dragged along the rotation of the black hole.

The Surface Gravity of the Schwarzschild Spacetime

The time translation Killing vector is $K = \partial_t = (1, 0, 0, 0)$, therefore $\sigma^2 = -K^2 = (1 - 2GM/r)$ and so $\sigma = \sqrt{1 - 2GM/r}$. The Killing horizon is given by $K^2 = 0$ hence $r = 2GM$.

We now use (12.26)

$$\begin{aligned}\kappa^2 &= g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\Big|_{r=2GM} = g^{rr}\partial_r\sigma\partial_r\sigma\Big|_{r=2GM} \\ &= \left(1-\frac{2GM}{r}\right)\left[\frac{1}{2\sqrt{1-2GM/r}}\frac{2GM}{r^2}\right]^2\Bigg|_{r=2GM} \\ &= \frac{1}{(4GM)^2}\end{aligned} \tag{12.32}$$

and so

$$\kappa = \frac{1}{4GM} \qquad \text{(Schwarzschild spacetime)} \tag{12.33}$$

We can also compute the acceleration of a static observer. The static four-velocity is $u^\mu = (\omega, 0, 0, 0)$. Requiring $u^2 = -1$ gives $\omega = (1-2GM/r)^{-1/2}$. The acceleration is given by $a^\mu = u^\nu\nabla_\nu u^\mu$ which gives the components $(0, GM/r^2, 0, 0)$ and $a_\mu = (0, GM/[r^2(1-2GM/r)], 0, 0)$ and so

$$a = \sqrt{a^\mu a_\mu} = \frac{GM}{r^2\sqrt{1-2GM/r}} \tag{12.34}$$

and we see that it indeed diverges for the static observer at the horizon. However for an observer at infinity watching the static observer at the horizon, we need to multiply by $\sigma_a = \sqrt{-k^2} = \sqrt{1-2GM/r}$ and so we find for $a_\infty = \sigma a = GM/r^2$.

Note that the surface gravity (12.33) decreasing for increasing mass; the surface gravity of a heavy black hole is actually weaker than that of a lighter black hole. We can understand this from looking at the redshifted acceleration $\sigma_a = GM/r^2$. The event horizon is proportional to the mass and so provides a factor M^{-2} which dominates.

12.8 Charge, Mass, and Spin of a Black Hole

We have seen that the no-hair theorem says that a black hole can be fully described by a limited number of parameters such as its mass, charge and angular momentum. It is therefore important to have a thorough understanding of how these are defined in general relativity, as we will see that they are not without some subtleties.

The Charge of a Black Hole

To describe the charge of a black hole is easy. It is the conserved charge of the electromagnetic current. Maxwell's equation is $\nabla_\nu F^{\mu\nu} = J^\mu_e$, and the charge passing through a spacelike hypersurface Σ is given by the integral over that hypersurface

$$Q = -\int_{\Sigma} d^3x \sqrt{\gamma} n_\nu J_e^\mu \tag{12.35}$$

where γ_{ij} is the induced metric and n^μ is the unit normal vector of Σ. The minus signs ensures that if the charge density is positive, a future directed normal vector will give a positive charge. Using Maxwell's equation and Stokes theorem, we can write

$$Q = -\int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu F^{\mu\nu} \tag{12.36}$$

with $\gamma^{(2)}$ is the induced metric on the boundary of Σ, which is usually a two-sphere, and σ^μ is its outwards pointing unit normal vector.

Let us work this out for the simple example of a point charge in Minkowski spacetime to convince ourselves that this makes sense. The electric field of a point charge has only a radial component $E^r = q/4\pi r^2$, so the only non-vanishing field tensors are $F^{tr} = -F^{rt} = E^r$. The unit vector on spacelike infinity is $n^\mu = (1, 0, 0, 0)$ and the unit vector on the two-sphere points in the radial direction, hence is $\sigma^\mu = (0, 1, 0, 0)$. Thus $n_\nu \sigma_\nu F^{\mu\nu} = -E^r = -q/4\pi r^2$. The induced metric on the two-sphere is $\gamma^{(2)}_{ab} dx^a dx^b = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ and so the volume element is $d^2x\sqrt{\gamma^{(2)}} = r^2 \sin\theta\, d\theta\, d\phi$. Bringing everything together we find

$$Q = -\lim_{r\to\infty} \int_{S^2} \left(-\frac{q}{4\pi r^2}\right) \times (r^2 \sin\theta\, d\theta\, d\phi) = q \tag{12.37}$$

which is indeed exactly what we expect.

If we would wish to, we could likewise define a magnetic charge by replacing $F^{\mu\nu}$ by its dual $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

The Energy of a Black Hole

The concept of energy is quite subtle in general relativity. We can understand this from the fact that the energy-momentum is a tensor and that it only describes the properties of matter, not of the gravitational field itself. But we saw in (5.13) that we could define a conserved total energy in a stationary spacetime. If K^μ is a timelike Killing vector then we can define

$$E_T = \int_{\Sigma} J_T^\mu n_\mu \sqrt{\gamma}\, d^3x \qquad \text{with} \qquad J_T^\mu = K_\nu T^{\mu\nu} \tag{12.38}$$

Unfortunately there are some problems with this concept. Consider for example the Schwarzschild spacetime. It has a timelike Killing vector but it is a solution of the Einstein equations in vacuum so the energy-momentum tensor is equal to zero, therefore this energy E_T would be zero. But how can we be sure of this since the spacetime contains a singularity and it may not be clear how we can evaluate the integral. In addition, a black hole may be the end-point of the collapse of a massive star, which clearly has non-zero energy. If the end-point has no total energy then we are violating the conservation of energy.

It hence make sense to look for a new concept of energy. Once more we assume a stationary, asymptotically flat spacetime with timelike Killing vector K^μ and we construct the tensor

$$J_R^\mu = K_\nu R^{\mu\nu} \tag{12.39}$$

From the Einstein equation we can rewrite this as

$$J_R^\mu = 8\pi G K_\nu \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right) \tag{12.40}$$

Let us now evaluate the divergence of (12.39)

$$\nabla_\mu J_R^\mu = (\nabla_\mu K_\nu) R^{\mu\nu} + K_\nu (\nabla_\mu R^{\mu\nu}) = \frac{1}{2} K_\nu \nabla^\nu R = 0 \tag{12.41}$$

For the first term we have used the Killing equation and for the second we have used the Bianchi identity, $\nabla_\mu G^{\mu\nu} = 0$. For the last equation we have used the fact that the directional derivative of the curvature along any Killing vector vanishes as per the last equation of (5.12).

The advantage of defining a conserved quantity using J_R^μ is that it can be written as a surface integral over a two-sphere at spatial infinity. Indeed we have

$$\nabla_\nu (\nabla^\mu K^\nu) = \nabla_\nu \nabla^\mu K^\nu) = K_\nu R^{\mu\nu} = J_R^\mu \tag{12.42}$$

where we have used the second equation of (5.12). Thus, if we define an energy

$$E_R = \frac{1}{4\pi G} \int_\Sigma \sqrt{\gamma} n_\mu J_R^\mu d^3x \tag{12.43}$$

then we can use Stoke's theorem to write it as

$$E_R = \frac{1}{4\pi G} \int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu d^2x \tag{12.44}$$

This expression is known as the **Komar Integral** of the timelike Killing vector $K^\mu = \partial_t$.

Let us work out the example of the Schwarzschild metric to show that E_R has the interpretation of the total energy of spacetime. This is a similar calculation as the one we did for the charge of a point particle in Minkowski spacetime, so we will proceed swiftly. The unit vectors have non-zero components

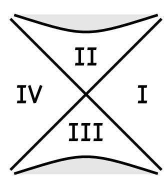
$$n_0 = -\left(1 - \frac{2GM}{r}\right)^{1/2} \qquad \text{and} \qquad \sigma_1 = \left(1 - \frac{2GM}{r}\right)^{-1/2} \tag{12.45}$$

note that n is timelike, $n^2 = -1$ and σ is spacelike, $\sigma^2 = +1$. Then $n_\mu \sigma_\nu \nabla^\mu K^\mu = -\nabla^0 K^1$, with $K^\mu = (1, 0, 0, 0)$. This gives $\nabla^0 K^1 = -GM/r^2$. Using the two-sphere volume element $d^2x\sqrt{\gamma^{(2)}} = r^2 \sin\theta \, d\theta \, d\phi$ we find

$$E_R = \frac{1}{4\pi G} \int \frac{GM}{r^2} r^2 \sin\theta \, d\theta \, d\phi = M \tag{12.46}$$

which indeed recovers the mass, i.e. energy of the Schwarzschild black hole.

Our motivation for turning to the Komar integral was that $E_T = \int_\Sigma K_\nu T^{\mu\nu} \, n_\mu \sqrt{\gamma} \, d^3x$ was identical to zero in vacuum. The attentive reader will notice that $E_R = \int_\Sigma K_\nu R^{\mu\nu} \, n_\mu \sqrt{\gamma} \, d^3x$ may have the same fate as $R_{\mu\nu} = 0$ in vacuum. But we performed this integral by using Stoke's theorem to integrate it over the boundary and the result was definitely not zero for non-vanishing mass! What's going on? The issue really is that we really only integrated over the spacelike region I of the Kruskal diagram (see the figure below) of the extended Schwarzschild metric, and hence only over that boundary. The spacelike region really extends via the wormhole over region IV as well. The contribution from that region would be the same and opposite. Hence both contributions together will indeed give a contribution of zero. What happens when the spacelike region ends in the singularity is a mystery, at least to me.



The Angular Momentum of a Black Hole

It is now entirely straightforward to define the total spin of a spacetime. Indeed the definition of the Komar integral for K^μ, only required K^μ to be a Killing vector, so if we have a Killing vector for rotational symmetry, $R^\mu = \partial_\phi$, we can define a conserved current

$$J^\mu_\phi = R_\nu R^{\mu\nu} \tag{12.47}$$

and a Komar integral

$$J = -1\frac{1}{8\pi G}\int_{\partial\Sigma}\sqrt{\gamma^{(2)}}\,n_\mu\sigma_\nu\nabla^\mu R^\nu d^2x \tag{12.48}$$

Working this out for the Schwarzschild metric is trivial. We have $R^\mu = \partial_\phi = (0,0,0,1)$ and thus $R_\mu = (0,0,0,1)$. We therefore have $n^\mu\sigma^\nu\nabla_\mu R_\mu = -\nabla_0 R_1 = 0$, hence $J_R = 0$.

12.9 ADM Gravity

The fact that E_R recovers the mass of the Black hole for the Schwarzschild metric, and, as we will see, also for othermetrics, is encouraging to say it is the energy. But the evidence seems somewhat meagre. Further and better evidence comes form a Hamiltonian approach to general relativity known as **ADM Gravity**.[2] In that approach one consider the metric and its "conjugate momentum" as independent variables and takes the Hamiltonian as generator of time translation. If one then splits the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with h very small at infinity, so that we have an asymptotically flat spacetime, then one finds that the **ADM Energy** is given by

$$E_{\mathsf{ADM}} = \frac{1}{16\pi G}\int_{\partial\Sigma} d^2x\,\sqrt{\gamma^{(2)}}\,\sigma^i(\partial_j h_i^j - \partial_i h_j^j) \tag{12.49}$$

If $h_{\mu\nu}$ is time-independent at infinity that one can show that the ADM energy and the Komar energy are the same.

One can further more prove a **Positive Energy Theorem** stating that

> In a nonsingular, asymptotically flat spacetime obeying the
> Einstein equations and with the dominant energy condition,
> the ADM energy is nonnegative, and only zero
> for Minkowski spacetime.

It is comforting that the ADM energy cannot be negative, or otherwise a zero energy solution could always spilt in the combination of a positive and negative energy solution. The Dirac sea all again!

[2] ADM stands for Arnowitt, Deser and Misner

Chapter 13

The Rotating or Kerr Black Hole

13.1 The Kerr Metric

The Schwarzschild metric is the unique spherically symmetric solution to Einstein's equations in vacuum. We now consider stationary and cylindrically symmetric solutions in vacuum. Stationary means that the object can rotate with constant angular momentum. Cylindrical symmetry means that the metric components only depend on r and θ, but not on t or ϕ. We also require the metric to be unchanged under $(t, \phi) \to -(t, \phi)$. This means that the only cross terms that can appear are $dt\, d\phi$ and $dr\, d\theta$, but the latter can be eliminated by a coordinate transformation. The corresponding metric is then of the form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dt\, d\phi \tag{13.1}$$

where the metric components are functions of r and θ only.

We will not derive the solution but just posit it:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2GMa^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \\ &- \frac{4GMar\sin^2\theta}{\rho^2}dt\, d\phi \end{aligned} \tag{13.2}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 + a^2 - 2GMr \\ a &= \frac{J}{GM} \end{aligned} \tag{13.3}$$

This solution is known as the **Kerr Metric** and this particular form is known amongst aficionados as the **Boyer-Lindquist Coordinates**. The Kerr metric has two free parameters M and J. In order to identify M and J with things we know, we establish a link to specific concepts in Newtonian gravity.

For $r \to \infty$ the Kerr metric reduces to the Minkowski metric and in particular $g_{tt} \to -(1-2GM/r)$, which becomes Newtonian gravity in the weak field limit, hence the identification of M with a mass. The $g_{t\phi}$ component then becomes

$$g_{t\phi} \to -\frac{2J\sin^2\theta}{r} dt\, d\phi = -\frac{2J\sin^2\theta}{r} dt\,(xdy - ydx) \tag{13.4}$$

Using this in the metric we find that for $r \to \infty$ it is of the form

$$ds^2 = \cdots + r^2 \sin^2\theta \left(d\phi - \frac{2J}{r^3} dt \right)^2 + \cdots \tag{13.5}$$

One can then show that J indeed corresponds to the angular momentum of a slowly rotating body.

In fact, the best way to show that M and J represent the mass and angular momentum of the black hole is by computing the respective Komar integrals.

13.2 The Komar Integrals for the Kerr Black Hole

Let us evaluate the energy and angular momentum Komar integrals for the Kerr metric. They are given by (12.44) and (12.48) respectively

$$\begin{aligned} E_R &= \frac{1}{4\pi G} \int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu d^2x \\ J_R &= -1\frac{1}{8\pi G} \int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu R^\nu d^2x \end{aligned} \tag{13.6}$$

The unit normal vectors are $n^\mu = (1,0,0,0)$ and $\sigma^\mu = (0,1,0,0)$. This implies that

$$n_\mu = \left(-1 + \frac{2GMr}{\rho^2}, 0, 0, -\frac{2GMar\sin^2\theta}{\rho^2} \right) \quad \text{and} \quad \sigma_\mu = \left(0, \frac{\rho^2}{\Delta}, 0, 0 \right) \tag{13.7}$$

Thus for a Killing vector k^μ

$$n_\mu \sigma_\nu \nabla^\mu k^\nu = n^\mu \sigma^\nu \nabla_\mu k_\nu = \nabla_0 k_1 = \partial_t k_r - \Gamma^\mu_{tr} k_\mu \tag{13.8}$$

Let us now first consider E_R. The Killing vector is $K^\mu = (1,0,0,0) = n^\mu$, hence

$$K_\mu = \left(-1 + \frac{2GMr}{\rho^2}, 0, 0, -\frac{2GMar\sin^2\theta}{\rho^2} \right) \tag{13.9}$$

Thus

$$n_\mu \sigma_\nu \nabla^\mu K^\nu = -\Gamma^\mu_{tr} K_\mu = -\Gamma^t_{tr} K_t - \Gamma^\phi_{tr} K_\phi \tag{13.10}$$

The relevant Christoffel symbols are

$$\Gamma^t_{tr} = \frac{GM(r^2+a^2)(r^2-a^2\cos^2\theta)}{\Delta\rho^4} \quad \text{and} \quad \Gamma^\phi_{tr} = \frac{GMa(r^2-a^2\cos^2\theta)}{\Delta\rho^4} \tag{13.11}$$

and this gives

$$n_\mu \sigma_\nu \nabla^\mu K^\nu = \frac{GM(r^2-a^2\cos^2\theta)}{\rho^4} \tag{13.12}$$

Combining with the volume element $d^2x\sqrt{\gamma^{(2)}} = r^2 \sin\theta \, d\theta \, d\phi$ we find

$$\begin{aligned} E_R &= \lim_{r\to\infty} \frac{1}{4\pi G} \int d\theta \, \frac{GM(r^2-a^2\cos^2\theta)}{\rho^4} r^2 \sin\theta \int d\phi \\ &= \lim_{r\to\infty} \frac{1}{4\pi G} \left(\frac{2GMr^2}{r^2+a^2} \right) (2\pi) = M \end{aligned} \tag{13.13}$$

and we find that the total energy of spacetime is indeed the mass of the black hole.

Turning to the angular momentum, we use the Killing vector $R^\mu = \partial_\phi = (0,0,0,1)$. This gives

$$R_\mu = \left(-\frac{2aGMr\sin^2\theta}{\rho^2}, 0, 0, r^2 + a^2 + \frac{2a^2GMr\sin^2\theta}{\rho^2} \right) \tag{13.14}$$

and we find similarly

$$n_\mu \sigma_\nu \nabla^\mu R^\nu = -\frac{aGM(r^2-a^2\cos^2\theta)\sin^2\theta}{\rho^4} \tag{13.15}$$

and

$$J_R = \lim_{r\to\infty} -\frac{1}{8\pi G} \int d\theta \, \left(-\frac{aGM(r^2-a^2\cos^2\theta)\sin^2\theta}{\rho^4} \right) r^2 \sin\theta \int d\phi \tag{13.16}$$

This integral is harder to compute[1], but we can evaluate it numerically for very large values of r and we find that it is equal to $-4aGM/3$. Thus

$$J_R = -\frac{1}{8\pi G} \left(-\frac{4aGM}{3} \right) (2\pi) = \frac{aM}{3} = \frac{J}{3G} \tag{13.17}$$

I am not sure about the strange denominator, but it certainly is the case that J_R is proportional to J and hence J is indeed a measure of the angular momentum of the rotating black hole.

[1] My Mathematica license runs out of time.

13.3 General Properties of a Stationary Cylindrically Symmetric Metric

It is possible to deduce some important properties of a stationary cylindrically symmetric metric, without going into the detailed from of the Kerr metric. Assume the metric is of the form (13.1), i.e.

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dt\,d\phi \tag{13.18}$$

with the components functions of r and θ only. This metric is invariant under the transformations $t \to t + \varepsilon_t$ and $\phi \to \phi + \varepsilon_\phi$ and thus has two corresponding conserved currents. These are most easily found from the corresponding equations of motion from the action $S = \int ds$ which are

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}\right) \\ 0 &= \frac{d}{d\tau}\left(g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}\right) \end{aligned} \tag{13.19}$$

The conserved quantities are thus

$$\begin{aligned} \epsilon &= -\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}\right) \\ \ell &= g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau} \end{aligned} \tag{13.20}$$

and these do not change along geodesics.

In the context of differential geometry we say that there are two conserved Killing vectors $k_\epsilon = k_\epsilon^\mu\partial_\mu$ and $k_\phi = k_\ell^\mu\partial_\mu$ with $k_\epsilon^\mu = (1,0,0,0)$ and $k_\phi^\mu = (0,0,0,1)$ so that $k_E = \partial_t$ and $k_\phi = \partial_\phi$. For a particle of mass m with momentum $p^\mu = m dx^\mu/d\tau$ the conserved quantities are then

$$\begin{aligned} -k_\epsilon \cdot p =& -k_\epsilon^\mu p_\mu = -p_t = -g_{t\mu}p^\mu = -(g_{tt}p^t + g_{t\phi}p^\phi) = \epsilon \\ k_\ell \cdot p =& k_\ell^\mu p_\mu = p_\phi = g_{\phi^\mu}p^\mu = g_{\phi t}p^t + g_{\phi\phi}p^\phi = \ell \end{aligned} \tag{13.21}$$

13.3 Frame Dragging

We see that the "angular momentum" ℓ has an extra term containing the non-diagonal component $g_{t\phi}$, which will give some new physics.

Take a particle[2] far way from the source of the gravitational field that has $\ell = 0$. By

[2] If the particle is massive then τ is the proper time; if it is massless then τ is an affine parameter.

far away we mean the the metric approaches a flat metric and thus $g_{t\phi} \to 0$ and $g_{\phi\phi} \to 1$. Thus $\ell = 0$ implies that $d\phi/d\tau \to 0$ far away.

As the particle approaches $r = 0$, the angular momentum remains zero as it is conserved, but the particle picks up a angular dependent velocity, defined w.r.t. to the coordinate time of an external observer, i.e. not w.r.t. the proper time of the particle.

$$\omega(r,\theta) \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = -\frac{g_{t\phi}}{g_{\phi\phi}} \tag{13.22}$$

Thus, an external observer will see as if the particles rotates, even though it has zero angular momentum. This is the phenomenon of **Frame Dragging** and can be interpreted as spacetime being deformed by the rotating source, as shown in fig. 13.1

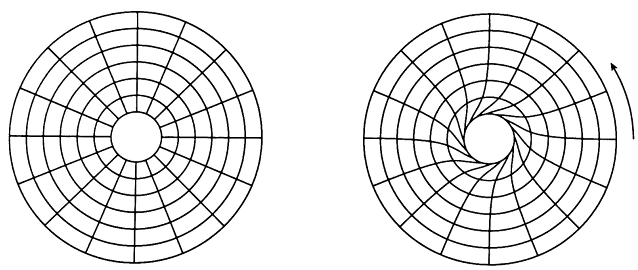


Figure 13.1: Frame dragging

13.3 Stationary Limit Surface

The frame dragging is experienced by any particle, massive or massless, in the neighbourhood of a rotating massive object, not necessarily a black hole and is due to the non-vanishing of $g_{t\phi}$. As another example of the consequence of this off-diagonal metric component, let us consider light rays. Solving the quadratic equation $ds^2 = 0$ for $d\phi/dt$ we find the two solutions

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{13.23}$$

We consider five regimes, depending on how far away we are from the black hole

(a) Very far away from the black hole we are approaching Minkowski spacetime and hence $g_{tt} < 0$ and $g_{\phi\phi} > 0$. Thus we have two roots

$$\Omega_+^{(a)} = \frac{d\phi}{dt} = \omega + \sqrt{\omega^2 + \left|\frac{g_{tt}}{g_{\phi\phi}}\right|} > 0$$
$$\Omega_-^{(a)} = \frac{d\phi}{dt} = \omega - \sqrt{\omega^2 + \left|\frac{g_{tt}}{g_{\phi\phi}}\right|} < 0 \tag{13.24}$$

where

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \tag{13.25}$$

These two two solutions describe light rays that for the external observer are **co-rotating** and **counter-rotating** w.r.t. the rotation of the source.[3]

(b) Suppose that as we get closer we come to a hypersurface that has $g_{tt} = 0$. on that surface we have for the two roots

$$\Omega_+^{(b)} = \frac{d\phi}{dt} = 2\omega$$
$$\Omega_-^{(b)} = \frac{d\phi}{dt} = 0 \tag{13.26}$$

The counter-rotating light ray stands still on that surface, which not surprisingly is called a **Stationary Surface**.

(c) Closer to the source g_{tt} may turn positive and

$$\Omega_+^{(c)} = \frac{d\phi}{dt} = \omega + \sqrt{\omega^2 - \left|\frac{g_{tt}}{g_{\phi\phi}}\right|} > 0$$
$$\Omega_-^{(c)} = \frac{d\phi}{dt} = \omega - \sqrt{\omega^2 - \left|\frac{g_{tt}}{g_{\phi\phi}}\right|} > 0 \tag{13.27}$$

From the point of view of an external observer, both light rays have a positive angular velocity. Inside the stationary surface everything, even light, is swept along with the flow of the rotation of the black hole.

(d) Even closer, we may come to a point where $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$. In this case

$$\Omega_+^{(d)} = \Omega_-^{(d)} = \omega \tag{13.28}$$

Co-rotating and counter-rotating light rays have the same angular velocity.

[3]We can always set the reference frame so that $\omega > 0$.

(e) Finally for $g_{t\phi}^2 - g_{tt}g_{\phi\phi} < 0$ there are no solutions and hence there is nothing the external observer can say about the light rays in that region

Not surprisingly $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$ will correspond to the event horizon.

Fig. 13.2 shows the values of the angular velocities $\Omega_\pm$ as a function of r for different values of the G, M, J and θ.

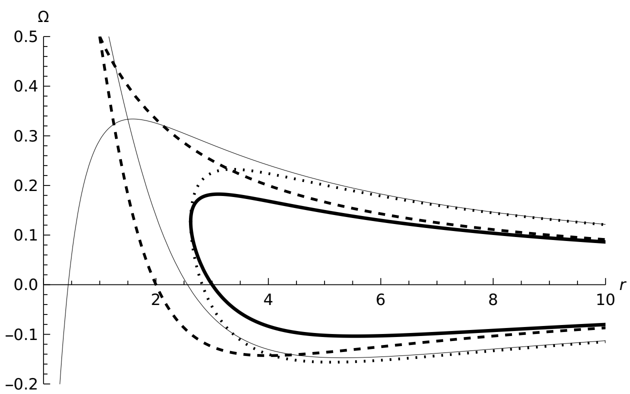


Figure 13.2: Angular velocities $\Omega_\pm$ of light rays in Kerr metric for varoius parameters (G, M, J, θ). Thin line: $(3/2, 1, 1, \pi/2)$; dashed line: $(1, 1, 1, \pi/2)$; dotted line: $(3/2, 1, 1, \pi/4)$; thin line: $(3/2, 1, 3/2, \pi/4)$;

Fig. 13.3 illustrates the different regimes of a stationary cylindrically symmetric black hole. Staring from the outside we have co-rotating and counter-rotating light rays. At the stationary surface $g_{tt} = 0$ the counter-rotating light rays stands still on that surface, then bot light rays rotate in the same direction as the gravitational source at different rotation speeds until the event horizon $g_{t\phi} - g_{tt}g_{\phi\phi} = 0$ where all light rays rotate at the same speed. Once inside the event horizon an external observer has no information about how the light rays may behave.

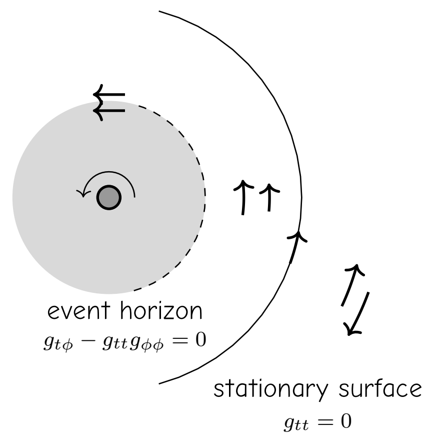


Figure 13.3: Light rays in a Kerr black hole

13.3 The Conserved Energy

Recall from (13.21) that the conserved energy is given by $\varepsilon = -p_t$. We can now solve the mass shell condition $-m^2 = p^2 = g^{\mu\nu}p_\mu p_\nu$ for p_t. This is a quadratic equation in p_t with solutions

$$-p_t^{\pm} = \frac{g_{t\phi}}{g_{\phi\phi}}p_\phi \pm \sqrt{\frac{1}{g_{\phi\phi}^2}\left[\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)p_\phi^2 + m^2(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\left(g_{rr}p_r^2 + g_{\theta\theta}p_\theta^2 + 1\right)g_{\phi\phi}\right]} \tag{13.29}$$

Far form the origin we expect flat space time and thus $g_{t\phi} \to 0$ which leads us to select the positive square root an thus the energy is given by

$$\epsilon = \frac{g_{t\phi}}{g_{\phi\phi}}p_\phi + \sqrt{\frac{1}{g_{\phi\phi}^2}\left[\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)p_\phi^2 + m^2(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\left(g_{rr}p_r^2 + g_{\theta\theta}p_\theta^2 + 1\right)g_{\phi\phi}\right]} \tag{13.30}$$

Note that we have the traditional square root of the form $\sqrt{\vec{p}^2 + m^2}$ plus a term outside of the square root that depends on the non-diagonal metric component $g_{t\phi}$.

13.4 General Properties of the Kerr Metric

For convenience, let us rewrite the Kerr metric

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2GMa^2 r \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2 - \frac{4GMar\sin^2\theta}{\rho^2} dt\, d\phi \tag{13.31}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta; \qquad \Delta = r^2 + a^2 - 2GMr; \qquad a = \frac{J}{GM} \tag{13.32}$$

and point out a number of properties

- For $r \to \infty$ we have $\rho^2 \approx \Delta \to r^2$ and $ds^2 \to \eta_{\mu\nu} dx^\mu dx^\nu$ and the metric becomes asymptotically flat.

- For $a = 0$ we recover the Schwarzschild metric, in fact

$$ds^2_{\text{Kerr}} = ds^2_{\text{Schwarzschild}} - \frac{4GMa\sin^2\theta}{r} dt\, d\phi + o(a^2) \tag{13.33}$$

- Keeping a fixed and letting $M \to 0$ we find

$$ds^2 \to -dt^2 + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2} dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2\right] \tag{13.34}$$

  Performing a coordinate transformation on the spatial part

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{13.35}$$

  we recover flat spacetime $-dt^2 + dx^2 + dy^2 + dz^2$. The ellipsoidal coordinates are shown in fig. 13.4

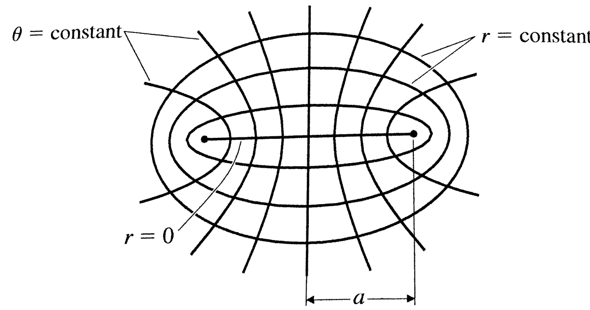


Figure 13.4: Ellipsoidal coordinates for the Kerr metric for a fixed and $M \to 0$

Note that the point $(r, \theta) = (0, \pi/2)$ corresponds to a ring of radius r, i.e. $x = a\cos\phi, y = a\sin\phi$.

- As we already saw, the Kerr metric is independent of t and ϕ and thus has two Killing vectors $k_\varepsilon = \partial_t$ and $k_\ell = \partial_\phi$, which correspond to the conserved quantities ε and ℓ along geodesics. The Kerr metric also has a rank two Killing tensor. Recall that a Killing tensor is defined by (5.11), i.e. $\nabla_{(\mu}K_{\nu_1\cdots\nu\ell)} = 0$. To obtain the Killing tensor we first define

$$\ell^\mu = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a) \qquad \text{and} \qquad n^\mu = \frac{1}{2\rho^2}(r^2 + a^2, -\Delta, 0, a) \tag{13.36}$$

These two vectors are null, $\ell^2 = n^2 = 0$, and $\ell \cdot n = -1$. We can now define the tensor

$$K_{\mu\nu} = \rho^2(\ell_\mu n_\nu + \ell_\nu n_\mu) + r^2 g_{\mu\nu} \tag{13.37}$$

and check that this is indeed a Killing tensor. By metric compatibility, also $K^{\mu\nu}$ is a Killing tensor.

- The frame dragging angular velocity (13.22) is for the Kerr metric

$$\omega(r, \theta) = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2GMar}{(r^2 + a^2)^2 - \Delta a^2 \sin^\theta} \tag{13.38}$$

We show different plots of ω as a function of r for different values of the parameters in fig. 13.5

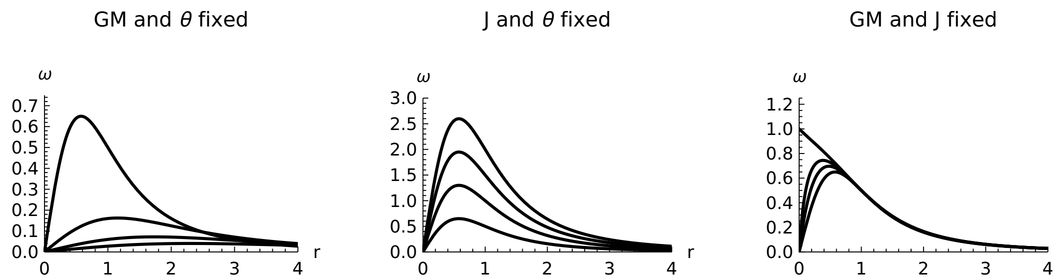


Figure 13.5: Angular velocities for frame dragging with the Kerr metric

- The stationary surface, where the counter-rotating light stands still for an external observer, is given by $g_{tt} = 0$ hence $\rho^2 = 2GMr$. This has two solutions

$$r_{S\pm} = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta} \tag{13.39}$$

Recall that $g_{tt=0}$ corresponds to infinite redshift. There is thus an inner and an outer surface of infinite redshift. These surfaces describe two ellipsoid-type surfaces in the x, y, z coordinates

$$\begin{aligned} x_\pm &= \left[\left(GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}\right)^2 + a^2\right]^{1/2} \sin\theta\cos\phi \\ y_\pm &= \left[\left(GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}\right)^2 + a^2\right]^{1/2} \sin\theta\sin\phi \\ z_\pm &= (GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta})\cos\theta \end{aligned} \tag{13.40}$$

13.5 Singularities of the Kerr Metric

The Kerr metric has singularities at $\rho = 0$ and at $\Delta = 0$. Taking $a \to 0$ brings us back to the Schwarzschild metric and in this limit $\rho \to 0$ corresponds to $r \to 0$ and $\Delta \to 0$ to $r \to 2GM$. We thus guess that $\rho = 0$ corresponds to a real singularity and $\Delta = 0$ is a coordinate singularity only. This is confirmed by calculating the Kretschmann invariant $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ which turns out to be

$$K = \frac{(GM)^2}{48}\,\frac{(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)}{\rho^2} \tag{13.41}$$

The physical singularity at $\rho = 0$ corresponds to $(r, \theta) = (0, \pi/2)$. We saw earlier that in ellipsoidal coordinates this corresponds to a ring of radius a.

The coordinate singularity $\Delta = 0$ has two solutions

$$r_{\pm} = GM \pm \sqrt{(GM)^2 - a^2} \tag{13.42}$$

These describe two ellipsoids

$$\begin{aligned} x_{\pm} &= \left[\left(GM \pm \sqrt{(GM)^2 - a^2}\right)^2 + a^2\right]^{1/2} \sin\theta\cos\phi \\ y_{\pm} &= \left[\left(GM \pm \sqrt{(GM)^2 - a^2}\right)^2 + a^2\right]^{1/2} \sin\theta\sin\phi \\ z_{\pm} &= (GM \pm \sqrt{(GM)^2 - a^2})\cos\theta \end{aligned} \tag{13.43}$$

Fig. 13.6 gives a side view of the structure of a Kerr black hole in the x, z plane. Note that we need $GM > a$ to have real solutions. We also see that $r_{S+} \geq r_+$ and $r_{S-} \leq r_-$, with the equality sign obtained when $\theta = \pi/2$. The region between the outer coordinate singularity and the outer stationary surface is known as the **Ergoregion**.

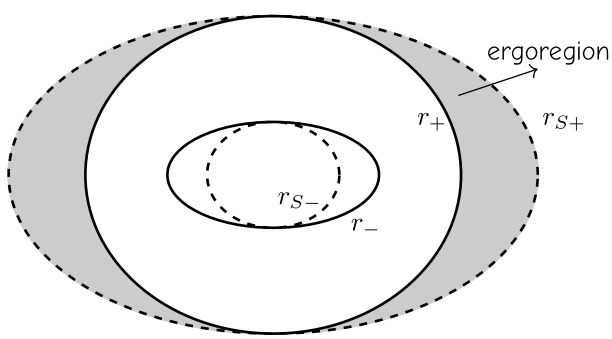


Figure 13.6: Side view of a Kerr black hole: stationary surface (dashed line) and coordinate singularity (thick line) of Kerr spacetime. The region between the outer coordinate singularity and the outer stationary surface is known as the ergoregion.

13.6 Extremal Kerr Black Holes

The location of the coordinate singularity involves a $\sqrt{(GM)^2 - a^2}$, which suggests that we have $|a| \leq M$ or equivalently

$$|J| \leq GM^2 \tag{13.44}$$

The situation where

$$|J| = GM^2 \tag{13.45}$$

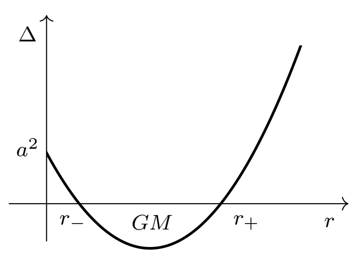
is called an **Extremal Black Hole**. We expect most astrophysical black holes to be (almost) extremal as in-falling matter tends to increase the angular momentum. In section 13.10 we will give a proof of that inequality.

13.7 The Coordinate Singularity and the Outer Horizon

Let us look at a radially outward going photon and see if it can escape to infinity. Start by assuming the photon is well outside the coordinate singularity r_+, so that $\Delta > 0$. Setting $ds^2 = 0$ in the Kerr metric and solving for dr whilst keeping the positive square root as we are considering an outgoing photon we have

$$dr = +\frac{\Delta}{\rho^2}\Bigg[\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2GMa^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 + \frac{4GMar\sin^2\theta}{\rho^2}dt\, d\phi\Bigg]^{1/2} \tag{13.46}$$

We can consider light rays with $d\theta = 0$, but because of the cross term $g_{t\phi}dtd\phi$ we cannot just set $d\phi = 0$. Consider now $\Delta = r^2 + a^2 - 2GMr$. Its general form is shown below.



Take the derivative $\partial_r \Delta = 2r - 2GM$. Thus at $r = GM$ the derivative changes sign; as long as $r > GM$ then Δ decreases as r decreases, but at a certain point Δ becomes zero and hence $dr = 0$. The light ray can no longer go to infinity, which is exactly what a horizon is. As $\Delta = 0$ corresponds to $r = r_\pm = GM + \sqrt{(GM)^2 - a^2}$, this is exactly the horizon, and it corresponds to the Schwarzschild horizon for $a = 0$ as it should.

Let us now consider what happens with a light ray at the outer horizon r_+, which for ease we will just call the horizon henceforth. Note first that clearly $r_+ > GM$ from its definition. In order to analyse what is going on it is easiest to write the Kerr metric in another form, viz.

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 \tag{13.47}$$

where as usual $\rho^2 = r^2 + a^2 cos^2\theta$ and $\Delta = r^2 + a^2 - 2GMr$ and

$$\begin{aligned} \omega &= -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2GMar}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \\ \Sigma &= (r^2+a^2)^2 - \Delta a^2 \sin^2\theta \end{aligned} \tag{13.48}$$

Assuming a light ray $ds^2 = 0$ and looking at the point where $dr = 0$, we can set $\Delta = 0$ to obtain

$$0 = \rho_+^2 d\theta^2 + \frac{\Sigma_+^2}{\rho_+^2}\sin^2\theta(d\phi - \omega dt)^2 \tag{13.49}$$

where the subscript $_+$ indicates this is to be evaluated at the horizon $r = r_+$. We can solve this equation to obtain $d\phi/dt = \omega_+$. As $\Delta_+ = 0$ we have $\Sigma+^2 = (r_+^2 + a^2)^2$. Filling in the value for r_+ this gives $\Sigma_+ = 2GMr_+$. Therefore

$$\frac{d\phi}{dt} = \frac{2GMar_+}{(r_+^2+a^2)^2} = \frac{a}{2GMr_+} \tag{13.50}$$

Thus on the horizon a light ray satisfies

$$(dt, dr, d\theta, d\phi) \propto \ell^\mu = \left(1, 0, 0, \frac{a}{2GMr_+}\right) \tag{13.51}$$

It so happens that ℓ^μ is a null vector on the horizon. In general

$$\begin{aligned} \ell^2 &= g_{\mu\nu}\ell^\mu\ell^\mu \\ &= -1 + \frac{2GMr}{\rho^2} - \frac{2a^2 r\sin^2\theta}{\rho^2 r_+} + \frac{a^2\sin^2\theta\left[\rho^2(r^2+a^2) + 2GMa^2\sin^2\theta\right]}{GM\rho^2 r_+} \end{aligned} \tag{13.52}$$

which on the horizon gives

$$\ell^2\Big|_{r=r_+} = 0 \tag{13.53}$$

Two other vectors spanning the horizon are $h^\mu = (0,0,1,0)$ and $k^\mu = (0,0,0,1)$. Tedious, but straightforward algebra tells us that on the horizon we indeed have.

$$\begin{aligned} h^2 &= \rho_+^2 \\ k^2 &= \frac{8(GM)^2(2GMr_+ - a^2)\sin^2\theta}{2GMr_+ - a^2\sin^2\theta} \\ h\cdot k &= h\cdot\ell = k\cdot\ell = 0 \end{aligned} \tag{13.54}$$

Thus ℓ is normal to the horizon and null, making the horizon a **null hypersurface**.

Consider a slice of the null hypersurface with $t = c^{\text{te}}$. Eq. (13.47) becomes

$$ds^2 = \rho_+^2 d\theta^2 + \frac{\Sigma_+^2}{\rho_+^2}\sin^2\theta d\phi^2 \tag{13.55}$$

This is clearly the metric of a squashed two-sphere. As an example, for an extremal black hole, one can calculate the circumference of a fixed circle along the longitude through the poles and this turns out to be $\approx 3.82(2GM)$, which is well below the circumference around the equator of $2\pi(2GM)$. This confirms our intuition that spacetime is squashed around a rotating black hole. One can work out the area of the squashed sphere

$$A = \int d\theta\, d\phi\, \sqrt{-g} = 4\pi(r_+^2 + a^2) \tag{13.56}$$

which gives

$$A = 8\pi\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right] \tag{13.57}$$

As a last comment, note that for the Kerr metric $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta\sin^2\theta$ and this vanishes at the horizon as $\Delta\big|_{r=r_+} = 0$. But as we saw in (13.28) that the horizon is also the location where the co-rotating and counter-rotating light rays have the same angular velocity, $\Omega_+^{(d)} = \Omega_-^{(d)} = \omega$, which turns out to be

$$\Omega_H = \omega\big|_{r_+} = \frac{a}{2GMr_+} = \frac{J}{2GM\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]} \tag{13.58}$$

13.8 The Surface Gravity of the Kerr Black Hole

Recall that we can associate with every Killing horizon, such as the event horizon, a surface gravity, which can then be interpreted as the acceleration of a static observer on the horizon as measured by an observer far away. In order to calculate the surface gravity we need a null Killing vector on the horizon. A general Killing vector of the Kerr metric is of the form $k^\mu = (1, 0, 0, \Omega)$. We then have

$$k^2 = g_{\mu\nu} k^\mu k^\nu = g_{tt} + \Omega^2 g_{\phi\phi} + 2\Omega g_{t\phi} \tag{13.59}$$

In order to proceed it is easiest to use the following form of the Kerr metric

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - a dt\right]^2 \tag{13.60}$$

from which we get

$$\begin{aligned} g_{tt} &= \frac{a^2\sin^2\theta - \Delta}{\rho^2} \\ g_{\phi\phi} &= \frac{\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\sin^2\theta}{\rho^2} \\ g_{t\phi} &= \frac{(\Delta - r^2 - a^2)a\sin^2\theta}{\rho^2} \end{aligned} \tag{13.61}$$

We then find

$$k^2 = \rho^{-2}\left\{-\Delta + a^2\sin^2\theta + 2a(\Delta - r^2 - a^2)\sin^2\theta\Omega + \left[(a^2+r^2)^2 - a^2\Delta\sin^2\theta\right]\sin^2\theta\Omega^2\right\} \tag{13.62}$$

On the horizon, this gets considerably simplified as $\Delta\big|_{r_+} = 0$. We then get an equation for Ω

$$0 = a^2 - 2a(r_+^2 + a^2)\Omega + (a^2 + r_+^2)^2\Omega^2 = \left[(a^2+r_+^2)\Omega - a\right]^2 \tag{13.63}$$

And thus we have a single solution, $\Omega = a/(r_+^2 + a^2)$. The null Killing vector on the horizon is thus

$$k^\mu = \left(1, 0, 0, \frac{a}{r_+^2 + a^2}\right) \tag{13.64}$$

We now use (12.25) to calculate the surface gravity

$$\kappa^2 = -\frac{g^{\mu\nu}\partial_\mu k^2 \partial_\nu k^2}{4k^2} \tag{13.65}$$

This is a tedious calculation, best performed by Mathematica. We show the relevant code for illustration purposes.

```
(* Surface gravity for Kerr metric *)
ClearAll[dim, x, gLinput, gUinput, gL, gU, G, J, a, M, k2, ρ2, Δ, sol,
  KUinp, kU, kL, rS, rp, rsp, a3];
dim = 4;
Δ = x[2]^2 + a^2 - rS x[2];
ρ2 = x[2]^2 + a^2 Cos[x[3]]^2;
sol = {x[1] → t, x[2] → r, x[3] → θ, x[4] → ϕ};
gLinput = {{-(1 - x[2] rS / ρ2), 0, 0, -rS a x[2] × Sin[x[3]]^2 / ρ2},
{0, ρ2 / Δ, 0, 0}, {0, 0, ρ2, 0},
{- rS a x[2] × Sin[x[3]]^2 / ρ2, 0, 0,
    (x[2]^2 + a^2 + rS a^2 x[2] × Sin[x[3]]^2 / ρ2) Sin[x[3]]^2}};
gUinput = Inverse[gLinput];
gL[i_, j_] := gLinput[[i, j]]
gU[i_, j_] := gUinput[[i, j]]
G[i_, b_ , c_] :=
 Simplify[
  (1/2) *
   Sum[gU[i, d] * (D[gL[c, d], x[b]] + D[gL[b, d], x[c]]  - D[gL[c, b], x[d]]),
    {d, dim}]]
rp = rS / 2 + (1 / 2) Sqrt[rS^2 - 4 a^2];
kUinp = {1, 0, 0, Ω};
kU[i_] := Simplify[kUinp[[i]]]
kL[i_] := Simplify[Sum[gL[i, j] × kU[j], {j, dim}]]
k2 =  Simplify[Sum[gL[i, j] × kU[i] × kU[j], {i, dim}, {j, dim}] ];
(* surface gravity general r *)
sg2 =
  -Simplify[Sum[gU[i, j] × D[k2, x[i]] × D[k2, x[j]] / (4 k2), {i, dim}, {j, dim}]] /.
   sol;
(* surface gravity on horizon *)
sg1 = Simplify[Simplify[sg2 /. r → rP /. Ω → a / (rP^2 + a^2) /. rP → (1/2) (rS + V)] /.
    V → Sqrt[rS^2 - 4 a^2]];
sg = Simplify[sg1 /. rS → 2 G M /.  a → J / (G M)]
```

The result on the horizon $r = r_+$ turns out to be

$$\kappa^2 = \frac{(GM)^4 - J^2}{4(GM)^2 \left[2(GM)^4 - J^2 + 2(GM)^2\sqrt{(GM)^4 - J^2}\right]} \tag{13.66}$$

From this we find that

$$\kappa = \frac{\sqrt{(GM)^4 - J^2}}{2GM \left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]} \tag{13.67}$$

Note that the surface gravity is independent of θ. This would be an obvious result for a spherically symmetric metric such as the Schwarzschild, but it is not a priori obvious for a cylindrically symmetric solution such as the Kerr metric. In fact there is a general theorem that the surface gravity is constant over the Killing horizon.

13.8 The Penrose Process

Let us briefly review what we have discovered so far about the Kerr black hole. There are two types of hypersurfaces of importance.

1. **The Stationary Surface** of infinite redshift, defined by $g_{tt} = 0$ and given by

$$r_{S\pm} = GM \pm \sqrt{(GM)^2 - a^2 \cos^2\theta} \tag{13.68}$$

   where the light rays stand still as viewed by a static observer far away.

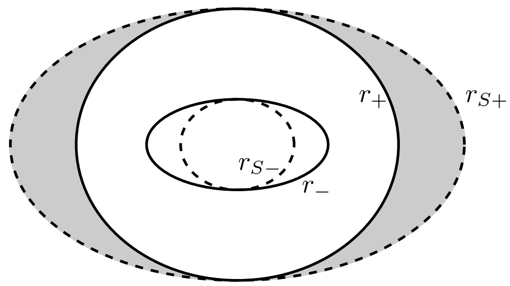
2. **The Event Horizon**, defined by $g^{rr} = 0$ and given by[4]

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2} \tag{13.69}$$

   beyond which all particles and light remain trapped.

Nota that for the Schwarzschild metric $g_{tt} = -g^{rr}$ and so both types of surfaces coincide, $r_S = 2GM$ and $r = 0$. Let us now focus on the outer surfaces as these are the ones we reach first when we move towards the black hole from far away. Clearly $r_{S+} \geq r_+$ with the equality only valid at the poles, $\theta = 0$. A side view of the Kerr black hole was given in fig. 13.6 which we copy here for convenience.

[4]Recall that this is indeed the general condition the event horizon, see (12.2). It just so happens that this is equivalent for the Kerr metric to $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$.



The shaded area between the stationary surface and the event horizon is called the **Ergoregion**, or sometimes, somewhat abusively, the **Ergosphere**.

Energy

The existence of this region leads to new physics, such as the **Penrose Process** which is a mechanism by which we can extract energy from the black hole. As we will see, Hawking radiation is also a process to extract energy from a black hole, but that is an inherently quantum mechanical process. The Penrose process, on the contrary, is basically classical.

The key observation that allows the Penrose process is that g_{tt} flips signs when crossing into the ergosphere from outside. Thus the coordinate t becomes spacelike and energy, which is the conserved quantity associated with the time translation Killing vector $k_\varepsilon = \partial_t$ becomes rather a momentum, and can thus become negative in that region.

Let us therefore imagine a particle $\mathbf{1}$ with energy $E_\mathbf{1}$ that comes from far away and moves into the ergoregion and there decays into two particles $\mathbf{1} \to \mathbf{2}+\mathbf{3}$. We of course have momentum and energy conservation in that process. But as just explained, "energy" can be negative in the ergoregion, so let us assume that particle $\mathbf{2}$ has negative energy $E_\mathbf{2} < 0$ and that this particle then crosses the event horizon towards the singularity as $r = 0$, whilst particle $\mathbf{3}$ moves out of the stationary surface to infinity with $E_\mathbf{3} > 0$. Conservation of energy dictates that $E_\mathbf{1} = E_\mathbf{2} + E_\mathbf{3}$ or hence

$$E_\mathbf{3} = E_\mathbf{1} - E_\mathbf{2} > E_\mathbf{1} \tag{13.70}$$

so we are putting a particle with energy $E_\mathbf{1}$ into the black hole but are getting out a particle with a higher energy $E_\mathbf{3}$, hence we are extracting energy from the black hole.

Angular Momentum

We can follow a similar reasoning for angular momentum. Consider an observer inside the ergoregion with velocity

$$U^\mu = U^0(1, 0, 0, \Omega_{\text{obs}}) \tag{13.71}$$

Thus $dt/d\tau = U^0$ and $d\phi/d\tau = U^0\Omega_{\text{obs}}$. Since in the ergoregion the observer has to rotate in the same direction of the black hole, we must have that Ω_{obs} is positive. Let us now rewrite

$$U^\mu = U^0(k_\varepsilon + \Omega_{\text{obs}} k_\ell) \tag{13.72}$$

where $k_\varepsilon = (1, 0, 0, 0)$ and $k_\ell = (0, 0, 0, 1)$ are the time translation and angular rotation Killing vectors respectively.

Recall that the energy of a particle as seen by that observer is $-U \cdot p$, with p the momentum of the particle. For the decay process $\mathbf{1} \to \mathbf{2} + \mathbf{3}$ we have for particle $\mathbf{2}$ that

$$\begin{aligned} E_{\mathbf{2}\,\text{obs}} =& -U \cdot p_{\mathbf{2}} = -U^0(k_\epsilon + \Omega_{\text{obs}} k_\ell) \cdot p_{\mathbf{2}} \\ =& U^0(\varepsilon_{\mathbf{2}} - \Omega_{\text{obs}} L_{\mathbf{2}}) \end{aligned} \tag{13.73}$$

with $L_{\mathbf{2}} = +\Omega_{\text{obs}} k_\ell$ is the angular momentum of particle $\mathbf{2}$. The energy of particle $\mathbf{2}$ as measured by the observer in the ergoregion must be positive, thus so must be the RHS of (13.73) and hence $\varepsilon_{\mathbf{2}} - \Omega_{\text{obs}} L_{\mathbf{2}}$ must be positive too. As Ω_{obs} is positive by construction we thus have

$$\frac{\varepsilon_{\mathbf{2}}}{\Omega_{\text{obs}}} \geq L_{\mathbf{2}} \tag{13.74}$$

As a Penrose process has $\varepsilon_{\mathbf{2}}$ negative in the ergoregion, we thus necessarily also have $L_{\mathbf{2}}$ negative. If particle $\mathbf{2}$ moves across the event horizon, conservation of angular momentum teaches us that the the angular momentum of the black hole reduces. But also its mass reduces as $\varepsilon_{\mathbf{2}} < 0$. Thus for the black hole

$$\delta M = \varepsilon_{\mathbf{2}} < 0 \qquad \text{and} \qquad \delta J = L_{\mathbf{2}} < 0 \tag{13.75}$$

Thus by this process the black hole loses mass and angular momentum and this with a relation

$$\delta M \geq \Omega_{\text{obs}} \delta J \tag{13.76}$$

Thus by extracting energy from the black hole, we also reduce its angular momentum, eventually leading to a Schwarzschild black hole. Note the inequality (13.76) remains valid even if we have no Penrose process, i.e. if $\varepsilon_{\mathbf{2}}$ is positive.

13.9 The Second Law of Black Hole Thermodynamics

Recall that we worked out the area of a Kerr black hole in (13.57)

$$A = \int d\theta\, d\phi\, \sqrt{-g} = 4\pi(r_+^2 + a^2) = 8\pi \left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right] \tag{13.77}$$

How does this change in a Penrose process were both M and J decrease? Clearly

$$
\begin{aligned}
(8\pi)^{-1}\delta A &= G^2 2M\delta M + \frac{G^4 4M^3\delta M - 2J\delta J}{2\sqrt{(GM)^4 - J^2}} \\
&= \frac{2G^2M\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]\delta M - J\delta J}{\sqrt{(GM)^4 - J^2}} \\
&= \frac{2J}{\sqrt{(GM)^4 - J^2}}\left(\Omega_H^{-1}G\delta M - \delta J\right)
\end{aligned}
\tag{13.78}
$$

where we have used (13.58) to introduce Ω_H. But since (13.76) holds for any observer, it must also hold for an observer just outside the horizon, whose angular velocity is precisely Ω_H. And as we mentioned this inequality holds in general, not only for the Penrose process. We thus conclude that as a general rule

$$
\delta A \geq 0 \tag{13.79}
$$

the surface area of a Kerr black hole cannot decrease. Take a moment to consider this. Whatever happens the area of the black hole cannot decrease. Clearly this reminds us of the second law of thermodynamics, stating that entropy in a closed system cannot decrease. That is why the area law $\delta A \geq 0$ is known as the **Second Law of Black Hole Thermodynamics** and is an indication that the surface of a black hole could be considered as an entropy.

13.10 The First Law of Black Hole Thermodynamics

Now that we have found something that resembles the second law of black hole thermodynamics, it seems to make sense to ask whether there is also such a thing as the first law of black hole thermodynamics.

Our starting point is the formula for the conserved energy (13.30)

$$
-p_t = \frac{g_{t\phi}}{g_{\phi\phi}}p_\phi + \sqrt{\frac{1}{g_{\phi\phi}^2}\left[\left(g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right)p_\phi^2 + m^2(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\left(g_{rr}p_r^2 + g_{\theta\theta}p_\theta^2 + 1\right)g_{\phi\phi}\right]} \tag{13.80}
$$

which we rewrite as

$$
-p_t = \omega p_\phi + \sqrt{\frac{Dp_\phi^2}{g_{\phi\phi}^2} + \frac{K}{g_{\phi\phi}}} \tag{13.81}
$$

where as usual $\omega = g_{t\phi}/g_{\phi\phi}$ and

$$D = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$$
$$K = Dm^2 \left[g_{rr} \left(\frac{dr}{d\tau}\right)^2 + g_{\theta\theta} \left(\frac{d\theta}{d\tau}\right)^2 + 1 \right] \tag{13.82}$$

Note that K should not be confused with the Kretschmann invariant.

Since this is a conserved quantity on geodesics, we can calculate it on any point of the geodesic, e.g. on the event horizon $r = r_+$. For the Kerr metric $D = \Delta \sin^2\theta$ so that $D\big|_{r_+} = 0$. But we need to be careful as other factors may blow up on the event horizon. Indeed $g_{rr} = \rho^2/\Delta$ does and hence $Dg_{rr}\big|_{r_+} = \rho_+^2 \sin^2\theta$. The other contributions in the square root all vanish so that, using $g_{\phi\phi} = (\Sigma^2/\rho^2)\sin^2\theta$ we have

$$-p_t = \Omega_H L + \sqrt{\left[\frac{\rho_+^2}{\Sigma_+} m \left(\frac{dr}{d\tau}\right)^2\right]} \tag{13.83}$$

where we have used $\omega\big|_{r_+} = \Omega_H$ and $L = p_\phi$. Taking the variation of that we find that

$$\delta M = \Omega_H \delta J + \frac{r_+^2 + a^2\cos^2\theta}{r_+^2 + a^2}\, m \left|\frac{dr}{d\tau}\right|_{r_+} \tag{13.84}$$

This gives us an interesting interpretation of the inequality $\delta M \geq \Omega_{\text{obs}}\delta J$ we derived in (13.76), and which led to the second law of black hole thermodynamics $\delta A \geq 0$. Here we see that this inequality is entirely driven by the radial movement $|dr/d\tau|$.

Eq. (13.84) also explains our observation in (13.44) that $J \leq GM^2$ with the equality valid for an extremal black hole. We easily work out that

$$\Omega_H = \frac{J}{2GM\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]} \tag{13.85}$$

For an extremal black hole this becomes

$$\Omega_H^e = \frac{1}{2GM} \tag{13.86}$$

Let us now consider the inequality (13.76) for an extremal black hole:

$$\delta M \geq \Omega_H^e \delta J = \frac{\delta J}{2GM} \qquad \Rightarrow \qquad \delta(GM^2) \geq \delta J \tag{13.87}$$

So if we have an extremal black hole, $J = GM^2$ and try as we may to increase both the mass and the angular momentum of the black hole from M and J by δM and δJ respectively, we will always have $GM^2 + \delta(GM^2) \geq J + \delta J$ and so we cannot violate the inequality $J \leq GM^2$.

Note now that we can write (13.78) as

$$\delta M = \frac{\sqrt{(GM)^4 - J^2}}{16\pi GM\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]}\,\delta A + \Omega_H \delta J \tag{13.88}$$

which we can write as

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \tag{13.89}$$

where κ turns out to be the surface gravity as computed in (13.67).

Eq. (13.89) is very suggestive of thermodynamics. We already know that we could identify the area of the black hole A with an entropy S and clearly we can identify the mass of the black hole M with its energy. This suggest that $\tilde{\kappa}/8\pi$ is to be viewed as a temperature and $\Omega_H \delta J$ as work dW we then get the usual

$$dE = TdS + dW = TdS - pdV \tag{13.90}$$

with the correspondence

black hole			**thermodynamics**
mass	M	E	energy
area	A	S	entropy
surface gravity	κ	T	temperature
angular velocity	$\Omega_H J$	W	work

Not surprisingly, (13.89) is known as the **First Law of Black Hole Thermodynamics**. . Note that Ω_H and J can be viewed as conjugate variables, just as p and V are in thermodynamics. Note also that the normalisation between the temperature and entropy definition is not yet fixed. Hawking radiation will give a natural definition for the temperature that will fix this normalisation.

13.11 Reversible Process for Black Holes

In thermodynamics a reversible process is defined by $dS = 0$. We can likewise define a process here as having $\delta A = 0$ to be reversible. It then follows from (13.89) that for such a process

$$\delta M = \Omega_H \delta J = \frac{J\delta J}{2GM\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right]} \tag{13.91}$$

We can then write this as

$$\left[(GM)^2 + \sqrt{(GM)^4 - J^2}\right] d(GM)^2 = \frac{1}{2G} dJ^2 \tag{13.92}$$

Setting $G = 1$ for convenience and $M^2 = x$ and $J^2 = j$ this becomes

$$\left(x + \sqrt{x^2 - j}\right) dx = \frac{1}{2} dj \tag{13.93}$$

We now show that this is solved by

$$x_0 = x + \sqrt{x^2 - j} \tag{13.94}$$

for a constant x_0. Indeed solving this for j gives $j = 2x_0 x - x_0^2$ so that $dj = 2x_0 dx$. But we can also write (13.93) as $x_0 dx = (1/2)dj = (1/2)2x_0 dx = x_0 dx$. Reverting back to M and J we thus see that a reversible process $dA = 0$ satisfies

$$2M_0^2 = M^2 + \sqrt{M^4 - J^2} \tag{13.95}$$

where M_0 is an integration constant and corresponds to the mass of the Schwarzschild black hole in the limit that J vanishes. In fact we can start with a Kerr black hole and use the Penrose process to reduce both M and J whilst ensuring that $|dr/d\tau|_{r_+}$ remains positive at all times. When $J = 0$ we will then end with a Schwarzschild black hole of mass M_0. Conversely, we can start with a Schwarzschild black hole and throw in mass that increases its angular momentum, and we will end with an extremal Kerr black hole with mass $\sqrt{2}M_0$.

13.12 Closed Timelike Curves and the Extended Kerr Spacetime

Another surprising property of the ergoregion is the existence of **Closed Timelike Curves** or **CTC**s. Indeed in the Kerr spacetime there is a region that observers can access by following a timelike geodesic that passes through $r = 0$ but with $\theta \neq \pi/2$, hence avoiding the singularity, corresponding to $r < 0$ in the original coordinates. Let us now imagine the observer is in that region and on a geodesic with constant t and r and with $\theta = \pi/2$. His metric then becomes

$$ds^2 = \left(r^2 + a^2 + \frac{2GMa^2}{r}\right) d\phi^2 \tag{13.96}$$

For $r < 0$ and small enough this metric component can become negative and hence is timelike. But ϕ is a periodic coordinate $\phi \equiv \phi + 2\pi$, which means that moving along such a curve, would bring one back to the same time!

Unfortunately, all this happens to an observer who has crossed the event horizon, so she will not be able to tell us mere mortals. But it is an interesting fact that CTCs are theoretically possible. However, common sense, or maybe healthy conservatism, reigns amongst most general relativists as there is a general belief, that, just as for naked singularities, one should not have CTCs when starting with benign initial conditions. This belief is known as the **Chronology Protection Conjecture**.

Note that in the extended Kerr spacetime we still have

$$g^{rr} = \frac{r^2 + a^2 - 2GMr}{\rho^2} \tag{13.97}$$

and as we now have $r < 0$ we see that the numerator is strictly positive so there is no solution for $g^{rr} = 0$ and hence this extension of the Kerr spacetime has no event horizon.

13.13 Superradiance

We now show that it is possible for a matter field to scatter of a Kerr black hole and end up with more energy that it started with. We consider a scalar field $\Phi(x)$ with energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \Phi \nabla_\rho \Phi \tag{13.98}$$

From (5.13) we know that given a Killing vector K^μ we can construct a new conserved current

$$J^\nu = K_\mu T^{\mu\nu} \tag{13.99}$$

with conserved charge

$$E_\Sigma = \int_\Sigma J^\mu \, n_\mu \sqrt{\gamma} \, d^3x \tag{13.100}$$

where γ_{ij} is the induced metric on a spacelike hypersurface Σ and n^ν is the normal to Σ.

Let us now integrate $\nabla_\mu J^\mu = 0$ over a region bounded by two spacelike hypersurfaces Σ_1 and Σ_2, a part of the event horizon $\mathcal{H}$ and lightlike future infinity i^0 as shown in fig. 13.7.

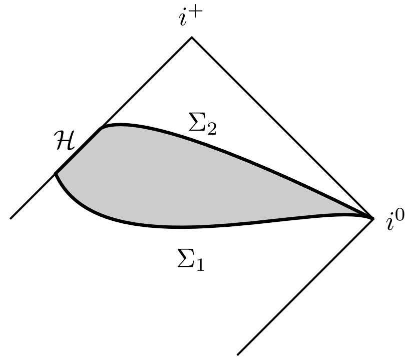


Figure 13.7: Spatial integration region outside a Kerr black hole

Because $\nabla_\mu J^\mu = 0$ we thus have, using Stokes theorem and assuming that $\nabla_\mu \Phi$ vanishes at infinity,

$$0 = \int d^4x \sqrt{-g}\, \nabla_\mu J^\mu = \int_{\Sigma_2} d^3x \sqrt{\gamma_2}\, n_1^\mu J_\mu - \int_{\Sigma_1} d^3x \sqrt{\gamma_1}\, n_1^\mu J_\mu + \int_{\mathcal{H}} d^3 S_\mu J^\mu \qquad (13.101)$$

where γ_i and n_i^μ are the induced metric and normal to the hypersurface Σ_i and $\mathcal{H}$ is the relevant part of the event horizon. Using the definition of the conserved charge (13.100) we can thus write

$$E_{\Sigma_2} - E_{\Sigma_1} = -\int_{\mathcal{H}} d^3 S_\mu J^\mu \qquad (13.102)$$

We now introduce **Kerr Coordinates** (v, r, θ, χ) defined by[5]

$$\begin{aligned} v &= t + r_* \\ dr_* &= \frac{r^2 + a^2}{\Delta} dr \\ d\chi &= d\phi + \frac{a}{\Delta} dr \end{aligned} \qquad (13.104)$$

[5]In These Kerr coordinates the metric becomes

$$ds^2 = -\frac{\Delta}{\rho^2}(dv - a\sin^2\theta d\chi)^2 + \frac{\sin^2\theta}{\rho^2}[adv - (r^2+a^2)d\chi)^2 + (dv - a\sin^2\theta d\chi)dv + \rho^2 d\theta^2 \quad (13.103)$$

and null in-falling geodesics are now characterised by $dv = d\chi = d\theta = 0$. Recall that in the (t, r, θ, ϕ) coordinates light rays do not follow strictly radial geodesics as they get twisted by the rotation of the black hole. Notice also that with the Kerr coordinates the coordinate singularity at $\Delta = 0$ is not present.

This allows us to rewrite (13.102) as

$$E_{\Sigma_2} - E_{\Sigma_1} = -\int d^2A\, dv\, k_\mu J^\mu \tag{13.105}$$

where k^μ is the null Killing vector along the horizon and d^2A is the spatial cross-section of the horizon.

The power absorbed by the black hole per unit null time v is given by

$$\mathcal{P} = -\int d^2A\, k_\mu J^\mu \tag{13.106}$$

with

$$k_\mu J^\mu = -(K^\mu \nabla_\mu \Phi)(k^\nu \nabla_\nu \Phi) + \frac{1}{2}(\nabla_\rho \Phi)(\nabla^\rho \Phi)(k_\mu K^\mu) \tag{13.107}$$

Now $k_\mu K^\mu = 0$ as k is a null Killing vector orthogonal to the horizon and K is the Killing vector defining the horizon. As we saw in (13.64) the null Killing vector on the horizon is given by

$$k^\mu = (1, 0, 0, \Omega) \tag{13.108}$$

with $\Omega = a/(r_+^2 + a^2)$. We can replace the covariant derivatives by ordinary derivatives as Φ is a scalar and find for $\mathcal{P}$

$$\mathcal{P} = \int d^2A\,(K^\mu \nabla_\mu \Phi)(k^\nu \nabla_\nu \Phi) = \int d^2A\, \partial_v \Phi(\partial_v \Phi + \Omega \partial_\chi \Phi) \tag{13.109}$$

Let us now expand the scalar field in angular momentum modes and consider one mode

$$\Phi = \Phi_0(r, \theta)\, \cos(\omega v - \nu \chi) \tag{13.110}$$

with ω positive. As χ is a periodic coordinate, the angular momentum ν is quantised, $\nu \in \mathbb{Z}$. We then have

$$\begin{aligned} \mathcal{P} &= \int d^2A\, \Phi_0^2\, [-\omega \sin(\omega v - \chi\nu)]\, [-\omega \sin(\omega v - \chi\nu) + \Omega\nu \sin(\omega v - \chi\nu)] \\ &= \int d^2A\, \Phi_0^2\, \omega(\omega - \Omega\nu) \sin^2(\omega v - \chi\nu) \end{aligned} \tag{13.111}$$

and we can take the time averaged power absorbed by the black hole by integrating over v

$$\bar{\mathcal{P}} = \frac{1}{2}\int d^2A\,\Phi_0^2\,\omega(\omega - \Omega\nu) \tag{13.112}$$

We see that for high frequencies ω the power $\bar{\mathcal{P}}$ is positive, so the black hole absorbs energy. However $\bar{\mathcal{P}}$ can be negative for small frequencies ω and so the field Φ has extracted energy from the horizon. This is the process of **Superradiance**.

Note that superradiance can only occur if $\nu \neq 0$. This is because the amplified field also needs to take angular momentum from the black hole, as we have seen earlier.

Note also that this process is reminiscent of stimulated emission in atomics physics. One might then well wonder if there also exists a process of spontaneous emission. This is possible in the quantum theory and it turns out that any black hole with an ergoregion cannot be quantum mechanically stable.

Note that we have not incorporated the impact of the field Φ on the spacetime, i.e. we have assumed the Kerr metric to be valid, It turns out that when corrected for this the metric can only be stationary if $\partial_\phi \Phi = 0$, but then $J^\mu = 0$ and the black hole energy doesn't change at all. This means that in reality superradiance is incompatible with a stationary metric.

Chapter 14

The Charged or Reissner-Nordström Black Hole

We now turn our attention to the extension of the Schwarzschild black hole to include an electric charge. This is actually a purely theoretical exercise as it it unlikely that a black hole with a considerable electric charge exists, as it would quickly engulf particles of the opposite charge to make it neutral.

14.1 The Reissner-Nordström Metric

We need a solution of the Einstein equations not in vacuum, but in the presence of an electromagnetic field. The relevant action is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \tag{14.1}$$

and the equations of motion are the Maxwell equations

$$\nabla^\mu F_{\mu\nu} = 0 \tag{14.2}$$

together with the Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T^{(e)}_{\mu\nu} \tag{14.3}$$

where $T^{(e)}_{\mu\nu}$ is the energy-momentum tensor for the electromagnetic field

$$T^{(e)}_{\mu\nu} = F_\mu{}^\sigma F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \tag{14.4}$$

We are looking for static, spherically symmetric solutions, hence a metric of the form

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{14.5}$$

Solving the Maxwell Equations

Use as Ansatz for a static spherical symmetric solution of the Maxwell equations, we take as non-zero components

$$F_{tr} = -F_{rt} = E(r) \qquad \text{and} \qquad F_{\theta\phi} = -F_{\phi\theta} = B(\theta) \tag{14.6}$$

so that

$$F^{tr} = -F^{rt} = \frac{E(r)}{a(r)b(r)} \qquad \text{and} \qquad F^{\theta\phi} = -F^{\phi\theta} = \frac{B(\theta)}{r^4 \sin^2\theta} \tag{14.7}$$

Using $\sqrt{-g} = r^2 \sin\theta\,\sqrt{ab}$ and $\nabla_\mu F^{\mu\nu} = (-g)^{-1/2}\partial_\mu(\sqrt{-g}F^{\mu\nu})$ we then find for the Maxwell equation with $\nu = t$

$$0 = \partial_\mu(\sqrt{-g}F^{\mu t}) = \partial_r(\sqrt{-g}F^{rt}) = \partial_r\left[r^2 \sin\theta\,\sqrt{ab}\frac{E(r)}{a(r)b(r)}\right] \tag{14.8}$$

or hence $0 = \partial_r(r^2 E/\sqrt{ab})$. We can easily write a solution for this as

$$E(r) = \frac{Q_e\sqrt{ab}}{r^2} \tag{14.9}$$

Similarly for $\nu = \phi$

$$0 = \partial_\mu(\sqrt{-g}F^{\mu\phi}) = \partial_\theta(\sqrt{-g}F^{\theta\phi}) = \partial_\theta\left[r^2 \sin\theta\,\sqrt{ab}\frac{B(\theta)}{r^4 \sin^2\theta}\right] \tag{14.10}$$

or $\partial_\theta(B(\theta)/\sin\theta) = 0$. Which has as solution

$$B(\theta) = Q_m \sin\theta \tag{14.11}$$

Here Q_E and Q_m are constants whose interpretation has to be determined, but you will be correct if you guess them to be related to the electric and magnetic charge.

Solving The Einstein-Maxwell Equations

It is actually not too hard to solve the Einstein-Maxwell equations, but will just give the result, leaving it to the reader to check. The solution is

$$a(r) = b(r)^{-1} = 1 - \frac{2GM}{r} - \frac{4\pi G e^2}{r^2} \tag{14.12}$$

with

$$e^2 = Q_e^2 + Q_m^2 \tag{14.13}$$

Thus the **Reissner-Nordström Metric** is given by

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{4\pi Ge^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{4\pi Ge^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{14.14}$$

The non-zero components of the electromagnetic tensor are given by

$$F_{tr} = -F_{rt} = \frac{Q_e}{r^2} \quad \text{and} \quad F_{\theta\phi} = -F_{\phi\theta} = Q_m \sin\theta \tag{14.15}$$

and we recognise that at infinity, where spacetime is asymptotically flat, Q_e represents the electric charge and Q_m the magnetic charge.[1]

The non-zero components of the energy-momentum tensor are given by

$$\begin{aligned} T_{tt}^{(e)} &= \frac{e^2[4\pi Ge^2 + r(r-2GM)]}{2r^6} \\ T_{rr}^{(e)} &= -\frac{e^2}{2r^2[4\pi Ge^2 + r(r-2GM)]} \\ T_{\theta\theta}^{(e)} &= \frac{e^2}{2r^2} \\ T_{\phi\phi}^{(e)} &= \frac{e^2\sin^2\theta}{2r^2} \end{aligned} \tag{14.16}$$

One easily checks that the energy-momentum tensor is traceless, $g^{\mu\nu}T_{\mu\nu}^{(e)} = 0$.

The curvature of the metric is zero and the Kretschmann invariant is given by

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{16G^2(56\pi^2e^2 - 24\pi e^2 Mr + 3M^2r^2)}{r^8} \tag{14.17}$$

This shows that $r = 0$ is a real singularity.

[1] One may set $Q_m = 0$ is one is convinced no magnetic monopoles exist.

14.2 The Komar Integrals for the Reissner-Nordström Black Hole

Let us evaluate the electric charge, energy and angular momentum Komar integrals for the Reissner-Nordström metric. They are given by (12.36), (12.44) and (12.48) respectively

$$\begin{aligned} Q =& -\int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu F^{\mu\nu} d^2x \\ E_R =& \frac{1}{4\pi G}\int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu d^2x \\ J_R =& -1\frac{1}{8\pi G}\int_{\partial\Sigma} \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu R^\nu d^2x \end{aligned} \tag{14.18}$$

We refer to section 13.2 for details.

Charge

We find

$$n_\mu \sigma_\nu F^{\mu\nu} = \frac{Q_e}{r^2} \tag{14.19}$$

and therefore

$$Q_E = \lim_{r\to\infty}\int r^2 \sin\theta d\theta d\varphi \frac{Q_e}{r^2} = 2\pi Q_e \tag{14.20}$$

So Q_E is proportional to the electric charge of the black hole.

To deduct the magnetic charge, we need to find the Komar integral for the dual field tensor ${}^*F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. This leads to

$$n_\mu \sigma^*_\nu F^{\mu\nu} = -\frac{2Q_m}{r^4 \sin\theta} \tag{14.21}$$

giving

$$Q_M = \lim_{r\to\infty}\int r^2 \sin\theta d\theta d\varphi \left(-\frac{2\pi Q_m}{r^2}\right) = 0 \tag{14.22}$$

At infinity the magnetic charge disappears, but is proportional to Q_m.

Energy

We find

$$n_\mu \sigma_\nu \nabla^\mu K^\nu = -\frac{4\pi G e^2}{r^2} + \frac{2GM}{r^2} \tag{14.23}$$

from which we get

$$E_R = \lim_{r\to\infty}\left(2GM - \frac{8\pi Ge^2}{r}\right) = 2GM \tag{14.24}$$

The energy at very large is given by its rest mass plus the electromagnetic energy of the charges, the latter vanishing at infinity.

Angular Momentum

We find

$$n_\mu \sigma_\nu \nabla^\mu R^\nu = 0 \tag{14.25}$$

and thus

$$J_R = 0 \tag{14.26}$$

The Reissner-Nordström has no angular momentum.

14.3 The Horizon of the Reissner-Nordström Black Hole

The condition $g^{rr} = 0$ for a horizon is

$$0 = 1 - \frac{2GM}{r} + \frac{Ge^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2} \tag{14.27}$$

where we have for convenience rescaled the charges $e^2/4\pi \to e^2$ and

$$r_\pm = GM \pm \sqrt{(GM)^2 - Ge^2} \tag{14.28}$$

It is clear that there are three regimes depending on the sign of $(GM)^2 - Ge^2$.

The Naked Black Hole: $e^2 > GM^2$

In this case $r_\pm$ are not real and there is no horizon. The time coordinate remains timelike all the way from infinity to the singularity at $r = 0$. We thus have a naked singularity. Note that this does not violate the cosmic censorship conjecture that we discussed in section 12.5. Indeed that conjecture states that a spacetime obeying the dominant energy condition cannot form a naked singularity. But in this case the naked singularity is not formed, it has always been there.

At large distances the solution approaches flat spacetime. The conformal diagram will thus be that of Minkowski space fig. 12.1 but with a singularity at the origin, depicted by a wavy line.

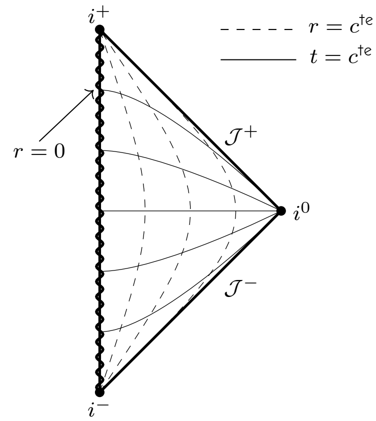


Figure 14.1: Conformal diagram for a Reissner-Nordström black hole: $e^2 > GM^2$

Further analysis of geodesics reveals that the singularity is repulsive: timelike geodesics never intersect $r = 0$, they just scatter of it. Null geodesics can reach the singularity, and, of course, so can timelike curves that are non-geodesic.

The Subextremal Black Hole: $e^2 < GM^2$

Start with considering $e = 0$, which gives a Schwarzschild black hole. As we approach the singularity from infinity g_{tt} changes sign as we cross the horizon at $r = 2GM$, hence the coordinate t changes from timelike to spacelike. Let now $e \neq 0$. The time coordinate becomes spacelike as we cross $r = r_+$, but changes again to timelike when we cross $r = r_-$ and the singularity at $r = 0$ is a timelike point.

Let us consider the road travelled by an observer falling from far way towards the singularity.

- As the observer approaches r_+ it will be just like a Schwarzschild black hole. She will cross it and r will become a timelike coordinate and she will fall further towards the singularity. An external observer far away will never see her cross the event horizon as is the case with the Schwarzschild black hole.
- As the in-falling observer reaches r_-, the coordinate r becomes spacelike again; the movement towards the origin can be stopped and she does not have to hit the singularity.
- In fact, she can move back towards and cross the horizon at r_-. The coordinate r becomes timelike once more, but with opposite orientation – i.e. moving in the

lightcone facing downwards – and so she can cross the r_+ boundary as well. For an external observer it seems like she is appearing out of nowhere, i.e. from white hole.

- The whole process can repeat itself any number of times.

This is illustrated in the conformal diagram below.

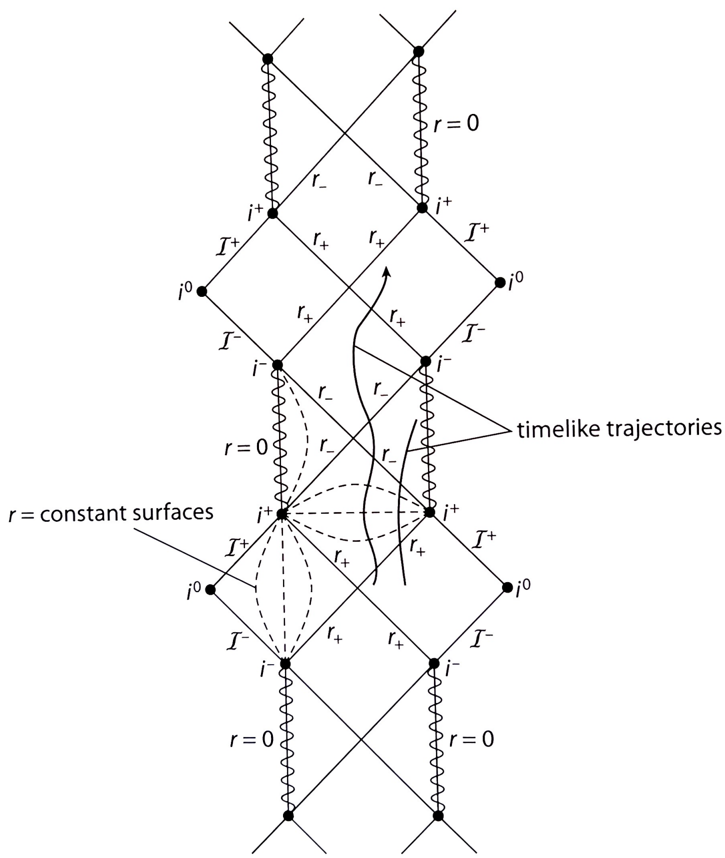


Figure 14.2: Conformal diagram for a Reissner-Nordström black hole: $e^2 < GM^2$

The Extremal Black Hole: $e^2 = GM^2$

In this case $r_+ = r_- = GM$ and there is only one horizon. Consider now two such extremal black holes with equal mass M and electric charge Q_e and no magnetic charge Q_m. Their gravitational attraction GM^2/r^2 exactly matches their electromagnetic repulsion $+Q_e^2/r^2$ so there is no net force between them. So we could have a situation where there are any number of static extremal Reissner-Nordström black holes scattered around spacetime at random locations and they just remain static. This is quite a surprise as the gravitational force is so much weaker than the electric force.

Whilst extremal black holes are inherently unstable because the slightest addition of mass will bring it to the subextremal black hole, it is of some interest in the study of black holes in quantum gravity and certain supersymmetric theories.

Let us consider the geodesic of an in-falling observer. On both sides of the single horizon $r = 2GM$ the r coordinate is spacelike, so you can avoid the singularity at $r = 0$ and move back out and in as you wish. This is illustrated in the conformal diagram below.

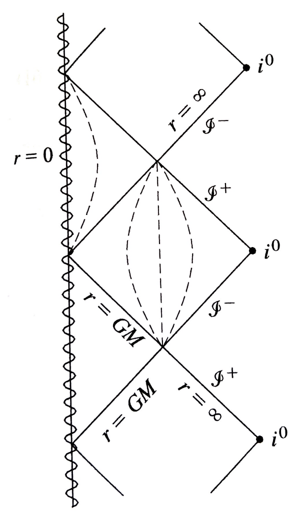


Figure 14.3: Conformal diagram for a Reissner-Nordström black hole: $e^2 = GM^2$

14.4 Rotating Charged Black Holes

Finally, let us mention that there are also solutions to the Einstein-Maxwell equations that describe rotating black holes. We leave the details to the industrious reader.

Chapter 15

More Black Holes

There are many other solutions to the Einstein field equations that display a black hole. Here we summarise some of them.

15.1 The Kottler Black Hole

The metric is

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{15.1}$$

This is a solution of the EFE with cosmological constant $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Note that this spacetime is not asymptotically flat, but is asymptotically de Sitter or anti-de Sitter depending on the sign of Λ. This metric is then known as the **Schwarzschild–de Sitter Metric** for $\Lambda > 0$ and the **Schwarzschild–anti-de Sitter Metric** for $\Lambda < 0$. It has constant curvature

$$R = 4\Lambda \tag{15.2}$$

and the Kretschmann invariant is

$$K = \frac{48(GM)^2}{r^6} + \frac{8\Lambda^2}{3} \tag{15.3}$$

showing that there is a singularity at $r = 0$.

The Komar integrals are

$$E_R = 2GM - \frac{2\Lambda r^3}{3} \qquad \text{and} \qquad J_R = 0 \tag{15.4}$$

The cosmological term gives an extra contribution to the energy that is proportional to the volume of spacetime, here infinite.

This solution can be generalised by adding a charge

$$1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \to 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} + \frac{e^2}{r^2} \tag{15.5}$$

to give a solution of the Einstein-Maxwell field equations, with a constant cosmological constant.

15.2 Topological Black Holes

A somewhat surprising solution to the EFE is given by a generalisation of

$$ds^2 = -\left(k - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(k - \frac{2m}{r} + \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2_{(k)} \tag{15.6}$$

where $k = 0, \pm 1$. For $k = +1$ we recover the Kottler metric. For $k = 0$ and $k = -1$ we need to make some change to the two-sphere metric. For $k = 0$ we need to replace S^2 by flat space $\mathbb{R}^2$ or the torus T^2

$$d\Omega^2_{(0)} = d\theta^2 + d\phi^2 \tag{15.7}$$

and for $k = -1$ we need to replace S^2 by H^2, the two-dimensional hyperbolic plane.

$$d\Omega^2_{(-1)} = d\theta^2 + \sinh^2\theta d\phi^2 \tag{15.8}$$

In order to have the standard interpretation of the coordinates we need $\Lambda < 0$ and so these solutions describe black holes immersed in AdS spacetime with horizons that have non-spherical topology. Confusingly these solutions are knows as topological black holes.[1]

The curvature, Kretschmann invariant and Komar integrals are the same for the $k = 0$ and $k = -1$ cases as for the $k = +1$ case.

15.3 Black Hole Solutions of Einstein-Yang-Mills Equations

Having found the Reissner-Nordström solutions for gravity coupled to electromagnetism, it is logical to look for solutions for gravity coupled to Yang-Mills fields. These are mostly of

[1]Confusingly because topological usually means metric independent.

theoretical interest because, contrary to electromagnetism, Yang-Mills interaction is short range only.

There are however such solutions and they do bring about some new properties. In particular it seems that the no-hair theorems have no good analogues. For example there are solutions with non-trivial Yang-Mills fields but zero Yang-Mills charges. There are also static solutions that are not spherically symmetric

15.4 Regular Black Holes

We have seen in section 12.5 that singularity theorems say that under certain benign conditions event horizons imply the existence of a singularity. Somewhat surprising, if we let go these conditions, even weakly, then it is possible to have event horizons without singularities. We present two such solutions.

Bardeen Black Holes

This describes a metric coupled to a non-linear version of an electromagnetic field

$$ds^2 = -\left[1 - \frac{2mr^2}{(r^2+e^2)^{3/2}}\right] dt^2 + \left[1 - \frac{2mr^2}{(r^2+e^2)^{3/2}}\right]^{-1} dr^2 + r^2 d\Omega^2 \tag{15.9}$$

The curvature and Kretschmann invariants are given by

$$\begin{aligned} R &= \frac{6GMe^2(4e^2 - r^2)}{(e^2+r^2)^{7/2}} \\ K &= \frac{12(GM)^2(4r^8 - 12e^2r^6 + 47e^4r^4 - 4e^6r^2 + 8e^8)}{(e^2+r^2)^{7/2}} \end{aligned} \tag{15.10}$$

and these are indeed an indication that there is no singularity present.

For large r it approaches the Schwarzschild metric, and for small r it approaches de Sitter spacetime, confirming the absence of a singularity

Hayward Black Holes

Another example is provided by the metric

$$ds^2 = -\left[1 - \frac{2mr^3}{(r^4+2mrL^2)}\right] dt^2 + \left[1 - \frac{2mr^3}{(r^4+2mrL^2)}\right]^{-1} dr^2 + r^2 d\Omega^2 \tag{15.11}$$

which solves the Einstein field equations not in vacuum, but with matter having an energy-momentum tensor

$$\begin{aligned} T_{tt} &= \frac{12(GML)^2(r^3 - 2GMr^2 + 2GML^2)}{(2GML^2 + r^3)^3} \\ T_{rr} &= -\frac{12(GML)^2(r^3 - 2GMr^2 + 2GML^2)}{(2GML^2 + r^3)(r^3 - 2GMr^2 + 2GML^2)} \\ T_{\theta\theta} &= -\frac{24(GML)^2 r^2 (GML^2 - r^3)}{(2GML^2 + r^3)^3} \\ T_{\phi\phi} &= -\frac{24(GML)^2 r^2 (GML^2 - r^3)}{(2GML^2 + r^3)^3} \sin^2\theta \end{aligned} \tag{15.12}$$

The singularity at $r = 0$ is regularised by the cut-off L that can be thought of as related to the Planck length. The curvature and Kretschmann invariant are given by

$$\begin{aligned} R &= \frac{24(GML)^2(4GML^2 - r^3)}{(2GML^2 + r^3)^3} \\ K &= \frac{48(GM)^2(r^{12} - 8GML^2 r^9 + 72(GML^2)^2 r^6 - 16(GML^2)^3 r^3 + 32(GML^2)^4)}{(2GML^2 + r^3)^3} \end{aligned} \tag{15.13}$$

Here too we see that there seems to be no singularity.

Solving $g^{rr} = 0$ gives three solutions, one of which is real and determines the horizon.

For $r \to \infty$ this reduces to the Schwarzschild metric and for $r \to 0$ we find that $g_{tt} \to 1 - (r/L)^2$ and corresponds to de Sitter spacetime, and hence there is no singularity.

15.5 Non-static Black Holes: Vaidya Metrics

So far we have only considered static solutions of the Einstein equations. There are, of course, a plethora of time dependent solutions, and we give one simple generalisation of the Schwarzschild metric. This is best done by going to the Eddington-Finkelstein co-ordinates (10.3) $v = t + r^*$ with $r^* = r + 2GM\,\ln(r/2GM - 1)$and making the mass v dependent

$$ds^2 = -\left(1 - \frac{2GM(v)}{r}\right) dv^2 + 2dv\,dr + r^2 d\Omega^2 \tag{15.14}$$

Likewise we can use the coordinate $u = t - r^*$ in stead of v and have similarly

$$ds^2 = \left(1 - \frac{2GM(u)}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2 \qquad (15.15)$$

Vaidya metrics provide useful toy models to study the evolution of black holes.

15.6 Higher Dimensional Black Holes

It is easy to generalise the four-dimensional black holes to higher dimensions or to find new solutions in more than four dimensions. We keep it very brief, barely enough to give a flavour of the landscape.

Schwarzschild-Tangherlini Black Holes

These are the natural extensions of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-2}}\right) dt^2 + \left(1 - \frac{\mu}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega_{d-1} \qquad (15.16)$$

The parameter μ is proportional to the mass of the black hole. There is a natural extension to the Reissner-Nordström black hole and asymptotically flat solutions satisfy certain uniqueness theorems.

Topological Black Holes

These have metric

$$ds^2 = -\left(k - \frac{\mu}{r^{d-2}} \pm \frac{r^2}{\ell^2}\right) dt^2 + \left(k - \frac{\mu}{r^{d-2}} \pm \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1,k} \qquad (15.17)$$

where the $(d-1)$-dimensional space can be $S^{d-1}, \mathbb{R}^{d-1}, H^{d-1}$ or any other manifold satisfying Einstein's equations with a metric h_{ij} and Ricci tensor $R_{ij}(h) = (d-2)kh_{ij}$.

Myers-Perry Black Holes

These are the analogues for the rotating or Kerr black holes to $d > 4$. For a d-dimensional spacetime with $d = 2n+1$ or $d = 2n+2$, hence corresponding to a spatial rotation of $SO(d-1)$, i.e. of $SO(2n)$ or $SO(2n+1)$ respectively, such a black hole is characterised by n independent parameters. I.e. the number of parameters is equal to the rank of the corresponding Lie algebra. As an example, for $d = 4$ we have $SO(3)$ which has rank one and hence one parameter, corresponding to the angular momentum. This is of course the Kerr metric we studied in detail.

Black Strings and Black Branes

Black strings are straightforward extensions of the Schwarzschild metric by adding an extra dimension, which can be $\mathbb{R}$ or S^1;

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2 + dy^2 \tag{15.18}$$

Here the black hole is actually a string extended along the fifth dimension.

More generally we can consider black p-branes with metric

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-2}}\right) dt^2 + \left(1 - \frac{\mu}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega_{d-1} + dy_p^2 \tag{15.19}$$

These black strings and black p-branes appear in supergravity and string theories.

Exotic Black Holes

Somewhat surprisingly the uniqueness theorems in four dimensions do not generalise to higher dimensions, where the landscape of black holes solutions to the Einstein equations is much more varied. As an example, in $d = 5$ there are asymptotically black rings solutions where the horizon has the topology of $S^1 \times S^1$.

Chapter 16

Cosmology and the State of the Universe

16.1 The Robertson-Walker Metric

We consider spacetime to be spatially homogeneous and isotropic, but evolving in time, i.e. it is of the form $\mathbb{R} \times \Sigma$ with Σ a three dimensional maximally symmetric space with metric g_{ij}. The spacetime metric is then

$$ds^2 = -dt^2 + a(t)^2 g_{ij}(\vec{x}) dx^i dx^j \tag{16.1}$$

The coordinates x^i used here are chosen such that the metric is independent of cross terms $dtdx^i$, that g_{tt} is independent of x^i and g_{ij} are independent of t. They are known as **Comoving Coordinates**.

As we are assuming Σ to be maximally symmetric, we know from (5.14) that the curvature tensor is given by

$$R^{(3)}_{ijkl} = \frac{\kappa}{6}(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{16.2}$$

The superscript $^{(3)}$ is there to remind us that this tensor is for the three dimensional space Σ and κ is some constant, which turns out to be the three dimensional curvature. The Ricci tensor is the given by

$$R^{(3)}_{ij} = 2\kappa g_{ij} \tag{16.3}$$

A maximally symmetric space will also be spherically symmetric so we can write

$$d\sigma^2 = \tilde{g}_{ij}(\vec{x}) dx^i dx^j = e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \tag{16.4}$$

Note that one could add a factor $e^{2\alpha(r)}$ to the last term, but this can always be removed by rescaling r. For this metric we find the non-zero components of the Ricci tensor to be

$$\begin{aligned} R^{(3)}_{11} &= \frac{2}{r}\partial_r\beta \\ R^{(3)}_{22} &= e^{-2\beta}(r\partial_r\beta - 1) + 1 \\ R^{(3)}_{33} &= \Big[e^{-2\beta}(r\partial_r\beta - 1) + 1\Big]\sin^2\theta \end{aligned} \tag{16.5}$$

Setting $R^{(3)}_{11} = 2\kappa g_{rr}$ we find

$$\frac{2}{r}\partial_r\beta = 2\kappa e^{2\beta} \tag{16.6}$$

which is solved by

$$\beta = -\frac{1}{2}\ln\big(1 - \kappa r^2\big) \tag{16.7}$$

One easily checks that this solution also satisfies the other equations. Thus the three dimensional metric is

$$d\sigma^2 = \frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2 \tag{16.8}$$

As κ is the curvature of Σ it sets its scale and we can always rescale it such that it is ± 1 or 0. We then also rename it to k and have found that the most general homogeneous and isotropic spacetime that can evolve in time is of the form

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{16.9}$$

This is the **Robertson-Walker Metric** also known as the **Friedman-Lemaître-Robertson-Walker** or **FLRW Universe** and is a key building block of virtually all cosmological models.

Depending on k we have three cases

$\boldsymbol{k = +1}$: the curvature is positive and the universe is **closed**;

$\boldsymbol{k = 0}$: the curvature is zero and the universe is **flat**;

$\boldsymbol{k = -1}$: the curvature is negative and the universe is **open**.

For further use, we give the non-zero components of the Ricci tensor

$$\begin{aligned} R_{00} &= -\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta \end{aligned} \tag{16.10}$$

the curvature

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{16.11}$$

and the Kretschmann invariant

$$K = \frac{12(a^2\ddot{a}^2 + (\dot{a}^4 + 2k\dot{a}^2 + k^2)}{a^4} \tag{16.12}$$

16.2 The Friedmann Equations

The next step is to fill the universe with something. For this we assume the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} \tag{16.13}$$

With the isotropic metric that we consider the velocity of the perfect fluid is by construction of the reference frame at rest, so that $U^\mu = (1, 0, 0, 0)$ and

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & Pg_{ij} & \\ 0 & & & \end{pmatrix} \tag{16.14}$$

From this we find that

$$T^\mu_\nu = \text{diag}\,(-\rho, P, P, P) \tag{16.15}$$

and

$$T = T^\mu_\mu = -\rho + 3P \tag{16.16}$$

Let us consider the conservation equation $\nabla_\mu T^{\mu\nu} = 0$. For $\nu = 0$ this is

$$0 = \partial_\mu T^\mu_{\ 0} + \Gamma^\mu_{u\lambda} T^\lambda_{\ 0} - \Gamma^\lambda_{\mu 0} T^\mu_{\ \lambda} = -\partial_t \rho - \frac{3\partial_t a}{a}(\rho + P) \tag{16.17}$$

We can rewrite this as

$$\partial_t(\rho a^3) = -P \partial_t a^3 \tag{16.18}$$

One can check that the conservation equations for $\nu \neq 0$ are identically satisfied

We now plug this energy-momentum tensor into the Einstein equations

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \tag{16.19}$$

For $\mu = \nu = 0$ this gives

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3P) \tag{16.20}$$

and for $\mu = i$ and $\nu = j$ we find a common factor g_{ij} so that there is only one equation

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - P) \tag{16.21}$$

It is common to combine these equations as

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \end{aligned} \tag{16.22}$$

These are known as the **Friedmann Equations** .

It is convenient to parametrise time such that at present time t_0 we have $a(t_0) = 1$. We can then define a variable that measures the actual scale of the universe by

$$R(t) = R_0 a(t) \tag{16.23}$$

so that $R(t_0) = R_0$ is the present scale of the universe. This variable is not to be confused with the curvature! Note that we have $\dot{a}/a = \dot{R}/R$.

The upshot of all this is that with only three simple equations we will be able to say already quite a lot about what types of universes are possible. These are (1) the equation of state relating energy density and pressure; (2) the energy-momentum conservation equation and (3) one of Einstein field equations. Together they can be written as

$$\begin{aligned}\rho &= wP \\ \frac{d\rho}{\rho+P} &= -3\frac{dR}{R} \\ \dot{R}^2+k &= \frac{8\pi G}{3}\rho R^2\end{aligned} \tag{16.24}$$

16.3 Common Cosmological Parameters

In order to make the link with cosmological observations, it is convenient to define some new variables

- The **Hubble Parameter H** characterises the rate of expansion and is defined as

  $$H = \frac{\dot{a}}{a} = \frac{\dot{R}}{R} \tag{16.25}$$

  At present time t_0 we have the **Hubble Constant H_0**. It is currently believed to be around $H_0 \approx 70 \pm 10\,$km/s/Mpc $\approx 2 \times 10^{-18}$s.

- Cosmological scales are often defined using the **Hubble length** $d_H = H_0^{-1} \approx 13.3 \times 10^{25}$ m and the **Hubble Time** $t_H \approx 4.4 \times 10^{17}$ s.

- The **Deceleration Parameter** is defined as

  $$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{16.26}$$

- For every type of material we fill the universe with we define a **Density Parameter**

  $$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_c} \tag{16.27}$$

  where ρ_c is the **Critical Density**

  $$\rho_c = \frac{3H^2}{8\pi G} \tag{16.28}$$

16.4 Filling up the Universe

Let us now assume an **Equation of State** relating P and ρ of the form

$$P = w\rho \tag{16.29}$$

We can rewrite the conservation equation as

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{16.30}$$

In general w could be a complicated function of ρ but as we saw in (2.15) it takes a constant value for the cases of importance:

dust/matter $w = 0$: these are just collisionless non-relativistic particles that exert no pressure;

radiation $w = 1/3$: these are photons or highly relativistic particles. An easy way to understand this to to notice that the energy-momentum tensor of the electromagnetic field is traceless (in four dimensions) so that the value for w follows directly from (16.16).

vacuum energy $w = -1$: this just gives a non-zero cosmological constant

When $w = c^{\text{te}}$ we can solve the equation of state

$$\rho \propto a^{-3(1+w)} = a^{-\gamma_j} \tag{16.31}$$

and we find

matter: $\rho \propto a^{-3} \quad \Rightarrow \quad \gamma_m = 3$

radiation: $\rho \propto a^{-4} \quad \Rightarrow \quad \gamma_r = 4$

vacuum: $\rho = c^{\text{te}} \quad \Rightarrow \quad \gamma_\Lambda = 0$

These results should not come as a surprise. For dust the energy sits entirely in the rest mass of the particles and so the energy density is proportional to the inverse volume of space, which in this spacetime has a time dependent scale factor, a. For radiation we have this same volume factor but an additional factor a^{-1} due to the increasing wavelength – and corresponding decreasing energy – as space expands. For the vacuum the energy density is constant by assumption.

We now fill up the universe with matter, radiation and vacuum energy with energy density ρ_m, ρ_r and ρ_Λ respectively. We can then rewrite the last equation of (16.24) in terms of the cosmological parameter as

$$H^2 = \frac{8\pi G}{3} \sum_{j=m,r,\Lambda} \rho_j - \frac{k}{R^2} \tag{16.32}$$

It is convenient to introduce a density linked to the geometry of spacetime

$$\rho_k = -\frac{3}{8\pi G k} R^2 \tag{16.33}$$

so that (16.34) becomes

$$H^2 = \frac{8\pi G}{3} \sum_{n=m,r,\Lambda,k} \rho_n \tag{16.34}$$

Using the definition of critical density (16.28) we thus get the simple equation

$$1 = \sum_{n=m,r,\Lambda,k} \Omega_n = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k \tag{16.35}$$

i.e. the sum of all the density parameters corresponding to what fills the universe Ω_m, Ω_r and Ω_Λ and what comes from geometry Ω_k should add to one.

We can now write

$$\frac{\Omega_j}{\Omega_{j,0}} = \frac{8\pi G \rho_j / 3H^2}{8\pi G \rho_{j,0} / 3H_0^2} = \left(\frac{H_0}{H}\right)^2 \frac{\rho_j}{\rho_{j,0}} = \left(\frac{H_0}{H}\right)^2 a^{-\gamma_j} \tag{16.36}$$

where we have used $a(t_0) = 1$. Thus

$$\Omega_j = \left(\frac{H_0}{H}\right)^2 a^{-\gamma_j} \Omega_{j,0} \tag{16.37}$$

Using this in (16.35) we find

$$H^2 = H_0^2 \left(\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda,0} + \frac{\Omega_{k,0}}{a^2} \right) \tag{16.38}$$

The $\Omega_{j,0}$ are the density parameters at current time t_0 so they can be measured and are given by

$$\Omega_{m,0} \approx 0.3; \qquad \Omega_{r,0} \approx 5 \times 10^{-5}; \qquad \Omega_{\Lambda,0} \approx 0.7 \tag{16.39}$$

We see that 70% of the energy density comes from vacuum energy and 30% from matter (of which 4% is from the usual baryonic matter and the remainder 25% is dark matter, of which know nothing, except that it must exist). Energy from radiation plays no significant role in our current universe. Note however that because $\rho_m \propto a^{-3}$, $\rho_r \propto a^{-4}$ and $\rho_\Lambda = c^{\text{te}}$, radiation dominated the universe at its very beginning when it was very small, i.e. right after the Big Bang. This is due to the fact that at that time, all matter particles were ultrarelativistic and their masses were negligible so that they behaved as radiation.

16.5 A Cosmic Diagram of the Universe

At this point we can start building a "phase diagram" for the universe. In theory we should consider a three parameter space $(\Omega_m, \Omega_r, \Omega_\Lambda)$, but as radiation is negligible, we will look at a two dimensional phase diagram spanned by Ω_m and Ω_Λ. We can identify the following features

- Consider the line $\Omega_m + \Omega_\Lambda = 1$. Because we have $1 = \Omega_m + \Omega_\Lambda + \Omega_k = \Omega_m + \Omega_\Lambda - k/\dot{R}^2$ we need $k > 0$ if $\Omega_m + \Omega_\Lambda > 1$ and $k < 0$ if $\Omega_m + \Omega_\Lambda < 1$. Thus, above the line the universe is closed $(k > 0)$ and below that line the universe is open $(k < 0)$.

- Next we consider the deceleration parameter q defined in (16.26)

$$
\begin{aligned}
q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{R}/R}{\dot{R}^2/R^2} &= \frac{1}{2}\sum_{j=m,r,\Lambda}(1+3w_j)\Omega_j \\
&= \frac{1}{2}(\Omega_m + 2\Omega_r - 2\Omega_\Lambda)
\end{aligned}
\tag{16.40}
$$

Here we have used

$$
\begin{aligned}
\frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}(\rho + 3P) = -\frac{4\pi G}{3}\sum_{j=m,r,\Lambda}(1+3w_j)\rho_j \\
&= -\frac{4\pi G}{3}\sum_{j=m,r,\Lambda}(1+3w_j)\frac{3H^2}{8\pi G}\Omega_j \\
&= -\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2\sum_{j=m,r,\Lambda}(1+3w_j)\Omega_j
\end{aligned}
\tag{16.41}
$$

If the cosmological constant is zero or negative then q is positive and the expansion of the universe will slow down. Ignoring radiation (16.40) becomes $-q = \Omega_\Lambda - \frac{1}{2}\Omega_m$ we see that cosmic expansion accelerates above the line $\Omega_\Lambda = \frac{1}{2}\Omega_m$ and decelerates below that line.

- Let us now rewrite (16.38) as

$$\frac{\dot{a}^2}{H_0^2} - \left(\frac{\Omega_{m,0}}{a} + \frac{\Omega_{r,0}}{a^2} + \Omega_{\Lambda,0}a^2 + \Omega_{k,0}\right) = 0 \tag{16.42}$$

This is an equation that we should be familiar with. It looks like the Newtonian description of a particle with mass $2/H_0^2$ with zero total energy in a potential

$$\begin{aligned} V(a) &= -\left(\frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2\right) - \Omega_{k,0} \\ &= -\frac{\Omega_{m,0}}{a} - \Omega_{\Lambda,0}a^2 - (1 - \Omega_{m,0} - \Omega_{\Lambda,0}) \end{aligned} \tag{16.43}$$

where we have again ignored the radiation energy and used $1 = \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0}$. Fort small a we have $V(a) \propto -\Omega_{m,0}/a$ and the potential is attractive like an inverse square law. For large a we have $V(a) \propto -\Omega_{\Lambda,0}a$ and is attractive for $\Omega_{\Lambda_0} < 0$ and repulsive for $\Omega_{\Lambda_0} > 0$, like a harmonic oscillator, with negative Hooke's constant. The constant term just moves the potential up and down. At present we have by convention $a = 1$ and by observation $\dot{a} > 0$.

Let us first consider the case of negative cosmological constant $\Omega_{\Lambda,0} < 0$. The potential is then

$$V(a) = -\frac{\Omega_{m,0}}{a} + |\Omega_{\Lambda,0}|a^2 - (1 - \Omega_{m,0} - \Omega_{\Lambda,0}) \tag{16.44}$$

The potential is shown as the thick line in fig. 16.1 and attractive everywhere. The "particle" is at present at $a(t_0) = 1$ and has $\dot{a} > 0$ so moves up the potential to the right. But as it reaches $V(a) = 0$ its "kinetic energy" and hence speed becomes zero and the particle falls down to the left, reducing $a(t)$, thus the universe that started as a Big Bang will eventually contract and end in a Big Crunch.

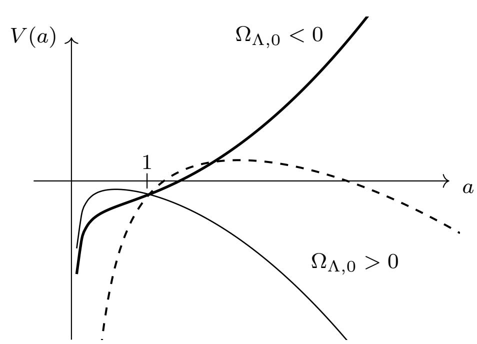


Figure 16.1: Cosmic potential for different values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$

Let us now consider the case $\Omega_{\Lambda,0} > 0$. Here both for $a \to 0$ and $a \to +\infty$ we have $V(a) \to -\infty$. Two cases are shown as the thin and shaded lines in fig. 16.1. The thin line is such that the universe has sufficient energy to go the top as the maximum potential is negative. The universe started in a Big Bang and will continue to expand forever. For the dashed line we need to consider whether the value a_{max} where $V(a)$ reaches is maximum is greater or smaller than one. If $a_{\text{max}} > 1$ then we will not have enough energy to reach the top and the universe will recollapse. If, however, $a_{\text{max}} < 1$, which is the current situation, we have already passed the peak of the hill and the universe will expand forever. In fact because the total energy of our particle model was zero, we could never have crossed the hill and come from a situation where a was zero. This is a universe where there was no Big Bang at all.

The dividing line between these scenarios are found by finding the maximum of $V(a)$, i.e. by solving $V'(a_{\text{max}}) = 0$, and then setting $V(a_{\text{max}}) = 0$. These two equations gives a relation between $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. To see what the two sides of these curves imply we need to check whether a_{max} is smaller or larger than one. We will not perform this straightforward calculation in detail.

All this above information is summarised in the **Cosmic Diagram** shown in fig. fig:grcosmdiag. The present value of the universe is somewhere in the grey circle. Somewhat surprisingly this sits on or very close to the line determining whether our universe is closed or open, i.e. it is nearly flat. We conclude that

Our universe is nearly flat, originated from a Big Bang and will expand forever.

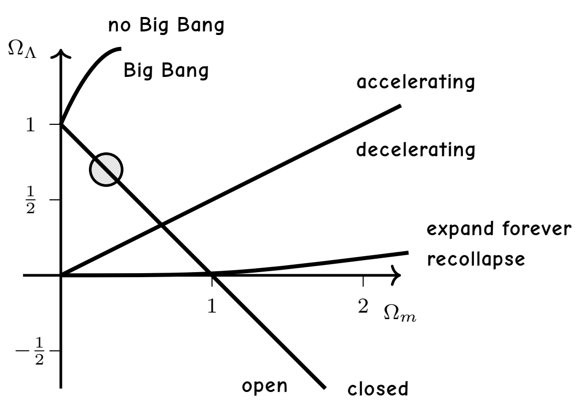


Figure 16.2: Cosmic diagram for a universe consisting of matter and vacuum energy only. The current universe is somewhere in the grey circle.

16.6 A Static Universe and the Cosmological Constant

When there is no vacuum energy, $\Omega_{\Lambda_0} = 0$ the Einstein equations admit no solution for a static universe. This bothered Einstein considerably as at that time it was generally believed that the universe was static. Einstein solved this by introducing the cosmological constant. As we now show with such a cosmological constant it is possible to have a static universe.

We start with the Einstein equation for R_{00} in (16.20) and set $\ddot{a} = 0$ to get $\rho + 3P = 0$. Using the equation of state $P = w\rho$ with $w = 0$ for matter and $\omega = -1$ for vacuum energy this gives $(1+0)\rho_m + (1-3)\rho_\Lambda = \rho_m - 2\rho_\Lambda = 0$. The total energy density is thus $\rho = \rho_m + \rho_\Lambda = 3\rho_\Lambda = 3\Lambda$.

Setting both $\ddot{a} = 0$ and $\dot{a} = 0$ in (16.21) gives

$$2k = a^2 4\pi G(\rho - P) = a^2 4\pi G(\rho_m + 2\rho_\Lambda) = 16a^2\pi G\Lambda \tag{16.45}$$

We thus find that there is a static solution for a closed universe with $k = 1$ as long as $\rho_\Lambda = \frac{1}{2}\rho_m$ and then a is a constant given by

$$a^2 = \frac{1}{8\pi G\Lambda} \tag{16.46}$$

It is rumoured that Einstein called his introduction of the cosmological constant to ensure a static universe his greatest blunder. In fact, the universe is not static but expanding, so had Einstein not introduced the cosmological constant, he could have predicted that the universe was expanding. Ironically enough , the universe is indeed expanding and does have a cosmological constant.

16.7 The Universe Flow in the Cosmic Diagram

We may now ask the question how a universe with a given set of density parameters Ω_j evolves over time, i.e. how universes flow in the cosmic diagram. We thus need an expression for $\dot{\Omega}_j$.

Our starting point is the definition of the density parameter (16.27) from which we find

$$\begin{aligned}\dot{\Omega}_j &= \frac{d}{dt}\left(\frac{8\pi G}{3H^2}\rho_j\right) = \frac{8\pi G}{3}\frac{\dot{\rho}_j}{H^2} + \frac{8\pi G}{3}\left(-\frac{2\dot{H}}{H^3}\right)\rho_j \\ &= \Omega_j\left(\frac{\dot{\rho}_j}{\rho_j} - 2\frac{\dot{H}}{H}\right) = -\Omega_j\left[3(1+w_j)H + 2\frac{\dot{H}}{H}\right]\end{aligned} \tag{16.47}$$

where we have also used (16.30). We now work out

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\frac{d}{dt}\frac{\dot{R}}{R} = \frac{1}{H^2}\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right) = -q - 1$$
$$= -\frac{1}{2}\sum_{j=m,r,\Lambda}(1+3w_j)\Omega_j - 1 \tag{16.48}$$

where we have introduced the deceleration parameter q as defined in (16.26) and its worked out formula in (16.40). This means that we have a simple equation for how the density parameters evolve

$$\dot{\Omega}_j = H\Omega_j\left(-3w_j - 1 + \sum_{i=m,r,\Lambda}(1+3w_i)\Omega_i\right) \tag{16.49}$$

If we ignore radiation, then these equations become

$$\begin{aligned}\dot{\Omega}_m &= H\Omega_m(\Omega_m - 2\Omega_\Lambda - 1)\\ \dot{\Omega}_\Lambda &= H\Omega_\Lambda(\Omega_m - 2\Omega_\Lambda + 2)\end{aligned} \tag{16.50}$$

We can view this as a vector field $\vec{v} = (\dot{\Omega}_m, \dot{\Omega}_\Lambda)$ in the two-dimensional space spanned by $(\Omega_m, \Omega_\Lambda)$ and the corresponding flow is shown in fig. 16.3.

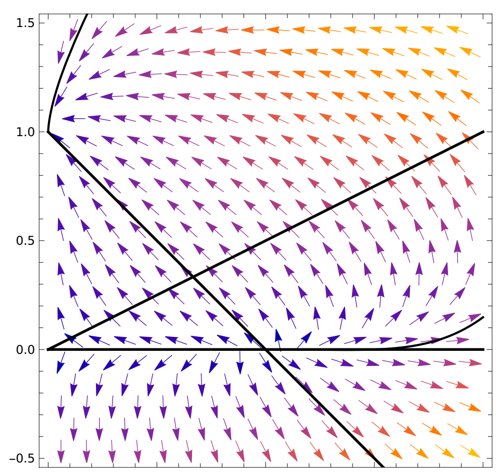


Figure 16.3: Flow of universes in the cosmic diagram with Ω_m and Ω_Λ and $\Omega_r = 0$

There are three fixed point where $\dot{\Omega}_m = \dot{\Omega}_\Lambda = 0$, viz $(\Omega_m, \Omega_\Lambda) = (0,1), (0,0)$ and $(1,0)$. The former, $(0,1)$ is an attractor: universes in the neighbourhood evolve towards that point. The latter two, $(0,0)$ and $(1,0)$, are unstable fixed points. Any perturbation from it will ensure that the universe evolves away from it. Our universe with $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ will evolve towards $(0,1)$, a universe fully dominated by vacuum energy.

Note that without cosmological constant we would be moving on the horizontal axis only. If $\Omega_m < 1$ then the universe will flow towards $\Omega_m = 0$ and the fixed point is stable; If $\Omega_m > 1$ then it will flow towards very large values. However small a perturbation Ω_Λ there is, the universe will always flow to an larger absolute value of Λ.

Let us now consider the case where we cannot neglect the radiation. The flow equations then become

$$\begin{aligned}
\dot{\Omega}_m &= H\Omega_m(\Omega_m + 2\Omega_r - 2\Omega_\Lambda - 1) \\
\dot{\Omega}_r &= H\Omega_r(\Omega_m + 2\Omega_r - 2\Omega_\Lambda - 2) \\
\dot{\Omega}_\Lambda &= H\Omega_\Lambda(\Omega_m + 2\Omega_r - 2\Omega_\Lambda + 2)
\end{aligned} \tag{16.51}$$

We now have four fixed points, the origin where all density parameters are zero and three fixed points where all density parameters are zero, bar one that is equal to one. The 3D flow diagram is shown in fig. 16.4.

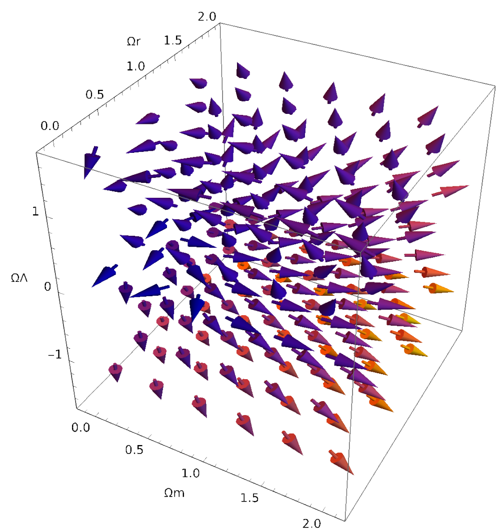


Figure 16.4: Flow of universes in the cosmic diagram with Ω_m, Ω_r and Ω_Λ

Finally, let us consider the case with no cosmological constant but with a universe that contains both radiation and matter. The cosmic flow diagram is then shown in fig. 16.5.

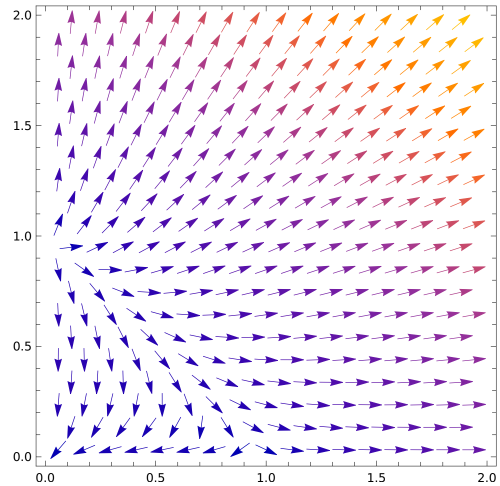


Figure 16.5: Flow of universes in the cosmic diagram with Ω_m and Ω_r and $\Omega_\Lambda = 0$

16.8 The Flat Universe

We can apply the previous analysis as well to Ω_k, the density parameter that is related to the geometry of spacetime. A similar calculation gives

$$\begin{aligned}\dot{\Omega}_k &= \frac{d}{dt}\left(-\frac{k}{H^2R^2}\right) = \frac{2k}{H^2R^2}\left(\frac{\dot{H}}{H} + \frac{\dot{R}}{R}\right) = -2\Omega_k\left(\frac{\dot{H}}{H} + H\right) = 2\Omega_k H q \\ &= \Omega_k H(\Omega_m + 2\Omega_r - 2\Omega_\Lambda) \end{aligned} \tag{16.52}$$

Let us assume a completely flat universe. Then $k = 0$ and $\dot{\Omega}_k = 0$ and Ω_k stays at zero: a completely flat universe remains flat. But let us assume that universe is nearly flat, i.e. k and by consequence Ω_k is very close to, but not equal to zero. If on the one hand $\Omega_m + 2\Omega_r - 2\Omega_\Lambda > 0$, which is certainly the case if the cosmological constant is negative, then Ω_k will grow and the universe will become less and less flat. If on the other hand $\Omega_m + 2\Omega_r - 2\Omega_\Lambda < 0$ then Ω_k is a stable fixed point. Given enough time, Ω_m and Ω_r will become very small and Ω_Λ will dominate so that for positive cosmological constant this condition will eventually be satisfied and the universe will end out flat.

Chapter 17

The Unruh Effect and Hawking Radiation

17.1 The Accelerated Observer and Bogolioubov Transformations

Consider the action for a free scalar field theory in curved spacetime

$$S[\varphi] = \int d^4x \, \mathcal{L} = \int d^4x \, \sqrt{-g} \, \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right] \tag{17.1}$$

Following the minimal coupling, we have replaced the ordinary partial derivatives by covariant derivatives.

Let us now proceed with the canonical quantisation of this theory. First we need the equation of motion. This is straightforward:

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi - m^2 \varphi = 0 \tag{17.2}$$

Canonical quantisation consists of expanding the field φ into a basis of solutions of this equation, the coefficients of this expansion are then promoted to operators and we can split the field into a positive energy frequency and a negative energy frequency part that corresponds to creation and annihilation operators respectively. In the case of flat spacetime, the equation of motion allows simple plane wave solutions $e^{ik \cdot x}$ and everything is pretty simple.

In a curved spacetime background, plane waves are not a solution of the equations of motion and this leads to interesting physics. We will therefore assume that we have a set of functions $\{f_i\}$ that form an orthonormal basis of solutions for the equation of motion, i.e. we have

$$\left[g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2 - \xi R \right] f_i(x) = 0 \tag{17.3}$$

We now expand the field φ in terms of these solutions and their conjugate

$$\varphi(x) = \sum_i (a_i f_i + a_i^* f_i^*) \tag{17.4}$$

At this point i may consist of discrete or continuous parameter. The functions f_i are orthonormal:[1]

$$(f_i, f_j) = \delta_{ij} \qquad \text{and} \qquad (f_i^*, f_j^*) = -\delta_{ij} \tag{17.6}$$

We follow the standard procedure for canonical quantisation. Define the conjugate momentum of the scalar field:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\nabla_0 \varphi)} = \sqrt{-g}\, \nabla_0 \varphi \tag{17.7}$$

and impose the equal-time canonical commutation relations (ETCCR)

$$\begin{aligned} [\varphi(t, \boldsymbol{x}), \varphi(t, \boldsymbol{x'})] &= [\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{x'})] = 0 \\ [\varphi(t, \boldsymbol{x}), \pi(t, \boldsymbol{x'})] &= \frac{i}{\sqrt{-g}} \delta^{(d-1)}(\boldsymbol{x} - \boldsymbol{x}') \end{aligned} \tag{17.8}$$

The coefficients a_i and a_i^* are promoted to operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ and satisfy the commutation relations

$$\begin{aligned} [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] &= 0 \\ [\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{ij} \end{aligned} \tag{17.9}$$

We define the vacuum state $|0\rangle_f$ by the fact that it is annihilated by the $\hat{a}_i$:

$$\hat{a}_i\, |0\rangle_f = 0 \qquad \text{for all } i \tag{17.10}$$

From this vacuum we define the entire Fock space by repeated action of creation operators $\hat{a}_i^\dagger$:

$$|n_1 n_2 \cdots\rangle_f = \frac{1}{\sqrt{n_1!}} \left(\hat{a}_1^\dagger\right)^{n_1} \frac{1}{\sqrt{n_2!}} \left(\hat{a}_2^\dagger\right)^{n_2} \cdots |0\rangle_f \tag{17.11}$$

We can also define a number operator for each mode

$$\hat{n}_{f_i} = \hat{a}_i^\dagger \hat{a}_i \tag{17.12}$$

[1]The inner product of two functions F and G is defined as:

$$(F, G) = -i \int_\Sigma (F \nabla_\mu G^* - G^* \nabla_\mu F) n^\mu \sqrt{-\gamma} d^{d-1}x \tag{17.5}$$

Here Σ is a space-like hyper-surface with induced metric γ_{ij} and unit normal vector n^μ.

that counts the number of excitations in mode i, i.e. $\hat{n}_{f_i} \left| n_1 n_2 \cdots \right\rangle_f = n_i \left| n_1 n_2 \cdots \right\rangle_f$.

So far, there is nothing unusual. The only point an attentive reader may have noticed is that we have here and there added a suffix $_f$ to make clear that we are working in the basis of the modes f_i.

We could of course have chosen a different basis of solutions of the equation of motion, say g_i. All the equations would then have been similar with due replacements. The field is expanded as

$$\varphi(x) = \sum_i b_i g_i + b_i^* g_i^* \tag{17.13}$$

and canonical quantisation imposes the commutation relations

$$\begin{aligned} [\hat{b}_i, \hat{b}_j][\hat{b}_i^\dagger, \hat{b}_j^\dagger] &= 0 \\ [\hat{b}_i, \hat{b}_j^\dagger] &= \delta_{ij} \end{aligned} \tag{17.14}$$

We define a vacuum $\left|0\right\rangle_g$ by $\hat{b}_i \left|0\right\rangle_g = 0$ for all i and can construct the Fock space by repeated application of the creation operators $\hat{b}_i^\dagger$. We can also define the number operator $\hat{n}_{g_i} = \hat{b}_i^\dagger \hat{b}_i$.

In flat spacetime we are able to pick out the plane waves as preferred basis. This allows us to split the field in a positive energy frequency part φ^+ and a negative energy frequency part φ^-

$$\varphi(x) = \varphi^+(x) + \varphi^-(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} e^{-ip\cdot x} + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}}^\dagger e^{+ip\cdot x} \tag{17.15}$$

The concept of positive/negative energy frequency part comes from the time-derivative of the plane wave solution

$$\begin{aligned} \frac{\partial}{\partial t} e^{-ip\cdot x} &= \frac{\partial}{\partial t} e^{-i(p^0 t - \boldsymbol{p}\cdot\boldsymbol{x})} = -ip^0 e^{-ip\cdot x} \\ \frac{\partial}{\partial t} e^{+ip\cdot x} &= \frac{\partial}{\partial t} e^{+i(p^0 t - \boldsymbol{p}\cdot\boldsymbol{x})} = +ip^0 e^{-ip\cdot x} \end{aligned} \tag{17.16}$$

The time coordinate is not unique as we can perform Lorentz transformation that will change the time coordinate. However, the vacuum $\left|0\right\rangle$ and the number operator $\hat{n}_{\boldsymbol{p}} = a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}$ are Lorentz-invariant. Let us indeed see what happens if we go to another inertial frame. Clearly if we do a three-dimensional rotation, this isn't going to affect time, so we need to consider a boost. These are of the general form

$$t' = \gamma(t - \boldsymbol{v}\cdot\boldsymbol{x}) \qquad \text{and} \qquad \boldsymbol{x}' = \gamma(\boldsymbol{x} - \boldsymbol{v}t) \tag{17.17}$$

where $\boldsymbol{v}$ is the boost velocity. The inverse transformation is

$$t = \gamma(t' + \boldsymbol{v} \cdot \boldsymbol{x}') \qquad \text{and} \qquad \boldsymbol{x} = \gamma(\boldsymbol{x}' + \boldsymbol{v}t') \tag{17.18}$$

We are interested in the time-derivative in the new coordinate system:

$$\partial_{t'} = \partial_{t'} x^\mu \partial_\mu = \partial_{t'} t \partial_t + \partial_{t'} x^i \partial_i = \gamma(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial}) \tag{17.19}$$

and hence looking at the positive energy mode $f_{\boldsymbol{p}} = e^{-ip\cdot x}$ we find

$$\partial_t' f_{\boldsymbol{p}} = \gamma(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial}) e^{-i(p^0 t - \boldsymbol{p}\cdot\boldsymbol{x})} = \gamma(-ip^0 + i\boldsymbol{v}\cdot\boldsymbol{p}) e^{-i(p^0 t - \boldsymbol{p}\cdot\boldsymbol{x})} = -ip'^0 f_{\boldsymbol{p}} \tag{17.20}$$

and so in the other coordinate system $f_{\boldsymbol{p}}$ is equally well a positive energy frequency mode, but with boosted momentum p'. Hence the split between creation and annihilation operator is identical in both coordinate systems and the number operator for a given mode, and for the total number of particles, measure exactly the same number of particles in both inertial frames.[2]

In flat spacetime, Lorentz invariance, guarantees that all inertial frames will measure the same number of particles.

This immediately raises the question of if and how this generalises to non-inertial frames, i.e. do observers in all frames from general relativity see an identical split between positive and negative energy frequency parts. In plain english: would all observers count the same number of particles?

In order to answer this question, we expand the modes in one reference frame in terms of the modes of the other reference frame:

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right) \tag{17.21}$$

Using the orthonormality conditions of the modes f_i, we find

$$(g_i, f_j) = \sum_j \left[\alpha_{ik}(f_k, f_j) + \beta_{ik}(f_k^*, f_j)\right] = \sum_k \alpha_{ik}\delta_{jk} = \alpha_{ij} \tag{17.22}$$

and

$$(g_i, f_j^*) = \sum_j \left[\alpha_{ik}(f_k, f_j^*) + \beta_{ik}(f_k^*, f_j^*)\right] = -\sum_k \beta_{ik}\delta_{jk} = -\beta_{ij} \tag{17.23}$$

We can also expand f_i in terms of g_i:

$$f_i = \sum_j \left(\rho_{ij} g_j + \sigma_{ij} g_j^*\right) \tag{17.24}$$

[2] We should add for the free theory, as that is what we are looking at. But this is sufficient for our purposes of looking how this works in curved spacetime

and find similarly

$$(f_i, g_j) = \sum_k \left[\rho_{ik}(g_k, g_j) + \sigma_{ik}(g_k^*, g_j)\right] = \rho_{ij}$$
$$(f_i, g_j^*) = \sum_k \left[\rho_{ik}(g_k, g_j^*) + \sigma_{ik}(g_k^*, g_j^*)\right] = -\sigma_{ij} \tag{17.25}$$

Now we have $\rho_{ij} = (f_i, g_j) = (g_j, f_i)^* = \alpha_{ji}^*$ and $\sigma_{ij} = -(f_i, g_j^*) = -(g_j^*, f_i)^* = -(g_j, f_i^*) = -\beta_{ji}$ so that we can rewrite (17.24) as

$$f_i = \sum_j \left(\alpha_{ji}^* g_j - \beta_{ji} g_j^*\right) \tag{17.26}$$

The transformations (17.21) and (17.26) between the different basis modes are known as **Bogoliubov Transformations** and the coefficients α and β are known as Bogolioubov parameters. This transformation also links the creation and annihilation operators in both reference frames. Indeed:

$$\begin{aligned}\varphi(x) &= \sum_i \left[\hat{b}_i g_i + \hat{b}_i^\dagger g_i^*\right] \\ &= \sum_i \left[\hat{b}_i \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right) + \hat{b}_i^\dagger \sum_j \left(\alpha_{ij}^* f_j^* + \beta_{ij}^* f_j\right)\right] \\ &= \sum_j \left[\sum_i (\alpha_{ij}\hat{b}_i + \beta_{ij}^* \hat{b}_i^\dagger) f_j + \sum_i (\beta_{ij}\hat{b}_i + \alpha_{ij}^*\hat{b}_i^\dagger) f_j^*\right]\end{aligned} \tag{17.27}$$

And so we see that we can identify

$$\hat{a}_j = \sum_i (\alpha_{ij}\hat{b}_i + \beta_{ij}^*\hat{b}_i^\dagger) \qquad \text{and} \qquad \hat{a}_j^\dagger = \sum_i (\beta_{ij}\hat{b}_i + \alpha_{ij}^*\hat{b}_i^\dagger) \tag{17.28}$$

and similarly we can write

$$\begin{aligned}\varphi(x) &= \sum_i \left[\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*\right] \\ &= \sum_i \left[\hat{a}_i \sum_j \left(\alpha_{ji}^* g_j - \beta_{ji} g_j^*\right) + \hat{a}_i^\dagger \sum_j \left(\alpha_{ji} g_j^* - \beta_{ji}^* g_j\right)\right] \\ &= \sum_j \left[\sum_i (\alpha_{ji}^* \hat{a}_i - \beta_{ji}^* \hat{a}_i^\dagger) g_j + \sum_i (-\beta_{ji}\hat{a}_i + \alpha_{ji}\hat{a}_i^\dagger) g_j^*\right]\end{aligned} \tag{17.29}$$

and identify

$$\hat{b}_i = \sum_j (\alpha_{ij}^*\hat{a}_j - \beta_{ij}^*\hat{a}_j^\dagger) \qquad \text{and} \qquad \hat{b}_i^\dagger = \sum_j (-\beta_{ij}\hat{a}_j + \alpha_{ij}\hat{a}_j^\dagger) \tag{17.30}$$

We already see that the creation and annihilation operators get mixed up in another coordinate system.

Let us now consider a system that is in the vacuum state $|0\rangle_f$ as measured by the f-modes. I.e. a state defined by $\hat{a}_i |0\rangle_f = 0$ for all i. Clearly the f-mode number operator in this state measures no particles:

$$_f\langle 0| n_{f_i} |0\rangle_f =_f \langle 0| \hat{a}_i^\dagger \hat{a}_i |0\rangle_f = 0 \tag{17.31}$$

What would an observer in the g-mode reference frame measure for particles in his g_i modes? That is just his number operator sandwiched between the state of the system, i.e. $_f\langle 0| n_{g_i} |0\rangle_f$. But we can now work this out using the Bogoliubov transformations:

$$\begin{aligned} _f\langle 0| n_{g_i} |0\rangle_f &=_f \langle 0| \hat{b}_i^\dagger \hat{b}_i |0\rangle_f \\ &=_f \langle 0| \sum_j (-\beta_{ij}\hat{a}_j + \alpha_{ij}\hat{a}_j^\dagger) \sum_k (\alpha_{ik}^* \hat{a}_k - \beta_{ik}^* \hat{a}_k^\dagger) |0\rangle_f \\ &= \sum_{jk} \beta_{ij}\beta_{ik}^* \times_f \langle 0| \hat{a}_k^\dagger \hat{a}_j + \delta_{jk} |0\rangle_f = \sum_j \beta_{ij}\beta_{ij}^* \end{aligned} \tag{17.32}$$

where we have used the commutation relations. We conclude that the observer of the g-mode reference frame measures

$$_f\langle 0| n_{g_i} |0\rangle_f = \sum_j |\beta_{ij}|^2 \tag{17.33}$$

If any of the coefficients β_{ij} is not equal to zero than the observer in the g-mode reference frame will see a non-zero number of particles! Now the β coefficients are defined in (17.21), i.e. $g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right)$; so they are non-zero when the positive and negative energy frequency modes of the two reference frames get mixed up. In flat-spacetime we have seen that, due to Lorentz invariance, these modes do not get mixed up and all β's are equal to zero. As a result both observers will measure the same number of particles. In curved spacetime, there is no reason to assume that all the β's are equal to zero and so the observer in the g-mode reference frame will measure one or more particles. Now, due to the principle of equivalence of general relativity, a curved spacetime is equivalent to acceleration, so whereas one observer will see a vacuum with no particles, another observer that is accelerating vs. the first one will measure particles. Thus

observers who are accelerating vs one another
do not see the same number of particles.

This is a direct consequence of field theory in curved spacetime. We would obviously like to understand this better.

Discussion

This is a striking and a perhaps counter-intuitive result. Clearly a particle detector in a curved spacetime will measure a number of particles unaware of what modes is being used? But what is the definition of a particle used by such a detector? The answer is that the detector will have its own proper time τ along the trajectory it follows in the curved spacetime and that proper time will define the positive and negative energy frequency parts and then also the number operator corresponding to that proper time. So, whilst the detector may not be aware of the modes, the modes themselves are solutions to the equation

$$\frac{D}{d\tau}f_i = -i\omega f_i \tag{17.34}$$

for some frequency $\omega > 0$. In general it may not be possible to find such modes all over spacetime. But this may be possible in a static spacetime, that is a spacetime where the metric is independent of the time coordinates and there are no time-space cross terms: $\partial_0 g_{\mu\nu} = g_{0i} = 0$. In such a metric

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{00}\partial_0^2 + \frac{1}{2}g^{00}g^{ij}\partial_i g_{00}\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k ij\partial_k \tag{17.35}$$

and the equation of motion (17.2) can be written as

$$\left[g^{00}\partial_0^2 + \frac{1}{2}g^{00}g^{ij}\partial_i g_{00}\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k ij\partial_k - m^2\right]f(x) = 0 \tag{17.36}$$

We can rewrite this as

$$\partial_0^2 f(x) = -g_{00}^{-1}\left[\frac{1}{2}g^{00}g^{ij}\partial_i g_{00}\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k ij\partial_k - m^2\right]f(x) \tag{17.37}$$

The RHS contains no dependence on time as we are in a static spacetime and the LHS is just a time derivative. We can therefore perform a separation of variables:

$$f(x) = f_\omega(t, \boldsymbol{x}) = e^{-i\omega t}F_\omega(\boldsymbol{x}) \tag{17.38}$$

where $F(\boldsymbol{x})$ and ω satisfy

$$-g_{00}^{-1}\left[\frac{1}{2}g^{00}g^{ij}\partial_i g_{00}\partial_j + g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k ij\partial_k - m^2 - \xi R\right]F_\omega(\boldsymbol{x}) = -\omega^2 F_\omega(\boldsymbol{x}) \tag{17.39}$$

Eq. (17.38) then defines the positive energy frequency part

$$\partial_t f_\omega(t, \boldsymbol{x}) = -i\omega f_\omega(t, \boldsymbol{x}) \tag{17.40}$$

If the proper time of the detector will be proportional to the Killing time of the static spacetime then these modes will be the natural basis to describe the Fock space.

17.2 The Unruh Effect

Let us show that an accelerating observer in the vacuum of flat-spacetime will see a thermal spectrum of particles, with the temperature of the spectrum related to the acceleration rate. As gravity and acceleration are equivalent this will be a natural first step to then discuss Hawking radiation, i.e. the radiation of black holes.

To simplify the analysis, we will work in $1+1$ dimension and consider a massless real scalar with Klein-Gordon equation $\Box\varphi = 0$. We immediately have one set of modes, the plane waves of this equation. We now consider an observer moving along the space direction with constant acceleration α. We will first need to identify the modes that this observer will use and then use the Bogoliubov transformation to calculate the particle spectrum seen by the accelerating observer.

The trajectory of the accelerating observer is given by:

$$t(\tau) = \frac{1}{\alpha}\sinh\alpha\tau \qquad \text{and} \qquad x(\tau) = \frac{1}{\alpha}\cosh\alpha\tau \tag{17.41}$$

where τ is the proper time of the observer and α is a positive constant. The acceleration of this observer is $a^\mu = d^2x^\mu/d\tau^2$ and we have

$$\begin{aligned} a^0 &= \frac{d^2t}{d\tau^2} = \frac{d^2}{d\tau^2}\frac{1}{\alpha}\sinh\alpha\tau = \alpha\sinh\alpha\tau \\ a^1 &= \frac{d^2x}{d\tau^2} = \frac{d^2}{d\tau^2}\frac{1}{\alpha}\cosh\alpha\tau = \alpha\cosh\alpha\tau \end{aligned} \tag{17.42}$$

and hence

$$a^2 = \eta_{\mu\nu}a^\mu a^\nu = -(a^0)^2 + (a^1)^2 = \alpha^2(\cosh^2\alpha\tau - \sinh^2\alpha\tau) = \alpha^2 \tag{17.43}$$

We can easily eliminate the parameter τ from the trajectory and describe the path in t-x space as

$$x^2 - t^2 = \frac{1}{\alpha^2} \tag{17.44}$$

We see that the observer follows a hyperboloid with null paths $x = -t$ and $x = +t$ as the asymptotic paths in the past and the future. This should, of course, not surprise us as these null asymptotes are the paths where the observer has accelerated to maximum velocity, the velocity of light.

Let us now move to new coordinates η and ξ:

$$t = \frac{1}{\alpha}e^{\alpha\xi}\sinh\alpha\eta \qquad \text{and} \qquad x = \frac{1}{\alpha}e^{\alpha\xi}\cosh\alpha\eta \qquad \text{for} \qquad x > |t| \tag{17.45}$$

We will come back to the condition $x > |t|$ very soon. The range of the new coordinates is $[-\infty, +\infty]$ for both η and ξ. The path of the observer is now given by

$$\begin{aligned} t(\tau) &= \frac{1}{\alpha} e^{\alpha\xi} \sinh \alpha\eta = \frac{1}{\alpha} \sinh \alpha\tau \\ x(\tau) &= \frac{1}{\alpha} e^{\alpha\xi} \cosh \alpha\eta = \frac{1}{\alpha} \cosh \alpha\tau \end{aligned} \tag{17.46}$$

We can simplify this. Squaring both equations and subtract them from one another immediately yields $e^{\alpha\xi} = 1$ or $\xi = 0$. Plugging this into the first equation gives $\eta(\tau) = \tau$. In terms of the new coordinates the path of the accelerating observer is thus:

$$\eta = \tau \qquad \text{and} \qquad \xi = 0 \tag{17.47}$$

This is just a reference frame that is moving with the observer. To compute the metric in terms of the new coordinates we calculate the line element $ds^2 = -dt^2 + dx^2$. Straightforward algebra gives

$$ds^2 = e^{2\alpha\xi}(-d\eta^2 + d\xi^2) \tag{17.48}$$

Let us for a minute look back at the coordinate transformation (17.45). Clearly this covers only the region $x > 0$ and as

$$x - t = \frac{1}{\alpha} e^{\alpha\xi}(\cosh \alpha\eta - \sinh \alpha\eta) = \frac{1}{\alpha} e^{\alpha\xi} e^{-\alpha\eta} > 0 \tag{17.49}$$

we have that our coordinate transformation only covers the region $x > |t|$ and not the entire spacetime. This is the region **R I** in Fig.17.1. In order to have a coordinate system over the whole of spacetime we also need to cover the three other regions. **R II** and **R III** are space-like regions, so we are not to worried about them. In fact they can be reached by analytical continuation if we would so desire. For **R IV** we can simply flip the signs and we then have

$$t = -\frac{1}{\alpha} e^{\alpha\xi} \sinh \alpha\eta \qquad \text{and} \qquad x = -\frac{1}{\alpha} e^{\alpha\xi} \cosh \alpha\eta \qquad \text{for} \qquad x < |t| \tag{17.50}$$

Strictly speaking we are abusing notation as in both **R I** and **R IV** the coordinates go from $-\infty$ to $+\infty$. However we can solve this by carefully identifying which region we are working in. The only point we have not covered then is the origin, but that is a point of zero measure and should not bother us.

These new coordinates η and ξ are as **Rindler Coordinates** and **R I** is known as **Rindler Space**. The observer moving with constant acceleration is known as a **Rindler Observer**.

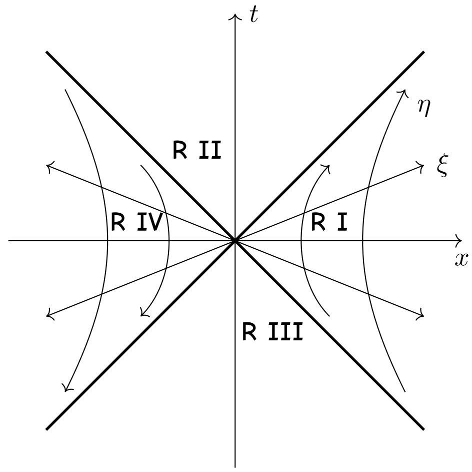


Figure 17.1: Minkowski spacetime in Rindler coordinates

Now that we have the metric in terms of Rindler coordinates (17.48) we can easily write the equation of motion $\Box\varphi = 0$ in terms of these coordinates:

$$\Box\varphi = e^{-2\alpha\xi}\big(-\partial_\eta^2 + \partial_\xi^2\big)\varphi = 0 \tag{17.51}$$

This, of course, has as solution the plane waves in terms or Rindler coordinates. But we need to be careful. In **R I** we have

$$\partial_\eta = \partial_\eta t\partial_t + \partial_\eta x\partial_x = e^{\alpha\xi}(\cosh\alpha\eta\partial_t + \sinh\alpha\eta\partial_x) = \alpha(x\partial_t + t\partial_x) \tag{17.52}$$

and this will be recognised as a boost in the x-direction, with ∂_t and ∂_η pointing in the same direction, the direction of time flowing forward. In **R IV** we have similarly $\partial_\eta = -a(x\partial_t + t\partial_x)$ and ∂_t and ∂_η point in opposite directions.

First consider **R I** and a normalised plane wave:

$$g^{(1)} = \frac{1}{\sqrt{4\pi\omega}} e^{i(-\omega\eta + k\xi)} \tag{17.53}$$

where, as we are in $1+1$ dimensions, $\omega = |k|$. This satisfies

$$\partial_\eta g^{(1)} = \partial_\eta \frac{1}{\sqrt{4\pi\omega}} e^{i(-\omega\eta + k\xi)} = -i\omega \frac{1}{\sqrt{4\pi\omega}} e^{i(-\omega\eta + k\xi)} = -i\omega g^{(1)} \tag{17.54}$$

and is indeed a positive energy frequency mode. In **R IV** we have to use $-\partial_\eta$ to ensure time flows in the same direction. And hence $g^{(1)}$ does not correspond to a positive energy frequency mode in that region. It is however straightforward to write one as

$$-\partial_\eta g^{(2)} = -\partial_\eta \frac{1}{\sqrt{4\pi\omega}} e^{i(+\omega\eta + k\xi)} = -i\omega \frac{1}{\sqrt{4\pi\omega}} e^{i(+\omega\eta + k\xi)} = -i\omega g^{(2)} \tag{17.55}$$

So we conclude that the Rindler observer can use the following modes:

$$\begin{aligned} \textbf{R I}: &\qquad g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}} e^{i(-\omega\eta + k\xi)} \qquad \text{and} \qquad g_k^{(2)} = 0 \\ \textbf{R IV}: &\qquad g_k^{(1)} = 0 \qquad \text{and} \qquad g_k^{(2)} = \frac{1}{\sqrt{4\pi\omega}} e^{i(+\omega\eta + k\xi)} \end{aligned} \tag{17.56}$$

These form a complete basis set for the Rindler observer and we can expand the field $\varphi(x)$ in terms of these modes:

$$\varphi(x) = \int dk \, \left[\hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)\dagger} g_k^{(1)*} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)\dagger} g_k^{(2)*} \right] \tag{17.57}$$

The observer in the inertial frame, let us call him the Minkowski observer, will have a mode expansion in plane waves in the t-x coordinate system:

$$\varphi(x) = \int dk \, \left[\hat{a}_k f_k + \hat{a}_k^\dagger g_k^* \right] \tag{17.58}$$

The Minkowski observer will have a vacuum state $|0\rangle_\mathsf{M}$ defined by

$$\hat{a}_k \, |0\rangle_\mathsf{M} = 0 \tag{17.59}$$

and sometimes called the **Boulware Vacuum**. The Rindler observer will have a vacuum $|0\rangle_\mathsf{R}$ defined by

$$\hat{b}_k^{(1)} \, |0\rangle_\mathsf{R} = \hat{b}_k^{(2)} \, |0\rangle_\mathsf{R} = 0 \tag{17.60}$$

and sometimes called the **Kruskal Vacuum**.

The Rindler observer will not see the Minkowski vacuum as containing zero particles as his annihilation operators are a mix of the Minkowski annihilation and creation operators. Two work out what the Rindler observer sees, we just need to find the Bogoliubov parameters that transform one set of modes into the other set of modes. This is straightforward algebra but rather lengthy. We will use an alternative way to achieve this, originally due to Unruh. We will find a set of modes that share the same vacuum state as the Minkowski modes, albeit with different description of the excited modes. But these new modes will have a simpler overlap with the Rindler modes and hence the Bogoliubov parameters will be easier to calculate. The way to do this is to analytically continue the Rindler modes over all of spacetime and express this extension in terms of the original Rindler modes.

In order to achieve this, first note that from (17.45) we find for **R I** that

$$\begin{aligned} \alpha(x - t) &= e^{\alpha\xi} \left(\cosh \alpha\eta - \sinh \alpha\eta \right) = e^{-\alpha(\eta - \xi)} \\ \alpha(x + t) &= e^{\alpha\xi} \left(\cosh \alpha\eta + \sinh \alpha\eta \right) = e^{+\alpha(\eta + \xi)} \end{aligned} \tag{17.61}$$

and similarly for **R IV**

$$\alpha(-x+t) = -e^{\alpha\xi}\left(-\cosh\alpha\eta + \sinh\alpha\eta\right) = e^{-\alpha(\eta-\xi)}$$
$$\alpha(-x-t) = -e^{\alpha\xi}\left(-\cosh\alpha\eta - \sinh\alpha\eta\right) = e^{+\alpha(\eta+\xi)} \tag{17.62}$$

If we now assume $k > 0$ and so $\omega = k$ then we can write for $g^{(1)}$ in **RI** :

$$\sqrt{4\pi\omega}g_k^{(1)} = e^{i(-\omega\eta+k\xi)} = \alpha^{i\omega/\alpha}(x-t)^{i\omega/\alpha} \tag{17.63}$$

We can analytically continue this expression for $g_k^{(1)}$ outside of **R I** to everywhere in Minkowski spacetime, by using the appropriate values of t and x. Similarly, we find, assuming $k > 0$ and so $\omega = k$ in **R IV** for the complex conjugate of $g^{(2)}$ with minus the momentum component:

$$\sqrt{4\pi\omega}g_{-k}^{(2)*} = e^{-i(+\omega\eta-k\xi)} = \alpha^{i\omega/\alpha}e^{\pi\omega/\alpha}(x-t)^{i\omega/\alpha} \tag{17.64}$$

Which again we can analytically continue to the whole Minkowski spacetime. We thus have a well defined combination over the whole of Minkowski spacetime given by:

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + e^{-\pi\omega/\alpha}g_{-k}^{(2)*}\right) = 2\alpha^{i\omega/\alpha}(x-t)^{i\omega/\alpha} \tag{17.65}$$

Now if we take $k < 0$ such that $\omega = -k$, we find similarly in **R I**:

$$\sqrt{4\pi\omega}g_k^{(1)} = e^{i(-\omega\eta+k\xi)} = \alpha^{i\omega/\alpha}(x+t)^{i\omega/\alpha} \tag{17.66}$$

and in **R IV**:

$$\sqrt{4\pi\omega}g_{-k}^{(2)*} = e^{-i(+\omega\eta-k\xi)} = e^{-i\omega(\eta+\xi)} = \alpha^{i\omega/\alpha}(-x-t)^{i\omega/\alpha}$$
$$= \alpha^{i\omega/\alpha}\left[e^{-i\pi}(x+t)\right]^{i\omega/\alpha} = \alpha^{i\omega/\alpha}e^{\pi\omega/a}(x+t)^{i\omega/a} \tag{17.67}$$

and the combination

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + e^{-\pi\omega/\alpha}g_{-k}^{(2)*}\right) = 2\alpha^{i\omega/\alpha}(x+t)^{i\omega/\alpha} \tag{17.68}$$

is again well defined over the whole of Minkowski spacetime. We can thus use this combination as a basis for the modes of the Rindler observer. In fact, it is better to take a properly normalised combination and define the modes $h_k^{(1)}$ as

$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left(e^{\pi\omega/2\alpha}g_k^{(1)} + e^{-\pi\omega/2\alpha}g_{-k}^{(2)*}\right) \tag{17.69}$$

We find the conjugate mode from a similar calculation. First for $k > 0$, combining the regions in one equation, hence upsetting some people who may prefer more rigour mortis,

$$\sqrt{4\pi\omega}(g_k^{(2)} + e^{-\pi\omega/\alpha}g_{-k}^{(1)*}) = e^{i\omega(\eta+\xi)}\Big|_{\text{R IV}} + e^{-\pi\omega/\alpha}e^{-i\omega(-\eta-\xi)}\Big|_{\text{R I}}$$
$$= 2\alpha^{i\omega/\alpha}e^{-\pi\omega/\alpha}(x+t)^{i\omega/\alpha} \tag{17.70}$$

Similarly for $k < 0$ we find

$$\sqrt{4\pi\omega}(g_k^{(2)} + e^{-\pi\omega/\alpha} g_{-k}^{(1)*}) = e^{i\omega(\eta-\xi)}\Big|_{\text{R IV}} + e^{-\pi\omega/\alpha} e^{i\omega(\eta-\xi)}\Big|_{\text{R I}}$$
$$= 2\alpha^{-i\omega/\alpha} e^{-\pi\omega/\alpha}(x-t)^{-i\omega/\alpha} \tag{17.71}$$

So that properly normalised we can define

$$h_k^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}} \left(e^{\pi\omega/2\alpha} g_k^{(2)} + e^{-\pi\omega/2\alpha} g_{-k}^{(1)*} \right) \tag{17.72}$$

as well over the entire Minkowski-space. We can easily check that the modes $h_k^{(1)}$ and $h_k^{(2)}$ form an orthonormal set, as log as the modes $g_k^{(1)}$ and $g_k^{(2)}$ form an orthonormal set. For example we have

$$\begin{aligned}
(h_{k_1}^{(1)}, h_{k_2}^{(1)}) &= \frac{1}{2\sqrt{\sinh(\pi\omega_1/a)\sinh(\pi\omega_2/\alpha)}} \\
&\quad \times \left(e^{\pi\omega_1/2\alpha} g_{k_1}^{(1)} + e^{-\pi\omega_1/2\alpha} g_{-k_1}^{(2)*}, e^{\pi\omega_1/2\alpha} g_{k_2}^{(1)} + e^{-\pi\omega_1/2\alpha} g_{-k_2}^{(2)*} \right) \\
&= \frac{1}{2\sqrt{\sinh(\pi\omega_1/a)\sinh(\pi\omega_2/\alpha)}} \left[e^{\pi(\omega_1+\omega_2)/2\alpha} - e^{-\pi(\omega_1+\omega_2)/2\alpha} \right] \delta(k_1 - k_2) \\
&= \frac{1}{2\sinh(\pi\omega_1/a)} \left[e^{\pi\omega_1/a} - e^{-\pi\omega_1/a} \right] \delta(k_1 - k_2) = \delta(k_1 - k_2)
\end{aligned} \tag{17.73}$$

A similar calculation shows that $(h_{k_1}^{(2)}, h_{k_2}^{(2)}) = \delta(k_1 - k_2)$ as well and that $(h_{k_1}^{(1)}, h_{k_2}^{(2)}) = 0$.

We now have a third set of modes, also expressed in terms of Rindler coordinates, but this time the formula is valid for the entire spacetime by analytical continuation. Again we can expand the field in these modes:

$$\varphi(x) = \int dk \, \left[\hat{c}_k^{(1)} h_k^{(1)} + \hat{c}_k^{(1)\dagger} h_k^{(1)*} + \hat{c}_k^{(2)} h_k^{(2)} + \hat{c}_k^{(2)\dagger} h_k^{(2)*} \right] \tag{17.74}$$

From our Bogoliubov transformation we know that the annihilation and creation operators of one set of modes can be written in terms of the other set of modes. Let us remind ourselves of this. If we have an expansion in two set of modes

$$\varphi(x) = \sum_i \left[\hat{a}_i f_i + \hat{a}_i^\dagger f_i^* \right] = \sum_i \left[\hat{b}_i g_i + \hat{b}_i^\dagger g_i^* \right] \tag{17.75}$$

and the different modes are related by (17.21):

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^* \right) \tag{17.76}$$

and its complex conjugate, then the annihilation and creation operators are related by (17.28)

$$\hat{a}_j = \sum_i (\alpha_{ij}\hat{b}_i + \beta^*_{ij}\hat{b}^\dagger_i) \qquad \text{and} \qquad \hat{a}^\dagger_j = \sum_i (\beta_{ij}\hat{b}_i + \alpha^*_{ij}\hat{b}^\dagger_i) \tag{17.77}$$

Let us rewrite the transformation rules for convenience:

$$\begin{aligned} h_k^{(1)} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left(e^{\pi\omega/2\alpha} g_k^{(1)} + e^{-\pi\omega/2\alpha} g_{-k}^{(2)*}\right) \\ h_k^{(2)} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left(e^{\pi\omega/2\alpha} g_k^{(2)} + e^{-\pi\omega/2\alpha} g_{-k}^{(1)*}\right) \end{aligned} \tag{17.78}$$

We see that the indices in (17.76) now run over the discrete values 1 and 2 and over the continuous k. The only non-zero Bogoliubov parameters can be read off straightforwardly:

$$\begin{aligned} \alpha_{kk}^{11} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}} e^{+\pi\omega/2\alpha}; \qquad \alpha_{kk}^{22} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}} e^{+\pi\omega/2\alpha} \\ \beta_{k-k}^{12} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}} e^{-\pi\omega/2\alpha}; \qquad \beta_{k-k}^{21} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}} e^{-\pi\omega/2\alpha} \end{aligned} \tag{17.79}$$

This gives the transformation rules for the annihilation operators

$$\begin{aligned} \hat{b}_k^{(1)} &= \alpha_{kk}^{11}\hat{c}_k^{(1)} + \beta_{k-k}^{12}\hat{c}_{-k}^{(2)\dagger} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left[e^{+\pi\omega/2\alpha}\hat{c}_k^{(1)} + e^{-\pi\omega/2\alpha}\hat{c}_{-k}^{(2)\dagger}\right] \\ \hat{b}_k^{(2)} &= \alpha_{kk}^{22}\hat{c}_k^{(2)} + \beta_{k-k}^{21}\hat{c}_{-k}^{(1)\dagger} = \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left[e^{+\pi\omega/2\alpha}\hat{c}_k^{(2)} + e^{-\pi\omega/2\alpha}\hat{c}_{-k}^{(1)\dagger}\right] \end{aligned} \tag{17.80}$$

and the creation operators follow from the hermitian conjugate

$$\begin{aligned} \hat{b}_k^{(1)\dagger} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left[e^{+\pi\omega/2\alpha}\hat{c}_k^{(1)\dagger} + e^{-\pi\omega/2\alpha}\hat{c}_{-k}^{(2)}\right] \\ \hat{b}_k^{(2)\dagger} &= \frac{1}{\sqrt{2\sinh(\pi\omega/\alpha)}}\left[e^{+\pi\omega/2\alpha}\hat{c}_k^{(2)\dagger} + e^{-\pi\omega/2\alpha}\hat{c}_{-k}^{(1)}\right] \end{aligned} \tag{17.81}$$

Let us look at the modes of the Minkowski observer. The original positive energy frequency modes are, for $k > 0$ given by

$$f_k \propto e^{-i(\omega t - kx)} = e^{-i\omega(t-x)} \tag{17.82}$$

If we look at this in the complex t-x plane than we note that it is analytical, We also see that if we write $t - x = \alpha_1 + i\alpha_2$ with α_1 and α_2 real we find

$$f_k \propto e^{-i\omega(\alpha_1 + i\alpha_2)} = e^{-i\omega\alpha_1} e^{\omega\alpha_2} \tag{17.83}$$

and so this analytical function is bounded as long as $\alpha_2 = \text{Im}\,(t-x) < 0$. Let us now look at $h_k^{(1)}$. From its definition (17.69) and the fact that we have worked out that for $k > 0$ it can be written as (17.65)

$$h_k^{(1)} \propto (x-t)^{i\omega/\alpha} = \exp\frac{i\omega}{\alpha}\ln(x-t) \qquad (17.84)$$

This function has a branch cut in the complex plane.[3] Let us again look at the imaginary part of $x-t$. Write $x-t = -i\beta$. We then have

$$\ln(x-t) = \ln -i\beta = \ln e^{-\frac{i\pi}{2}}\beta = -\frac{i\pi}{2} + \ln\beta \qquad (17.85)$$

This function is bounded as long as $\beta = \text{Im}\,(t-x) < 0$ and is analytical as long as we put the branch cut in the upper half complex place.

Similarly we find find that the modes $h_k^{(2)}$ are analytic and bounded in the lower half complex plane and so are the positive energy frequency modes of the inertial observer with $k < 0$.

This means that we can express the modes $h_k^{(1)}$ and $h_k^{(2)}$ in terms of the positive energy frequency modes f_k. In terms of the Bogoliubov parameters it means that the coefficients β_{ij} are zero and that the annihilation operators of the h-modes can be written exclusively in terms of the annihilation operators of the f-modes, and hence that the annihilation and creation operators don't get mixed up in this Bogoliubov transformation.

We thus have immediately that acting with the annihilation operators of the g-modes on the Minkowski vacuum $|0\rangle_{\text{M}}$ we find

$$\hat{c}_k^{(i)}\,|0\rangle_{\text{M}} \propto \hat{a}_k\,|0\rangle_M = 0 \qquad \text{for} \qquad i = 1,2 \qquad (17.86)$$

Let us ask how the Rindler observer experiences the Minkowski vacuum? A Rindler observer in **R I** will count particles of momentum k with his number operator $\hat{n}_R^{(1)}(k) = \hat{b}_k^{(1)}\hat{b}_k^{(1)\dagger}$. We can now simply use the transformation law (17.81) that expresses the b-operators in terms of the c-operators and use the fact that the Minkowski vacuum is

[3] $\ln z = \ln re^{i}\theta = \ln r + i\theta$ has a branch cut as $z(r,\theta+2\pi) = z(r,\theta)$ but $\ln z(r,\theta+2\pi) = \ln z(r,\theta) + 2\pi$. It is traditional to put this branch cut on the real axis, $\theta = 0$, but we can of course put it anywhere starting at the origin. The important point being that if we rotate θ by 2π we cross the branch cut.

annihilated by the c operators:

$$
\begin{aligned}
{}_{\mathsf{M}}\langle 0|\, \hat{n}_R^{(1)}(k)\, |0\rangle_{\mathsf{M}} &= {}_{\mathsf{M}}\langle 0|\, \hat{b}_k^{(1)} \hat{b}_k^{(1)\dagger}\, |0\rangle_{\mathsf{M}} \\
&= \frac{1}{2\sinh(\pi\omega/\alpha)}\; {}_{\mathsf{M}}\langle 0| \left[e^{+\pi\omega/2\alpha} \hat{c}_k^{(1)\dagger} + e^{-\pi\omega/2\alpha} \hat{c}_{-k}^{(1)} \right] \\
&\qquad \times \left[e^{+\pi\omega/2\alpha} \hat{c}_k^{(1)} + e^{-\pi\omega/2\alpha} \hat{c}_{-k}^{(1)\dagger} \right] |0\rangle_{\mathsf{M}} \\
&= \frac{1}{2\sinh(\pi\omega/\alpha)} e^{-\pi\omega/a}\; {}_{\mathsf{M}}\langle 0|\, \hat{c}_{-k}^{(1)} \hat{c}_{-k}^{(1)\dagger}\, |0\rangle_{\mathsf{M}} \\
&= \frac{1}{2\sinh(\pi\omega/\alpha)} e^{-\pi\omega/a}\; {}_{\mathsf{M}}\langle 0|\, \delta(0)\, |0\rangle_{\mathsf{M}} = \frac{1}{e^{2\pi\omega/\alpha} - 1} \delta(0)
\end{aligned}
\tag{17.87}
$$

The delta function may seem to compromise this result, but it is a consequence of the fact that we have used single-modes in our computation. The right analysis would have required us to use wave-packets concentrated around momentum k, but this would complicate the calculations needlessly. This is an important result, so let us rewrite it, dropping the inconvenient delta function

$$
{}_{\mathsf{M}}\langle 0|\, \hat{n}_R^{(1)}(k)\, |0\rangle_{\mathsf{M}} = \frac{1}{e^{2\pi\omega/\alpha} - 1} \tag{17.88}
$$

The accelerating Rindler observer does not see the Minkowski vacuum as empty but he sees a whole spectrum of particles with all possible momenta k and the number of particles he sees with momenta k is given by

$$
P(\omega) \propto \frac{1}{e^{2\pi\omega/a} - 1} \tag{17.89}
$$

with $\omega = |k|$. This precisely the Planck spectrum of black-body radiation, which for frequency ν and temperature T is given by

$$
P(\nu) \propto \frac{1}{e^{h\nu/k_B T} - 1} \tag{17.90}
$$

Setting dimensionless units $h = k_B = 1$ and equation ω with ν we find that the accelerating Rindler observer travelling through the Minkowski vacuum will see a Planck spectrum with temperature

$$
T = \frac{\alpha}{2\pi} \tag{17.91}
$$

This is the famous **Unruh Effect**. The fact that the radiation seen by the Rindler observer follows Planck's law (17.89) does, of course, not not necessarily mean that the radiation is effectively thermal. To be thermal one should check that there are no correlations in the observed particles. This is outside our scope, but it has indeed been checked.

The temperature $T = \alpha/2\pi$ is the temperature measured by an observer accelerating at a constant rate α. From (17.47) we know that this observer moves along a path with $\xi = 0$. Any other path with ξ constant feels a constant acceleration $\tilde{\alpha}$ given by

$$\tilde{\alpha} = \alpha e^{-\alpha\xi} \tag{17.92}$$

and this observer should measure thermal radiation of a temperature $T = \tilde{\alpha}/2\pi$. An alternative way to express this is to say that if an observer at $\xi = \xi_1 = 0$ detects a temperature $T_1 = \alpha/2\pi$ then an observer at $\xi = \xi_2$ will see this radiation redshifted to a temperature

$$T_2 = \frac{\tilde{\alpha}}{2\pi} = \frac{\alpha e^{-\alpha\xi}}{2\pi} = e^{-\alpha\xi} T_1 \tag{17.93}$$

To a naive physicist the Unruh effect is a paradox. Indeed a Minkowski observer sees a vacuum and more precisely she sees the expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$ to be identical to zero. However a Rindler observer sees a whole spectrum of particles. But if the energy-momentum tensor is zero how can particles be created? Does this not violate energy conservation? The answer is, of course, that to measure the particles the Rindler observer needs a detector. And to keep that detector moving at constant acceleration one needs to put in work. That work provides the energy for the particles observed by the Rindler observer and energy is conserved, as it should be.

17.3 Hawking Radiation

One can also turn the argument around and consider the situation from the point of view of an observer in flat spacetime, e.g. an observer very far away from a strong gravitational field, looking at a detector in that gravitational field. For the external observer that detector is accelerating and so it needs to use the Rindler vacuum. But the external observer will use her own Minkowski vacuum and thus will see a thermal radiation at the detector; gravitational fields produce thermal radiation. This is the cause of the so-called **Hawking Radiation**. A black hole creates a gravitational field and so will cause thermal radiation, also within the event horizon. Classically such radiation cannot go beyond the event horizon, but quantum mechanically there is always a non-zero probability that it will tunnel through it. Thus we should expect black holes to radiate.

We can make this relation between the accelerated Rindler observer and the horizon of a black hole more precise. Consider the metric for this observer (17.48), i.e. $ds^2 =$

$e^{2\alpha\xi}(-d\eta^2 + d\xi^2)$. Change variables $\rho = \alpha^{-1}e^{\alpha\xi}$ so that $d\rho = e^{\alpha\xi}d\xi$ and

$$ds^2 = -\alpha^2\rho^2 d\eta^2 + d\rho^2 \tag{17.94}$$

Let us now consider the Schwarzschild metric near the Schwarzschild radius $r_S = 2GM$. Ignoring the angular part

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1}dt^2 \approx -\frac{r - r_S}{r}dt^2 + \frac{r_S}{r - r_S}dr^2 \tag{17.95}$$

Change variables $\rho^2 = 4r_S(r - r_S)$ so that $\rho d\rho = 2r_s dr$ and the metric becomes

$$ds^2 = -\frac{\rho^2}{4r_s^2}dt^2 + d\rho^2 \tag{17.96}$$

The metric of the accelerating Rindler observer is thus of the same form as the metric near the event horizon of a black hole and we would thus expect Unruh radiation near the horizon of the black hole as well.

It is in fact possible to determine the equation for the temperature of the thermal radiation up to a numerical factor, purely on dimensional grounds. Consider the possible variables we have in the system. We have the mass of the black hole M with dimension $[M] = +1$, but this always comes in combination with Newton's constant as GM and has dimension $[GM] = -1$. In natural units temperature has units of energy and hence of mass, $[T] = +1$. Therefore necessarily[4] $T \propto 1/GM$. Filling in the numerical factors we find

$$T = \frac{\hbar c^3}{8\pi GMk_b} \tag{17.97}$$

Some remarks are in order.

- Hawking radiation is often presented as the creation of virtual pairs of photons near the event horizon, with one photon staying behind the horizon and one crossing it. This is not a tenable picture as virtual particles are a purely mathematical construct, following from our choice to solve the evolution equation in field theory via perturbation theory, and do not exist in reality, let alone that one could go left and the other could go right.
- As we enter into the event horizon the signs of g_{00} and g_{rr} flip. It looks like the role of time and distance are interchagned. Strange as the person falling crossing the event horizon does not feel anything particular.

[4]We are ignoring the fact that we could also have an arbitrary function of the dimensionless combination GMM_P with M_P the Planck mass.

- What is the entropy associated with this Hawking radiation? From thermodynamics we know that $dE = TdS - pDV - \mu dN$, which becomes, ignoring volume and particle number changes becomes $dE = Td$. But E is just the mass of the black holes, so $dM = TdS$ or $dS/dM = 1/T \propto GM$. Integrating we get $S \propto GM^2 \propto (M/M_p)^2$, where we have expressed the entropy in terms of a dimensionless combination. But the radius of the horizon is given by $R_S = 2GM$ so we find that $S \propto R^2$, i.e. the entropy is proportional to the area of the black hole. This is remarkable as entropy is an extensive quantity and is expected to be proportional to a volume and not an area. This observation has led to the **Holographic Principle** that states that many theories in a bulk, can be fully determined by another theory as a boundary.

- If a black hole radiates, it should radiate completely until nothing of it is left, but pure radiation. This is the origin of the **Information Paradox**. With the matter falling into the black hole is associated a certain amount of information: e.g. correlation functions. But if the whole of a black hole evaporates and becomes thermal radiation that is uncorrelated and hence carries no information, then the initial information is lost. Another way to look at this is that in quantum mechanics one assumes that a the evolution of a wavefunction to the future or in the past is determined by the action of a unitary operator. But if all black holes, irrespective of what has gone into them, eventually turn into the same state of thermal radiation, then this assumption is not valid. There are some possible solutions to this paradox though. String theory and the AdS/CFT correspondence e.g. predict that the thermal radiation is not exactly thermal but receives corrections that preserve the information.

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